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# CHARACTERS OF THE WEYL GROUP OF $SU(n)$ ON ZERO WEIGHT SPACES AND CENTRALIZERS OF PERMUTATION REPRESENTATIONS

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1. **Introduction.** If $G$ is a compact simple Lie group with maximal abelian subgroup $T$ and normalizer $N(T)$, then $W = N(T)/T$ is a finite group called the *Weyl group* of $G$. If $\mathfrak{G}$ is the Lie algebra of $G$ with $\mathfrak{T}$ the Cartan subalgebra corresponding to $T$, then the adjoint action of $G$ on $\mathfrak{G}$ has the property that $\mathfrak{T} = \{x \in \mathfrak{G} : t \cdot x = x \text{ for all } t \in T\}$. Thus $\mathfrak{T}$ is naturally a $W$-module and it is well-known that $W$ acts on $\mathfrak{T}$ as a group generated by reflections. A generalization of this situation is the following. Let $M$ be a complex $G$-module and let $M_0 = \{x \in M : t \cdot x = x \text{ for all } t \in T\}$, the *zero-weight space* of $M$. Then $M_0$ is naturally a $W$-module. It is the purpose of this paper to characterize the $W$-module structure of $M_0$ in case $G = SU(V)$ (where $V$ is $n$-dimensional unitary space) and $M$ is a finite dimensional simple $G$-module.

REMARK. The structure of $M_0$ as a $W$-module is closely related to the structure of $H$, the graded $G$-module of $G$-harmonic polynomials over $\mathfrak{G}$. For example, the multiplicity of $M$ in $H$ is exactly $k = \dim(M_0)$. Furthermore, if $m_1, \cdots, m_k$ are the homogeneous degrees of $H$ in which $M$ occurs (the *generalized exponents* of $M$), then the eigenvalues in $M_0$ of a Coxeter-Killing element in $W$ are just $\exp(2\pi i/m_j)$ $(j = 1, \cdots, k)$. See Kostant's paper [3] for a definition of the $G$-harmonic polynomials and more details.

Our results for $G = SU(V)$ depend heavily on the classical correspondence between the irreducible representations of $SU(V)$ and those of the symmetric groups $S_m$ as $m$ ranges over all positive integers. This correspondence is due to the fact that the linear span of the action of $S_m$ on $\otimes^m V$ is the full centralizer of the action of $SU(V)$ on $\otimes^m V$. In § 2, we will summarize this correspondence using a more general result about centralizing group representations. In § 3 we will prove a sharpened version of this result for permutation representations of finite groups. Finally, in § 4 we will obtain a formula for the character of $W$ on $M_0$ related to Littlewood's plethysm of S-functions.

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## 2. Centralizers of linear group representations.

PROPOSITION 1. *Let $G$ be a group and $M$ a complex, finite-dimensional semi-simple $G$-module with $M_i$ $(1 \leqq i \leqq k)$ its non-isomorphic simple summands. If $H$ is a group and $M$ is also an $H$-module such that $H$ centralizes $G$ on $M$ $(H \subset \mathrm{Hom}_G(M, M))$, then*

(a) $M^i = \mathrm{Hom}_G(M_i, M)$ *is an $H$-module, and*

$$(1) \qquad M \simeq \sum_{i=1}^{k} M_i \otimes M^i$$

*both as $G$- and as $H$-modules. Here the action of $G$ (resp. $H$) on $M_i \otimes M^i$ is given by $g \cdot (a \otimes b) = (g \cdot a) \otimes b$ (resp. $h \cdot (a \otimes b) = a \otimes (h \cdot b)$),*

(b) *the span of $H$ in* $\mathrm{Hom}_G(M, M)$ *is equal to* $\mathrm{Hom}_G(M, M)$ *iff the $M^i$ $(1 \leqq i \leqq k)$ are simple and non-isomorphic $H$-modules,*

(c) *the decomposition in* (1) *is unique in the sense that if $M \cong \sum M_i \otimes \overline{M}^i$ and the action of $G$ and $H$ on $M_i \otimes \overline{M}^i$ are given as in* (a), *then $M^i \simeq \overline{M}^i$ as $H$-modules.*

(This theorem is a consequence of Schur's Lemma. A version appears in [1] p. 23.)

We will call the decomposition of $M$ given by equation (1) the $G-H$ *splitting* for $M$.

Let $V^{(m)} = \otimes^m V$. It is well-known that $SU(V)$ spans $\mathrm{Hom}_{S_m}(V^{(m)}, V^{(m)})$ (see [1], p. 134) so that part (b) of the proposition is applicable. Thus, if $\Omega_m$ is the set of simple characters of $S_m$ and $V_\chi$ a simple $S_m$-module with character $\chi \in \Omega_m$, then there is a simple $SU(V)$-module $V^\chi$ (possibly trivial) so that

$$(1) \qquad V^{(m)} \simeq \sum_{\chi \in \Omega_m} V \otimes V^\chi$$

is the $S_m - SU(V)$ splitting of $V^{(m)}$. (Here it is understood that $V^\chi \neq \{0\}$ iff $\chi$ occurs in the character of $S_m$ on $V^{(m)}$.) It can also be shown that, if $M$ is any simple, finite-dimensional $SU(V)$-module, there exists a positive integer $m$ and a simple character $\chi \in \Omega_m$ so that $M \simeq V^\chi$.

An immediate consequence of part (c) of proposition 1 is the following.

PROPOSITION 2. *If $V^{(m)} \simeq \sum_{\chi \in \Omega_m} V_\chi \otimes V^\chi$ is the $S_m - SU(V)$ splitting of $V^{(m)}$, then $V_0{}^{(m)} \simeq \sum_{\chi \in \Omega_m} V_\chi \otimes V_0{}^\chi$ is the $S_m - W$ splitting of the zero-weight space of $V^{(m)}$.*

**3. Centralizers of Transitive permutation representations.** In this section we will prove a sharpened version of proposition 1 for permutation representations of finite groups. To state the theorem, let $X$ be a finite set, $S_X$ the set of all permutations of $X$, $G$ a subgroup of $S_X$ acting transitively on $X$ and $C$ the centralizer of $G$ in $S_X$. If $X = \{1, 2, \cdots p\}$, we form a complex $p$-dimensional vector space $V_X$ with basis $x_1, \cdots, x_p$ and let $\sigma \in S_X$ act on $V_X$ by $\sigma(x_i) = x_{\sigma(i)}$ $(i = 1, \cdots, p)$. Thus $V_X$ is both a $C$- and $G$- module with $C \subseteq \mathrm{Hom}_G(V_X, V_X)$. Thus proposition 1 applies. Indeed, let $\lambda_1, \cdots, \lambda_k$ (resp. $\mu_1, \cdots, \mu_q$ be the simple characters of $G$ (resp. $C$). Let

$$V_X \simeq \sum_i V_{\lambda_i} \otimes V_X{}^{\lambda_i} \tag{2}$$

be the $G - C$ splitting of $V_X$ as in (1), and let $\chi(X, \lambda_i)$ denote the $C$-character of $V_X{}^{\lambda_i}$ $(i = 1, \cdots, k)$. The theorem we will prove is the following.

THEOREM 1. *Let $G_x$ be the subgroup of $G$ fixing $x \in X$ and $N(G_x)$ its normalizer in $G$. Then*

(a) $C \simeq N(G_x)/G_x$.

(b) *For $i = 1, \cdots, \ell$, let $\hat{\mu}_i$ be the simple character of $N(G_x)$ (with kernel $G_x$) corresponding to $\mu_i$. Then for $j = 1, \cdots, k$, the multiplicity of $\lambda_j$ in the induced character $\hat{\mu}_i{}^G$ is equal to the multiplicity of $\mu_i$ in $\chi(X, \lambda_j)$.*

REMARK. In the next section we will show that that $V_0{}^{(m)}$ has a basis on which $S_m$ acts transitively and such that $W$ is the centralizer of $S_m$ in $S_X$. By Proposition 2, $\chi(X, \lambda)$ is the $W$-character of the zero weight space of $V^\lambda$ $(\lambda \in \Omega_m)$.

For $x \in X$, $\sigma \in S_X$ let $\sigma \cdot x$ denote the action of $\sigma$ on $x$, and let $F(x) = \{y \in X : g \cdot y = y \text{ for all } g \in G_x\}$. If $K$ is a subgroup of $S_X$, let $K \cdot x$ denote the $K$-orbit of $x$. We will prove Theorem 1 by the following series of lemmas.

LEMMA 1. (1) *$N(G_x)/G_x$ is faithful and regular on $F(x)$.*
(2) *$C$ is faithful and regular on $F(x)$.*
(3) $|C| = |F(x)| = |N(G_x)/G_x|$.

PROOF. (1) $N(G_x)$ is transitive on $F(x)$ by [**6**], 3.1 and 3.6, with normal isotropy subgroup $G_x$. It follows that $N(G_x)/G_x$ is regular and faithful on $F(x)$.

(2) $C$ is semi-regular on $X$ and $|C| = |F(x)|$ by [**6**], 4.5′. Thus, since $C \cdot x \subseteq F(x)$, $C \cdot x = F(x)$. Part (2) of the lemma follows.

(3) This follows from (1) and (2). ∎

LEMMA 2. *Let $H$ be a group acting faithfully and regularly on a set $Y$, and let $K$ be the centralizer of $H$ in $S_Y$. Let $y \in Y$. Then there is an isomorphism $\phi : H \to K$ such that $h \cdot (k \cdot y) = k\phi(h)^{-1} \cdot y$ for all $h \in H$, $k \in K$.*

PROOF. (That $H$ and $K$ are isomorphic is well known ([5], 10.3.6).) Define a function $\phi : H \to S_Y$ by $\phi(h)h' \cdot y = h'h^{-1} \cdot y$ for all $h, h' \in H$. It is clear that $\phi$ is an isomorphism into and that $\phi(H)$ commutes with $H$. Thus $\phi(H) \subseteq K$. By lemma 1 part (3), $|K| = |Y| = |H| = |\phi(H)|$ so that $\phi(H) = K$. Furthermore, $h \cdot (k \cdot y) = k \cdot (h \cdot y) = k \cdot \phi(h)^{-1} \cdot y$. ▮

LEMMA 3. $C \simeq N(G_x)/G_x$.

PROOF. By Lemma 1 parts (1) and (2), $N(G_x)/G_x$ and $C$ act faithfully and regularly on $F(x)$. It is clear that $C$ commutes with $N(G_x)/G_x$ on $F(x)$. By Lemma 1 part (3), $|C| = |N(G_x)/G_x|$. By Lemma 2, $C \simeq N(G_x)/G_x$. ▮

LEMMA 4. *Assume the notation of Theorem 1. $V_X$ is naturally a $C \times G$-module and suppose it has character $\sum_{i,j} m_{ij}\lambda_i\mu_j$. Then the $C$-character of $V_X{}^{\lambda_i}$* is $\sum_{j=1}^{q} m_{ij}\mu_j$ $(1 \leqq i \leqq p)$. (This follows easily from the fact that the simple characters of $C \times G$ are just $\lambda_i\mu_j$ $(1 \leqq i \leqq p$, $1 \leqq j \leqq q)$ and from the uniqueness of the $G - C$ splitting given by formula (2)).

Lemma 4 reduces the problem of determining the $C$-character of $V_X{}^{\lambda_i}$ to that of determining the $C \times G$-character of $V_X$.

LEMMA 5. *Assume the notation of Theorem 1. The $C \times G$-character of $V_X$ is just $\sum_{j=1}^{q} \bar{\mu}_j\hat{\mu}_j{}^G$, where $\bar{\mu}_j$ denotes the character conjugate to $\mu_j$.*

PROOF. We first observe that, since $G$ is transitive on $X$, so is $C \times G$. Thus the character $\chi$ of $C \times G$ on $V_X$ is induced up from the identity character $\chi_1$ of $(C \times G)_x$ (= isotropy subgroup of $C \times G$ at $x \in X$). Thus $\chi \times \chi_1{}^{C \times G}$. We claim that $(C \times G)_x \subset C \times N(G_x)$ so that, by transitivity of induction, $\chi_1{}^{C \times G} = (\chi_1{}^{C \times N(G_x)})^{C \times G}$. To prove this claim, let $(c, g) \in (C \times G)_x$ so that $(c, g) \cdot x = c \cdot (g \cdot x) = g \cdot (c \cdot x) = x$ implies $g \in N(G_x)$ by Lemma 1 parts (1) and (2).

Next, we compute $\chi_1{}^{C \times N(G_x)}$. To do this, we note that $C \times N(G_x)$ is transitive on $F(x)$, the set of fixed points of $G_x$. Consequently, the character $\zeta$ of $C \times N(G_x)$ on $V_{F(x)}$ is induced up from the identity character of $(C \times N(G_x))_x = (C \times G)_x$. Thus $\zeta = \chi_1{}^{C \times N(G_x)}$. Furthermore,

(3) $$\zeta = \sum_{j=1}^{q} \bar{\mu}_j \hat{\mu}_j.$$

To prove formula (3), let $\phi : N(G_x)/G_x \to C$ be the isomorphism of Lemma 2 such that $k \cdot (c \cdot x) = c\phi(k)^{-1} \cdot x$ for every $c \in C, k \in N(G_x)/G_x$. For every $c \in C$, pick a coset representative $\hat{c} \in \phi^{-1}(c)$ so that, consequently, every element of $C \times N(G_x)$ is of the form $(c_1, \hat{c}_2 g)$ with $c_1, c_2 \in C, g \in G_x$. Now $\zeta(c_1, \hat{c}_2 g)$ is equal to the number of fixed points of $(c_1, \hat{c}_2 g)$ in $F(x)$. But $(c_1, \hat{c}_2 g) \cdot (c_3 \cdot x) = c_1 c_3 c_2^{-1} \cdot x$, and the latter equals $c_3 \cdot x$ iff $c_3^{-1} c_1 c_3 = c_2$. Thus

(4) $$\zeta(c_1, \hat{c}_2 g) = \begin{cases} 0, \text{ if } c_1, c_2 \text{ are not conjugate} \\ \text{order of the centralizer of } c_1 \text{ in } C, \text{ otherwise.} \end{cases}$$

On the other hand, $\sum_{j=1}^{q} \bar{\mu}_j \hat{\mu}_j(c_1, \hat{c}_2 g) = \sum_{j=1}^{q} \bar{\mu}_j(c_1)\mu_j(c_2)$, which is equal to the right hand side of (4) by the orthogonality relations for simple characters. Thus formula (3) is proved.

To finish the proof, we need only complete the final step of induction up to $C \times G$. We have, therefore

$$\chi_1^{C \times G} = \zeta^{C \times G} = \left( \sum_{j=1}^{q} \bar{\mu}_j \hat{\mu}_j \right)^{C \times G} = \sum_{j=1}^{q} \bar{\mu}_j{}^C \hat{\mu}_j{}^G = \sum_{j=1}^{q} \bar{\mu}_j \hat{\mu}_j{}^G. \blacksquare$$

4. **A formula for the character of $W$ on $V_0{}^\chi$.** In this section we will find a basis for the zero-weight space $V_0{}^{(m)}$ of $V^{(m)}$ and show that $S_m$ and the Weyl group $W$ act on this basis in such a way that Theorem 1 applies. This will enable us to find a formula for the character of $W$ on $V_0{}^\chi$ when $\chi$ is a simple character of $S_m$.

Let $e_1, e_2, \cdots, e_n$ be a fixed unitarily orthogonal basis for $V$, and let $T$ be the maximal abelian subgroup of $SU(V)$ consisting of the diagonal transformations with respect to this basis. Then $N(T)$ is the group of $n \times n$ monomial matrices, and the Weyl group $W = N(T)/T$ is isomorphic to $S_n$.

A basis for $V^{(m)}$ consists of vectors of the form $e_{i_1} \otimes \cdots \otimes e_{i_m}$ with $i_j \in \{1, 2, \cdots, n\}$, all $j$. Each is an eigenvector for the action of $T$ on $V^{(m)}$. In particular, if $t \in T$ and $t = \operatorname{diag}(s_1, \cdots, s_n)$ where $s_j = \exp(2\pi i\, \theta_j(t))$ $(1 \leqq j \leqq n)$, then $t \cdot (e_{i_1} \otimes \cdots \otimes e_{i_m}) = \exp(2\pi i \sum_{j=1}^{m} \theta_{i_j})(e_{i_1} \otimes \cdots e_{i_m})$. Thus $e_{i_1} \otimes \cdots \otimes e_{i_m} \in V_0{}^{(m)}$ iff $\exp 2\pi i \sum_{j=1}^{m} \theta_{i_j}(t) = 1$ for all $t \in T$. Since $\exp(2\pi i \sum_{j=1}^{n} \theta_j(t)) = 1$ is the only relation on the $\theta_j(t)$'s, this means that $e_{i_1} \otimes \cdots \otimes e_{i_m} \in V^{(m)}$ iff the number of occurrences of $k$ and $\ell$ as subscripts $i_j$ is the same for all $k, \ell \in \{1, 2, \cdots, n\}$. Thus the zero-weight space $V_0{}^{(m)} \neq \{0\}$ iff $m = sn$ for some positive integer $s$. If $m = sn$, then $V_0{}^{(m)}$ has a basis $X$ consisting of

$$\underbrace{e_1 \otimes \cdots \otimes e_1}_{s\text{-times}} \otimes \underbrace{e_2 \otimes \cdots \otimes e_2}_{s\text{-times}} \otimes \cdots \otimes \underbrace{e_n \otimes \cdots \otimes e_n}_{s\text{-times}}$$

together with all images of this under the action of $S_m$. It is also clear that $S_m$ and the Weyl group $S_n$ act as permutations on this basis in such a way that the two actions commute and the action of $S_m$ is transitive.

We can now apply the analysis of § 3 to the situation in which $X$ is the basis of $V_0^{(m)}$ given above and $G$ is the group $S_m$. We claim that, in this case, the centralizer $C$ is just the Weyl group $W$. Clearly $W \subseteq C$. To show $W = C$, we need only show $|W| = |C|$. But $|C| = |F(x)|$ where

$$x = \underbrace{e_1 \otimes \cdots \otimes e_1}_{s\text{-times}} \otimes \cdots \otimes \underbrace{e_n \otimes \cdots \otimes e_n}_{s\text{-times}}$$

and

$$G_x = \underbrace{S_s \times \cdots \times S_s}_{n\text{-times}}$$

Furthermore, $y \in F(x)$ iff

$$y = \underbrace{e_{i_1} \otimes \cdots \otimes e_{i_1}}_{s\text{-times}} \otimes \cdots \otimes \underbrace{e_{i_n} \otimes \cdots \otimes e_{i_n}}_{s\text{-times}}$$

for some permutation $(i_1, \cdots, i_n)$ of $(1, 2, \cdots, n)$. Thus $|F(x)| = n! = |W|$ since $W = S_n$. The following is therefore an immediate consequence of the results of § 3.

THEOREM 2. *Let $m = sn$, and let*

$$H = \underbrace{S_s \times \cdots \times S_s}_{n\text{-times}}$$

*with normalizer $N(H)$ in $S_m$. Then $N(H)/H \simeq S_n = W$. For $\mu \in \Omega_n$, let $\hat{\mu}$ be the corresponding simple character of $N(H)$ with kernel $H$, and suppose $\hat{\mu}^{S_m} = \sum_{\chi \in \Omega_m} n_{\chi\mu}\chi$. Then the character of the Weyl group of $V_0{}^{\chi}$ $(\chi \in \Omega_m)$ is just $\sum_{\mu \in \Omega_n} n_{\chi\mu}\mu$.*

We would like to characterize $V_0{}^{\chi}$ in yet another way using a product of characters first defined by Littlewood (see [4], p. 66). To describe this product, let $\lambda \in \Omega_q$ and $\tilde{\lambda}$ the corresponding simple character of $SU(V)$ with space $V^{\lambda} \subseteq V^{(q)}$. Then $(V^{\lambda})^{(p)}$, the $p$-fold tensor product of $V^{\lambda}$ with itself, can be considered as a subspace of $V^{(pq)}$ having $S_p - SU(n)$ splitting $(V^{\lambda})^{(p)} = \sum_{\psi \in \Omega_p} V_{\psi} \otimes (V^{\lambda})^{\psi}$. The $SU(V)$-module $(V^{\lambda})^{\tilde{\psi}}$, not necessarily simple, has character denoted $\tilde{\lambda}^{\tilde{\psi}}$ and is called the *plethysm* of $\tilde{\lambda}$ with $\tilde{\psi}$. In [4], Robinson proves that the multiplicity of $\chi$ in $\hat{\mu}^{S^m}$ is equal to the multiplicity of $\tilde{\chi}$ in $\tilde{1}_s^{\tilde{\mu}}$. Thus we have the

COROLLARY. *Let $1_s$ denote the identity character of $S_s$, and let $\mu \in \Omega_n$ where $S_n$ is the Weyl group of $SU(V)$. If $\tilde{1}_s^{\tilde{\mu}} = \sum_{\chi \in \Omega_{sn}} n_{\chi\,\mu} \tilde{\chi}$ is the plethysm of $\tilde{1}_s$ with $\tilde{\mu}$, then the character of $W$ on $V_0{}^{\chi}$ is just $\sum_{\mu \in \Omega_n} n_{\chi\,\mu}\mu$.*

REMARKS. (1) Results concerning the decomposition of the plethysm can be found in many places. See [4] for a bibliography.

(2) Let $p = mn(n+1)/2$, and let $\chi$ be the character of $S_p$ corresponding to the partition $(m, 2m, \cdots, nm)$ of $p$. Then $V^{\chi} \simeq M$, the simple $SU(V)$-module with dominant weight $m\lambda$, where $\lambda$ is the highest root. By a previous result [2], the $W$-module $M_0$ is isomorphic to $S^m$, the homogeneous polynomials of degree $m$ over the Cartan subalgebra $\mathfrak{I}$. Thus if $\mu$ is a simple character of $W$, then the multiplicity of $\mu$ in $S^m$ is equal to the multiplicity of $\tilde{\chi}$ in $(\tilde{1}_{n(n+1)/2})^{\tilde{\mu}}$.

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