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SOME CONVERGENCE THEOREMS FOR MULTIPOINT BOUNDARY VALUE PROBLEMS IN $\lambda(n, k)$-PARAMETER FAMILIES

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1. **Introduction.** Assume n and k are integers, $n \geqq 2$ and $1 \leqq k \leqq n$. Let $\lambda(n, k) = (\lambda(1), \cdots, \lambda(k))$ be an ordered k-tuple of positive integers satisfying $\lambda(1) + \cdots + \lambda(k) = n$, which we call an *ordered k-partition* of n. Suppose $\lambda(n, k)$ is a fixed ordered k-partition of n and $F \subset C^j(I)$ where $I \subset R$ is an interval and $j > 0$ is large enough so that the following definitions make sense.

Definition 1.1. F is said to be a *$\lambda(n, k)$-parameter family* on I if for every set of k distinct points $x_1 < x_2 < \cdots < x_k$ in I and every set of n real numbers y_{ir} there exists a unique $f \in F$ satisfying

$$f^{(r)}(x_i) = y_{ir}, r = 0, 1, \cdots, \lambda(i) - 1, i = 1, \cdots, k. \tag{1.1}$$

Let $P(\mathrm{n})$ denote the set of all ordered k-partitions $\lambda(n, k)$ of n with k varying such that $1 \leqq k \leqq n$. If $1 \leqq m \leqq k$ is fixed we shall define $\{\lambda(n, k; m)\} = \{\mu(n, j) \in P(n) : \mu(n, j)$ is obtained from $\lambda(n, k)$ by writing $\lambda(m) - 1$ in the place of $\lambda(m)$ and inserting the integer 1 in any one of the $k + 1$ possible places in the ordered array $(\lambda(1), \cdots, \lambda(m - 1), \lambda(m) - 1, \lambda(m + 1), \cdots, \lambda(k))\} \cup \{\mu(n, j) \in P(n) : \mu(n, j)$ is obtained from $\lambda(n, k)$ by writing $\lambda(m) - 1$ in the place of $\lambda(m)$ and writing $\lambda(i) + 1$ in the place of $\lambda(i)$ for any one $i \neq m$, leaving all the other $\lambda(i)$'s fixed$\}$. (In case $\lambda(m) = 1$, the entry $\lambda(m) - 1 = 0$ is simply deleted so that the first of the two sets above will consist of k-tuples whereas the second one will consist of $(k - 1) -$ tuples).

Definition 1.2. F is said to be a *$\{\lambda(n, k; m)\}$-parameter family* in case F is a $\mu(n, j)$-parameter family for all $\mu(n, j) \in \{\lambda(n, k; m)\}$.

Suppose F is a $\lambda(n, k)$ and also a $\{\lambda(n, k; m)\}$-parameter family on $I = [a, b]$ and $f_0 \in F$ is determined by the conditions (1.1). Let $\{x_{mj} : 1 \leqq j < +\infty\} \subset (x_m, x_{m+1})$ be a strictly decreasing sequence of real numbers such that $x_{mj} \rightarrow x_m$ as $j \rightarrow +\infty$ (we consider in this paper only a strictly decreasing sequence $\{x_{mj}\}$ although similar results can be obtained for a strictly increasing sequence $\{x_{mj}\} \subset$

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(x_{m-1}, x_m) such that $x_{mj} \to x_m$ as $j \to +\infty$.) and $\{\alpha_j : 1 \leqq j < +\infty\} \subset R$ be a sequence such that $\alpha_j \to y_{m0}$ as $j \to +\infty$. Also for each $j \geqq 1$ let $f_j \in F$ be the unique function determined by

$$f_j(x_{mj}) = \alpha_j \tag{1.2}$$

and all the conditions of (1.1) except for $i = m$ and $r = \lambda(m) - 1$. We will show in Theorem 2.1 that if the sequence $\{\alpha_j\}$ satisfies certain additional convergence conditions then the sequence $\{f_j\}$ converges to f_0 uniformly on $[a, b]$.

Thus in hypothesizing that $\{x_{mj}\} \to x_m$ as $j \to +\infty$ we treat in this theorem a situation of degeneracy of boundary conditions that is not covered by Tornheim's convergence theorem (Theorem 5 of [5]) for n-parameter families. Moreover if we have an n-th order differential equation of the form $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ satisfying the assumed uniqueness and existence conditions, this theorem illustrates how a solution to a k-point boundary value problem can be approximated by a solution to a $(k+1)$-point boundary value problem with suitably chosen boundary values. We also give in Theorem 2.3 an alternate set of sufficient conditions that will guarantee that $f_j \to f_0$ as $j \to +\infty$ uniformly on $[a, b]$. There are several papers in the literature concerning $\lambda(n, k)$-parameter families or their special cases and [1] - [5] are a few such references.

2. **Main results.** We shall introduce the following notations concerning sequences of points in $R \times R$ in order to simplify the statement of our main theorem.

If $\{(t_j, \alpha_j) : 1 \leqq j < +\infty\} \subset R \times R$ is a sequence of points such that $(t_j, \alpha_j) \to (t_0, c_0) \in R \times R$ as $j \to +\infty$ with $\{t_j\}$ strictly decreasing and $c_i \in R$, $i = 1, \cdots, r$ are given, then define $D^i \alpha_j$, $i = 0, 1, \cdots, r$ recursively as follows:

$$D^0 \alpha_j \equiv \alpha_j$$

$$D^i \alpha_j \equiv (D^{i-1} \alpha_j - c_{i-1}/(i-1)!)/(t_j - t_0).$$

The following remarks which are easy consequences of the above definitions will be useful in the proof of our main theorem.

REMARK 1. If $\lim_{j\to+\infty} D^i \alpha_j$ exists and $=$ a constant d then $\lim_{j\to+\infty} D^p \alpha_j = c_p/p!$, $p = 0, 1, \cdots, i-1$.

REMARK 2. If $t_0 \in (a, b)$ and $g \in C^n[a, b]$ is such that $g^{(p)}(t_0) = c_p$, $p = 0, 1, \cdots, r(\leqq n)$ and $g(t_j) = \alpha_j$, $j \geqq 1$ then $\lim_{j\to+\infty} D^p \alpha_j = c_p/p!$, $p = 0, 1, \cdots, r$.

REMARK 3. If $\{(t_j, \beta_j) : 1 \leqq j < +\infty\} \subset R \times R$ is such that $\alpha_j \leqq \beta_j$ for all $j \geqq 1$, $\lim_{j\to+\infty} D^p \alpha_j = \lim_{j\to+\infty} D^p \beta_j = c_p/p!$ $p = 0, 1, \cdots, r-1$ and $\lim_{j\to+\infty} D^r \alpha_j$ and $\lim_{j\to+\infty} D^r \beta_j$ exist, then $\lim_{j\to+\infty} D^r \alpha_j \leqq \lim_{j\to+\infty} D^r \beta_j$.

We are now ready to state our main theorem.

THEOREM 2.1. *Let F be a $\lambda(n, k)$ and also a $\{\lambda(n, k; m)\}$-parameter family on an interval $[a, b]$ for some fixed integer m, $1 \leqq m \leqq k$. Let $\alpha \leqq x_1 < x_2 < \cdots < x_k < b$ and $y_{ir} \in R$ be arbitrary for $r = 0, 1, \cdots, \lambda(i) - 1$, $i = 1, \cdots, k$. Let $\{(x_{mj}, \alpha_j) : 1 \leqq j < +\infty\} \subset (x_m, x_{m+1}) \times R$ be a sequence of points such that i) $(x_{mj}, \alpha_j) \to (x_m, y_{m0})$ as $j \to +\infty$ with x_{mj} strictly decreasing (in case $m = k$ interpret $x_{k+1} = b$) and ii) $D^r \alpha_j \to y_{mr}/r!$ as $j \to +\infty$, $r = 0, 1, \cdots, \lambda(m) - 1$. Also suppose $f_0 \in F$ is the unique function determined by (1.1) and for $j \geqq 1$, $f_j \in F$ is the unique function determined by (1.2) and all conditions of (1.1) except for $i = m$ and $r = \lambda(m) - 1$. Then $f_j \to f_0$ as $j \to +\infty$ uniformly on $[a, b]$.*

PROOF: If the sequence $\{f_j\}$ is such that $f_j \equiv f_{j+1}$ for all $j \geqq q$ where q is a fixed positive integer, then we claim $f_j^{(\lambda(m)-1)}(x_m) = y_{m,\lambda(m)-1}$, $j \geqq q$. Setting $f_j \equiv g$ for $j \geqq q$ we have by virtue of our hypothesis and remark 2 that $g^{(\lambda(m)-1)}(x_m) = (\lambda(m) - 1)! \lim_{j\to+\infty} D^{\lambda(m)-1} \alpha_j = y_{m,\lambda(m)-1}$. Consequently $f_j \equiv f_0, j \geqq q$ and we are done. Now pick a sequence $\{n(i)\}$ of positive integers so that $n(1) = 1$ and for each $i \geqq 2, f_j \equiv f_{i-1}$ for $i - 1 \leqq j < n(i)$ and $f_{n(i)} \not\equiv f_{i-1}$. The sequence $\{f_{n(j)}\}$ clearly converges uniformly if and only if the sequence $\{f_j\}$ converges uniformly, so for simplicity of notation, we relabel $f_{n(j)}$ as f_j, $\alpha_{n(j)}$ as α_j and $x_{mn(j)}$ as x_{mj}. Then we have $f_j \not\equiv f_{j+1}$ for each $j \geqq 1$.

Further, for each $j \geqq 1$, $f_j - f_{j+1}$ has $\lambda(1), \cdots, \lambda(m-1), \lambda(m) - 1, \lambda(m+1), \cdots, \lambda(k)$ zeros at $x_1, \cdots, x_k$ respectively on $[a, b]$ and hence cannot have any more zeros on $[a, b]$. Thus $f_j - f_{j+1}$ must keep a constant sign on each of the intervals (x_i, x_{i+1}), $i = 0, 1, \cdots, k$ (where $x_0 = a$) and consequently $\{f_j\}$ is pointwise monotone on each of the intervals (x_i, x_{i+1}), $i = 0, 1, \cdots, k$. We can further assume without loss of generality that $f_j^{(\lambda(m)-1)}(x_m) \neq y_{m,\lambda(m)-1}$, $j \geqq 1$ for if equality holds for some $j = J$ then $f_J \equiv f_0$ and we can suppress f_J from $\{f_j\}$. Now at least one of the following two cases must occur.

Case 1: There exists an infinite number of functions f_j such that $f_j^{(\lambda(m)-1)}(x_m) < y_{m,\lambda(m)-1}$.

Case 2: There exists an infinite number of functions f_j such that $f_j^{(\lambda(m)-1)}(x_m) > y_{m,\lambda(m)-1}$.

Because of the similarity of the proofs involved we shall consider only the first case. In this case we claim that we can find a subsequence $\{f_{j(p)}\} \subset \{f_j\}$ such that $f_{j(1)}^{(\lambda(m)-1)}(x_m) < f_{j(2)}^{(\lambda(m)-1)}(x_m) < \cdots y_{m,\lambda(m)-1}$. For let $j(1)$ be any integer such that $f_{j(1)}^{(\lambda(m)-1)}(x_m) < y_{m,\lambda(m)-1}$. Then there must exist an integer $j(2) > j(1)$ such that $f_{j(1)}^{(\lambda(m)-1)}(x_m) < f_{j(2)}^{(\lambda(m)-1)}(x_m) < y_{m,\lambda(m)-1}$. If not, for every $j > j(1)$ we will have $f_j^{(\lambda(m)-1)}(x_m) < f_{j(1)}^{(\lambda(m)-1)}(x_m) < y_{m,\lambda(m)-1}$. This implies by virtue of our hypothesis on F that $f_j(x) < f_{j(1)}(x) < f_0(x)$ for all x, $x_m < x < x_{m+1}$ and all $j > j(1)$. In particular $\alpha_j < f_{j(1)}(x_{mj}) < f_0(x_{mj}), j > j(1)$. Consequently, in view of remarks 2 and 3 we have $y_{m,\lambda(m)-1} = (\lambda(m) - 1)! \lim_{j\to+\infty} D^{\lambda(m)-1}\alpha_j \leqq (\lambda(m) - 1)! \lim_{j\to+\infty} D^{\lambda(m)-1} f_{j(1)}(x_{mj}) = f_{j(1)}^{(\lambda(m)-1)}(x_m) < y_{m,\lambda(m)-1}$. This contradiction proves our claim.

For convenience of notation we shall now denote the subsequence $\{f_{j(p)}\}$ by $\{f_j\}$ and set $s_i = -1 + \sum_{p=i+1}^{m} \lambda(p)$, $i = 0, 1, \cdots, m-1$ and $S_i = \sum_{p=m+1}^{i} \lambda(p)$, $i = m, \cdots, k$ (where $S_m = 0$). Now the sequence $\{f_j\}$ has the property that $\{(-1)^{s_i} f_j\}$ is pointwise monotone increasing on (x_i, x_{i+1}), $i = 0, 1, \cdots, m-1$ and $\{(-1)^{S_i} f_j\}$ is pointwise monotone increasing on (x_i, x_{i+1}), $i = m, \cdots, k$. Furthermore $\{(-1)^{s_i}(f_0 - f_j)\}$ is positive on (x_i, x_{i+1}), $i = 0, 1, \cdots, m-1$ and $\{(-1)^{S_i}(f_0 - f_j)\}$ is positive on $(x_i, x_{i+1}), i = m, \cdots, k$.

We now claim $\lim_{j\to+\infty} f_j(x) = f_0(x), a \leqq x \leqq b$. We will first show $\lim_{j\to+\infty} f_j^{(\lambda(m)-1)}(x_m) = y_{m,\lambda(m)-1}$. By our choice of $\{f_j\}$ it is clear that $L \equiv \lim_{j\to+\infty} f_j^{(\lambda(m)-1)}(x_m) \leqq y_{m,\lambda(m)-1}$. Suppose $L < y_{m,\lambda(m)-1}$. Let $g \in F$ be determined by $g^{(\lambda(m)-1)}(x_m) = L$ and all the conditions in (1.1) except for $i = m$ and $r = \lambda(m) - 1$. Then $f_j(x) < g(x) < f_0(x)$ for all x, $x_m < x < x_{m+1}$ and all $j \geqq 1$. In particular, $\alpha_j < g(x_{mj}) < f_0(x_{mj})$ and as a result of remarks 2 and 3 it follows that $y_{m,\lambda(m)-1} = (\lambda(m) - 1)! \lim_{j\to+\infty} D^{\lambda(m)-1}\alpha_j \leqq g^{(\lambda(m)-1)}(x_m) < y_{m,\lambda(m)-1}$, a contradiction. Hence $L = y_{m,\lambda(m)-1}$.

Now suppose if possible $\lim_{j\to+\infty} f_j(x') \neq f_0(x')$ for some x', $x_t < x' < x_{t+1}$ where t is some fixed integer $0 \leqq t \leqq k$. Without loss of generality we can assume $m \leqq t \leqq k$ since the proof will be similar if $0 \leqq t \leqq m - 1$. So there exists an $\epsilon > 0$ and a subsequence of $\{f_j\}$ which we again call $\{f_j\}$ such that $|f_j(x') - f_0(x')| > \epsilon$. In particular $(-1)^{S_t}(f_0(x') - f_j(x')) > \epsilon$ for all $j \geqq 1$. Now choose $z' \in R$ such that $(-1)^{S_t} f_0(x') - \epsilon/2 > z' > (-1)^{S_t} f_j(x') + \epsilon/2$ and let $h \in F$ be the unique function determined by $h(x') = z'$ and all the conditions of (1.1) except for $i = m$ and $r = \lambda(m) - 1$. Then by our hypothesis on F we must have $f_j^{(\lambda(m)-1)}(x_m) < h^{(\lambda(m)-1)}(x_m) < y_{m,\lambda(m)-1}$. This contradicts our earlier assertion that $f_j^{(\lambda(m)-1)}(x_m) \to y_{m,\lambda(m)-1}$ as $j \to +\infty$. Hence $\lim_{j\to+\infty} f_j(x) = f_0(x), a \leqq x \leqq b$.

Now by Dini's theorem it follows that $\lim_{j \to +\infty} f_j(x) = f_0(x)$, uniformly on $[a, b]$. Since from every subsequence of the original sequence $\{f_j\}$ we can extract by the above process a further subsequence that converges to f_0 uniformly on $[a, b]$, it follows that $\{f_j\}$ converges to f_0 uniformly on $[a, b]$.

This completes the proof of the theorem.

COROLLARY 2.2. *Let* $p(x) = \sum_{r=0}^{\lambda(m)-1} (x - x_m)^r y_{mr}/r!$ *and* $(x_{mj}, \alpha_j) \to (x_m, y_{m0})$ *along the arc of the polynomial* $p(x)$. *Let* $f_j(x)$ *and* $f_0(x)$ *be as defined in Theorem* 2.1. *Then* $f_j(x) \to f_0(x)$ *as* $j \to +\infty$ *uniformly on* $[a, b]$.

In the next theorem we shall give an alternate set of sufficient conditions that will ensure the uniform convergence of f_j to f_0 on $[a, b]$.

THEOREM 2.3. *Assume* F, f_0 *and* $\{x_{mj}\}$ *are as in Theorem* 2.1. *Let* $\{\alpha_j : 1 \leqq j < +\infty\} \subset R$ *be a sequence such that* $\alpha_j \to y_{m0}$ *and* $(\alpha_j - f_0(x_{mj}))/(x_{mj} - x_m)^{\lambda(m)-1} \to 0$ *as* $j \to +\infty$. *For each* $j \geqq 1$, *let* $f_j \in F$ *be determined as in Theorem* 2.1. *Then* $f_j \to f_0$ *as* $j \to +\infty$ *uniformly on* $[a, b]$.

PROOF. If $\{f_j\}$ is such that $f_j \equiv f_{j+1}$ for all $j \geqq q$ then we claim $f_j^{(\lambda(m)-1)}(x_m) = f_0^{(\lambda(m)-1)}(x_m), j \geqq q$ for setting $f_j \equiv g, j \geqq q$ we have

$$(\alpha_j - f_0(x_{mj}))/(x_{mj} - x_m)^{\lambda(m)-1} = (g^{(\lambda(m)-1)}(x_m) - f_0^{(\lambda(m)-1)}(x_m))/(\lambda(m-1)! + \underline{o}(1).$$

On taking the limit as $j \to +\infty$ we obtain $g^{(\lambda(m)-1)}(x_m) = f_0^{(\lambda(m)-1)}(x_m)$ and consequently $f_j \equiv g \equiv f_0$, $j \geqq q$ and we are done.

Otherwise, arguing as in the proof of Theorem 2.1, we can assume without loss of generality that $f_j \not\equiv f_{j+1}$ for all $j \geqq 1$. Then for each $j \geqq$, $f_j - f_{j+1}$ has $\lambda(1), \cdots, \lambda(m-1), \lambda(m) - 1, \lambda(m+1), \cdots, \lambda(k)$ zeros at $x_1, \cdots, x_k$ respectively and hence cannot have any more zeros on $[a, b]$ and also must keep a constant sign on each of the intervals (x_i, x_{i+1}), $i = 0, 1, \cdots, k$. Further, we can assume without loss of generality as in the proof of Theorem 2.1 that $f_j^{(\lambda(m)-1)}(x_m) \neq y_{m,\lambda(m)-1}$, $j \geqq 1$. Now at least one of the following two cases must hold.

Case 1: There exists an infinite number of functions f_j such that $f_j^{(\lambda(m)-1)}(x_m) < y_{m,\lambda(m)-1}$.

Case 2: There exists an infinite number of functions f_j such that $f_j^{(\lambda(m)-1)}(x_m) > y_{m,\lambda(m)-1}$.

We shall consider only Case 1 since the proof for Case 2 is similar. We claim we can find a subsequence $\{f_{j(p)}\} \subset \{f_j\}$ such that $f_{j(1)}^{(\lambda(m)-1)}(x_m) < f_{j(2)}^{(\lambda(m)-1)}(x_m) < \cdots < y_{m,\lambda(m)-1}$, for let $j(1)$ be any integer such that $f_{j(1)}^{(\lambda(m)-1)}(x_m) < y_{m,\lambda(m)-1}$. Then there exists an integer $j(2) > j(1)$ such that $f_{j(1)}^{(\lambda(m)-1)}(x_m) < f_{j(2)}^{(\lambda(m)-1)}(x_m) < y_{m,\lambda(m)-1}$. If not, for all $j > j(1)$ we will have

$$\text{(A)} \qquad f_j^{(\lambda(m)-1)}(x_m) < f_{j(1)}^{(\lambda(m)-1)}(x_m) < f_0^{(\lambda(m)-1)}(x_m)$$

We also have

$$(\alpha_j - f_0(x_{mj}))/(x_{mj} - x_m)^{\lambda(m)-1} = (f_j^{(\lambda(m)-1)}(x_m) - f_0^{(\lambda(m)-1)}(x_m))/(\lambda(m) - 1)! + \underline{o}(1).$$

On taking the limit as $j \to +\infty$ by virtue of our hypothesis we obtain that $f_j^{(\lambda(m)-1)}(x_m) \to f_0^{(\lambda(m)-1)}(x_m)$ as $j \to +\infty$, a contradiction to assertion (A). This proves our claim.

For convenience of notation we shall again denote the subsequence $\{f_{j(p)}\}$ by $\{f_j\}$. Now the sequences $\{f_j\}$ and $\{f_0 - f_j\}$ have the properties of monotonicity and positiveness respectively on the intervals (x_i, x_{i+1}), $i = 0, 1, \cdots, k$ as in the proof of Theorem 2.1. Also from our choice of $\{f_j\}$ it follows that $\lim_{j\to+\infty} f_j^{(\lambda(m)-1)}(x_m) = f_0^{(\lambda(m)-1)}(x_m)$. The rest of the proof is similar to that of Theorem 2.1 and is omitted.

This completes the proof of the theorem.

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