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SINGULAR PERTURBATIONS IN OPTIMAL CONTROL*

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ABSTRACT. Optimal control is a new area for applications of singular perturbation methods. Two such applications to optimal regulators and trajectory optimization problems are briefly outlined.

Introduction. Although many control theory concepts are valid for any system order, their actual use is limited to low order models. In optimization of dynamic systems the "curse of dimensionality" is not only in a formidable amount of computation, but also in the ill-conditioned initial and two point boundary value problems. The interaction of fast and slow phenomena in high-order systems results in "stiff" numerical problems which require expensive integration routines.

The singular perturbation approach outlined in this survey alleviates both dimensionality and stiffness difficulties. It lowers the model order by first neglecting the fast phenomena. It then improves the approximation by reintroducing their effect as "boundary layer" corrections calculated in separate time scales. Further improvements are possible by asymptotic expansion methods. In addition to being helpful in design procedures, the singular perturbation approach is an indispensable tool for analytical investigations of robustness of system properties, behavior of optimal controls near singular arcs, and other effects of intentional or unintentional changes of system order.

Suppose that a dynamic system is modeled by

$$\dot{x} = f(x, z, u, t, \mu) \tag{1}$$

$$\mu\dot{z} = g(x, z, u, t, \mu) \tag{2}$$

where $\mu > 0$ is a scalar and x, z and u are n-, m-, and r-dimensional vectors, respectively. For $\mu = 0$, the order $n + m$ of (1), (2) reduces to n, that is (2) becomes

$$0 = g(\bar{x}, \bar{z}, \bar{u}, t, 0) \tag{3}$$

and the substitution of a root of (3),

$$\bar{z} = \varphi(\bar{x}, \bar{u}, t), \tag{4}$$

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into (1) yields a "reduced" model

$$\dot{\bar{x}} = f[\bar{x}, \varphi(\bar{x}, \bar{u}, t), \bar{u}, t, 0] \equiv \bar{f}(\bar{x}, \bar{u}, t). \tag{5}$$

Most of the available theory is restricted to models (4) corresponding to real and distinct roots of (3), along which $\partial g/\partial z$ is nonsingular. For a linear system

$$\dot{x} = A_{11}x + A_{12}z + B_1 u \tag{6}$$

$$\mu\dot{z} = A_{21}x + A_{22}z + B_2 u \tag{7}$$

the root (4) is

$$\bar{z} = -A_{22}^{-1} A_{21}\bar{x} - A_{22}^{-1} B_2\bar{u}, \tag{8}$$

yielding the reduced model

$$\dot{\bar{x}} = (A_{11} - A_{12}A_{22}^{-1} A_{21})\bar{x} + (B_1 - A_{12}A_{22}^{-1} B_2)\bar{u}. \tag{9}$$

In applications, models of various physical systems are put in form (1), (2) by expressing small time constants T_i, small masses m_j, large gains K_ℓ etc., as $T_i = c_i\mu$, $m_j = c_j\mu$, $K_\ell = (c_\ell/\mu)$ etc., where c_i, c_j, c_ℓ are known coefficients [7]. In power system models μ can represent machine reactances or transients in voltage regulators [16], in industrial control systems it may represent time-constants of drives and actuators [4], in biochemical models μ can indicate a small quantity of an enzyme [3], in a flexible booster model μ is due to bending modes [1], and in nuclear reactor models it is due to fast neutrons [13]. Singular perturbations are extensively used in aircraft and rocket flight models [5, 6].

Regulators and Riccati Equations. Among the most actively investigated singularly perturbed optimal control problems is the general linear-quadratic regulator problem. For brevity we consider only the time-invariant case. When the system (6), (7) is optimized with respect to

$$J = 1/2 \int_0^\infty (y'y + u'Ru)\, dt \tag{10}$$

where $y = C_1 x + C_2 z$ and $R > 0$, then to implement the optimal control

$$u = -R^{-1}[B_1{}'(1/\mu)B_2{}']\, K \begin{bmatrix} x \\ z \end{bmatrix} \tag{11}$$

we have to solve

$$(12)\qquad K\begin{bmatrix} A_{11} & A_{12} \\ \dfrac{A_{21}}{\mu} & \dfrac{A_{22}}{\mu} \end{bmatrix} + \begin{bmatrix} A'_{11} & \dfrac{A'_{21}}{\mu} \\ A'_{12} & \dfrac{A'_{22}}{\mu} \end{bmatrix} K$$

$$- K\begin{bmatrix} B_1 \\ \dfrac{B_2}{\mu} \end{bmatrix} R^{-1}\begin{bmatrix} B_1{}' & \dfrac{B_2{}'}{\mu} \end{bmatrix} K + C'C = 0,$$

where $C = [C_1\ C_2]$. To avoid unboundedness as $\mu \to 0$ the solution is sought in the form

$$(13)\qquad K = K(\mu) = \begin{bmatrix} K_{11}(\mu) & \mu K_{12}(\mu) \\ \mu K'_{12}(\mu) & \mu K_{22}(\mu) \end{bmatrix}$$

which permits us to set $\mu = 0$ in (12). At $\mu = 0$ an $m \times m$ equation for $\bar{K}_{22}$,

$$(14)\qquad \bar{K}_{22}A_{22} + A'_{22}\bar{K}_{22} - \bar{K}_{22}S_2\bar{K}_{22} + C_2{}'C_2 = 0,$$

where $S_2 = B_2R^{-1}B_2{}'$, separates from the $(n + m) \times (n + m)$ equation (12). If A_{22}, B_2 is a stabilizable pair, and if A_{22}, C_2 is a detectable pair, then a unique positive semidefinite solution $\bar{K}_{22}$ exists and the eigenvalues of $A_{22} - S_2\bar{K}_{22}$ have negative real parts. Another result of the substitution of (13) into (12) is that at $\mu = 0$ it is possible to express $\bar{K}_{12}$ in terms of $\bar{K}_{11}$ and $\bar{K}_{22}$, and to obtain an $n \times n$ equation for $\bar{K}_{11}$,

$$(15)\qquad \bar{K}_{11}\hat{A} + \hat{A}'\bar{K}_{11} - \bar{K}_{11}\hat{B}R^{-1}\hat{B}'\bar{K}_{11} + \hat{C}'\hat{C} = 0.$$

The expressions for $\hat{A}$, $\hat{B}$ and $\hat{C}$ are given in [8]. An interpretation of (14) and (15) is that (14) yields a "boundary layer regulator", and (15) yields the regulator for the reduced state variable $\bar{x}(t)$. For $\hat{A}$, $\hat{B}$ stabilizable and $\hat{A}$, $\hat{C}$ detectable, the implicit function theorem applied to (12) with (13) shows that

$$(16)\qquad K_{ij} = \bar{K}_{ij} + O(\mu) \qquad i, j = 1, 2.$$

Not only are the approximations $\bar{K}_{ij}$ calculated from lower order equations, but in addition the ill-conditioning of (12) has been removed.

The singularly perturbed regulator problem was posed in [14] with $C_2 = 0$ and A_{22} stable, which gave $\bar{K}_{22} = 0$. The general time-varying problem was treated in [8] using the notion of boundary layer controllability-observability. These results and extensions [19, 9] are based on the singularly perturbed differential Riccati equation. An alternative approach via boundary value problems is presented in [10], its relationship with the Riccati approach is analyzed in [11].

Trajectory Optimization. In trajectory optimization problems for the system (1), (2) some conditions are imposed on x, z at both $t = t_0$ and $t = T$, and a control $u(t)$ is sought to minimize the performance index

$$J = \int_{t_0}^{T} V(x, z, u, t)\, dt. \tag{17}$$

An optimal solution must satisfy $H_u = 0$ and

$$\dot{x} = H_p, \qquad \dot{p} = -H_x \tag{18}$$

$$\mu\dot{z} = H_q, \qquad \mu\dot{q} = -H_z, \tag{19}$$

with $2n + 2m$ boundary conditions. Here H_u, H_x, H_z, $H_p = f$, $H_q = g$, denote the partial derivatives of the Hamilton $H = V + p'f + q'g$, and the adjoint vairables for (1) and (2) are p and μq, respectively. At $\mu = 0$ we use $H_q = 0$ and $H_z = 0$ to eliminate z and q from (18) and to get the reduced system

$$\dot{\bar{x}} = \bar{H}_p, \qquad \dot{\bar{p}} = -\bar{H}_x \tag{20}$$

for which only $2n$ conditions can be imposed. Suppose that they are uniquely satisfied by a continuously differentiable reduced solution $\bar{x}(t)$, $\bar{p}(t)$. Since the reduced variables $\bar{z}(t)$, $\bar{q}(t)$ obtained from $H_q = 0$, $H_z = 0$ may not satisfy the remaining $2m$ conditions, corrections $\eta_L(\tau)$, $\eta_R(\sigma)$ for z, and $\rho_L(\tau)$, $\rho_R(\sigma)$ for q, are to be determined from appropriately defined layer systems

$$\frac{d\eta_L}{d\tau} = \tilde{H}_q(\eta_L, \rho_L), \qquad \frac{d\rho_L}{d\tau} = -\tilde{H}_z(\eta_L, \rho_L), \quad \tau = (t - t_0)/\mu \tag{21}$$

$$\frac{d\eta_R}{d\sigma} = \tilde{H}_q(\eta_R, \rho_R), \qquad \frac{d\rho_R}{d\sigma} = -\tilde{H}_z(\eta_R, \rho_R), \quad \sigma = (t - T)/\mu \tag{22}$$

where (21) is used at $t = t_0$ and (22) at $t = T$. To be specific consider the problem with fixed end points,

$$z(t_0) = z^0, \qquad z(T) = z^T. \tag{23}$$

Then the initial values for η_L and η_R are

$$\eta_L(0) = z^0 - \bar{z}(t_0), \qquad \eta_R(0) = z^T - \bar{z}(T) \tag{24}$$

and the additional boundary conditions are

(25) $\eta_L, \rho_L \to 0, \tau \to \infty; \qquad \eta_R, \rho_R \to 0, \sigma \to -\infty.$

Existence of optimal solutions and their approximation by reduced solutions have been investigated in [2] and extended in [12] by a construction of asymptotic expansions. Unfortunately the applicability of these results is restricted by the requirement that $\eta_L(0)$ and $\eta_R(0)$ be sufficiently small. To what extent such restrictions can be avoided in a general nonlinear problem (1), (2) and (17) is still an open question. Results without restrictions on z^0, z^T are available for linear time-varying systems [18] and for a special class of nonlinear systems [15]. The results of [18] are briefly outlined here.

Let the performance index be (10), but on the interval $[t_0, T]$, and consider the trajectory optimization problem for (6), (7) allowing that the matrices in (6), (7) and (10) be time-varying. Using a "dichotomy transformation" proposed in [17]

(26) $$x = \ell_1 + r_1, \qquad z = \ell_2 + r_2$$

(27) $$\begin{bmatrix} p \\ q \end{bmatrix} = P(t) \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} + N(t) \begin{bmatrix} r_1 \\ r_2 \end{bmatrix},$$

where $P(t)$ is a positive definite and $N(t)$ is a negative definite solution of a differential equation analogous to (12), we transform (21), (22) into two separate "layer regulator systems"

(28) $$\frac{d\eta_L}{d\tau} = [A_{22}(t_0) - S_{22}(t_0)P_{22}(t_0)]\eta_L$$

(29) $$\frac{d\eta_R}{d\sigma} = [A_{22}(T) - S_{22}(T)N_{22}(T)]\eta_R$$

where $\eta_L = \ell_2 - \bar{\ell}_2$, $\eta_R = r_2 - \bar{r}_2$ and $P_{22}(t_0)$, $N_{22}(T)$ are the positive and the negative definite roots of (14) at t_0 and T. If for all $t \in [t_0, T]$

(30) $$\operatorname{rank}[B_{22}, A_{22}B_2, \cdots, A_{22}^{m-1} B_2] = m$$

(31) $$\operatorname{rank}[C_2', A_{22}'C_2', \cdots, A_{22}'^{m-1} C_2'] = m$$

then the approximations

(32) $$x(t) = \bar{x}(t) + O(\mu)$$

(33) $$z(t) = \bar{z}(t) + \eta_L(\tau) + \eta_R(\sigma) + O(\mu)$$

(34) $$p(t) = \bar{p}(t) + O(\mu)$$

(35) $$q(t) = \bar{q}(t) + P_{22}(t_0)\eta_L + N_{22}(T)\eta_R + O(\mu)$$

hold for arbitrary boundary values z^o, z^T since (28), (29) satisfy the dichotomy condition [17]. A less restrictive stabilizability-detectability condition can be used instead of (30), (31). This result of [18] delineates a class of well posed singularly perturbed trajectory optimization problems. The use of $-R^{-1}B_2{}'P_{22}z + u^0$, results in a stable feedback realization of the initial layer and $u^0 = R^{-1}B_2{}'(P_{22} - N_{22})r_2$ is open-loop control of the endlayer.

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