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SOME LOGICAL PROBLEMS CONCERNING FREE AND FREE PRODUCT GROUPS*

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The problems discussed in this paper are all concerned with attempts to understand the first order theories of free groups and the closely related groups obtained by the free product construction. Some of these problems are as old as model theory itself, and appear in the earliest writings on that subject by Tarski [31, 33] and Malc'ev [19]. All of these problems appear to be very difficult, and it is only in the last several years that substantial progress has been made toward their solutions. Already these results have uncovered interesting new areas of model theory, such as projective model theory [26, 27, 28] and new techniques in combinatorial group theory, and further work on these problems promises substantial new insights into both algebra and model theory.

The *first order* (or *elementary*) *language* L of group theory can be described as follows: the symbols of L include the constant 1, the variables $x_1, x_2, \cdots$, which range over group elements, the symbols $\cdot$ and $^{-1}$ for the two group operations, the equality symbol $=$, and the logical symbols $\sim$ (negation), & (conjuction), $\vee$ (disjunction), $\forall$ (for all group elements $\cdots$), and $\exists$ (there is a group elements such that $\cdots$). The *atomic formulas* of L include all formulas of form $W = W'$ where W and W' are products of the variables, the constant, and their inverses. The class C of *well-formed formulas* of L is the least collection of formulas such that C contains all atomic formulas, and if α and β are in C and v is any variable, then $(\sim\alpha)$, $(\alpha \,\&\, \beta)$, $(\alpha \vee \beta)$, $\forall v\alpha$ and $\exists v\alpha$ are all in C. The sentences of L are the well-formed formulas α such that if v is any variable which occurs in α, then v only occurs in well-formed subformulas of α of form $\forall v\beta$ or $\exists v\beta$. Every sentence of L is logically equivalent to a sentence of form $Q_1x_1 \cdots Q_nx_nM$, where M is a Boolean combination of atomic formulas and each Q_i is either $\forall$ or $\exists$. In particular, if all Q_i's are $\forall$, then the formula is termed *universal.*

Two groups are *elementarily equivalent* if they satisfy precisely the same sentences of L. If G is any group, let L_G be the language ob-

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tained from L by adding new constants which name the elements of G. If G is a subgroup of H, G is an *elementary subgroup of* H if G and H satisfy precisely the same sentences of L_G.

In this survey of logical problems about free and free-product groups, I have chosen six principal questions and several closely related subsidiary questions for discussion. These are as follows:

QUESTION I: Are the non-abelian free groups elementarily equivalent?

QUESTION IA: In what sense can one axiomatize the notion of freeness?

QUESTION II: Can one give a decision procedure for the set of sentences of elementary logic which are true of one (or all) non-abelian free group(s)?

QUESTION III: Can one give a decision procedure for the set of strictly universal sentences true of all non-abelian free groups?

QUESTION IIIA: Can one give an algorithm to solve equations over a given non-abelian free group?

QUESTION IV: What group theoretic properties are preserved under the free product construction?

QUESTION V: What properties hold of all or nearly all free products of groups?

QUESTION VI: If the groups A and B are elementarily equivalent, and so are the groups C and D, is it necessarily the case that the free products $A * C$ and $B * D$ are also elementarily equivalent?

In general I have sought to discuss all of the known results, on these questions. However, in the case of problem IIIA, there are many fragmentary results and I have only selected certain typical results in order to convey to the reader some of the flavor of current work on this problem. For the rest, I apologize in advance to anyone whose work I have overlooked, and I hope that my plea of ignorance will be accepted in the case of a major oversight.

This paper will be divided into sections in such a way that each of the preceding questions will be discussed in the section bearing the same number.

I. **The Elementary Theory of Free Groups.** Almost as soon as this question was posed by Tarski [31, 33], Vaught proved that any two free groups of infinite rank are elementarily equivalent [34]. His argument proceeds by induction on the complexity of a first order sentence ϕ in the vocabulary of the free group F of rank ω to show that $F_\omega \models \phi$ if and only if $F_\kappa \models \phi$, for any infinite cardinal κ. If ϕ is atomic, then his assertion is clear since F_ω is embedded into F_κ; the inductive arguments in the cases that ϕ is of form $\theta \,\&\, \psi$, or $\sim\theta$, are straightforward. If ϕ is of form $(\exists x)\,\theta(x)$, and $F_\omega \models \phi$, then $F_\omega \models \theta(t)$ for some closed term t, and $F_\kappa \models \theta(t)$ by inductive hypothesis; consequently, $F_\kappa \models \phi$. Conversely, if $F_\kappa \models \phi$, then $F_\kappa \models \theta(\bar{t})$ for some closed term $\bar{t}$ which is not necessarily in the vocabulary of F_ω. By choosing an automorphism of F_κ which fixes the elements of F_ω named in ϕ and carries the element $\bar{t}$ into F_ω (considered as some fixed subtroup of F_κ of countable rank), one can find an element t' of F_ω such that $F_\omega \models \theta(t')$, and thus show that $F_\omega \models \phi$. Vaught's argument, however, does not apply to free groups of finite rank.

Much later Merzlyakov [20] and I [24] (in independent work) showed that the non-abelian free groups satisfy precisely the same positive and negative sentences; positive sentences are those which do not involve the negation symbol, and negative sentences are their negations. My method was to give an elimination of quantifiers for the positive sentences in which the key lemma showed that if F_n is the free group of rank $n \geqq 2$ and $\phi(c_1, \cdots, c_p)$ is a positive sentence in the vocabulary of F_n (here $c_1, \cdots, c_p$ can be considered part of the generating set for F_n) of form $Qx\,\theta(c_1, \cdots, c_p, x)$, where Q is a quantifier, and if $p > 1$, then $F_n \models \phi(c_1, \cdots, c_p)$ if and only if $F_n \models \theta(c_1, \cdots, c_p, t)$ where t is a word on the letters $c_1, \cdots, c_p$ which satisfies no "short" relations with the letters $c_1, \cdots, c_p$.

This key lemma can also be used to prove that if $n > m > 2$, and ϕ is a sentence in the voacabulary of F_m of form $(\forall\vec{x})(\exists\vec{x})(\forall\vec{z})M(\vec{x}, \vec{y}, \vec{z}, \vec{c})$, where M is a quantifier-free formula, and if $F_n \models \phi$, then $F_m \models \phi$. An immediate consequence is that F_n and F_m cannot be distinguished by any first order sentence whose quantifier structure is $\forall\exists$ or $\exists\forall$. A reference in the work of Mal'cev [19] to further results in this direction by another mathematician is apparently erroneous.

Another effort to solve the elementary equivalence problem, by using the equivalence for negative sentences, has led to the important model-theoretic notions of negatively complete structures and coforcing [26, 27, 28]. A model M of an elementary theory T is negatively complete if any negative sentence in the vocabulary of M which is satisfiable in a homorphic preimage of M (among the models of T) is already satisfiable in M. For example, the negatively closed groups are precisely the free groups of rank $\geqq 2$.

Co-forcing gives a means by which to determine all properties of certain structures from their negative properties. These certain structures are called cogeneric. The co-forcing relation $P \dashv\!\Vdash \phi$ says, in essence, that the finite set P of negative sentences determines that ϕ is true in all cogeneric structures which satisfy P. The (finitely) cogeneric structures turn out to be the ones in which the true sentences are precisely those sentences co-forced by some finite set of negative sentences in their vocabulary. It can be shown that the cogeneric groups are all free of rank $\geqq 2$, that all free groups of infinite rank are cogeneric, and that all cogeneric groups are elementarily equivalent. If one can show that every negatively closed homomorphic image of a cogeneric structure is again cogeneric, then one will have completely answered question I. It is to be noted that the dual to this last statement is easily proved for Robinson's forcing.

IA. **Axiomatizing the Notion of Freeness.** The language $L_{\omega_1\omega}$ differs from elementary logic in that we permit conjunctions and disjunctions of countable sets of formulas. A class of structures is an elementary class for $L_{\omega_1\omega}$, ($EC_{\omega_1\omega}$) if it is definable by a single sentence of $L_{\omega_1\omega}$. An example of this notion is provided by the class of ordered abelian groups. A class of structures is a projective elementary class for $L_{\omega_1\omega}$ ($PC_{\omega_1\omega}$) if it is the image of an $EC_{\omega_1\omega}$ under a forgetful functor which ignores some of the relations. The class of orderable abelian groups is a $PC_{\omega_1\omega}$. Vaught [35] has proved that the free groups of rank $\geqq 2$ are a $PC_{\omega_1\omega}$.

The language $L_{\infty\lambda}$ allows conjunctions and disjunctions over arbitrary sets of formulas and also permits use of the quantifiers $(\forall x_1x_2 \cdots x_\mu \cdots)$ and $(\exists x_1x_2 \cdots x_\mu \cdots)$ for $\mu < \lambda$. Eklof [6] and Kueker [12] have shown that the sentence "There is a set X such that the group G is free on X" can not be formalized in any of the languages $L_{\infty\aleph_n}$, where n is a positive integer, and for certain other cardinals. Their technique is as follows: A group is $\aleph_n$-free if every subgroup generated by fewer than $\aleph_n$ elements is free. For example, the $\aleph_n$-free groups are locally free. First they generalize a construction of Higman [8] to show the existence of $\aleph_n$-free groups which are not free, for all n. Next they show that the groups which they have constructed cannot be distinguished from a free group of rank $\aleph_{n+1}$ in $L_{\infty\aleph_n}$.

II. **Decision Procedures for the Elementary Theories of the Free Groups.** Verena Dyson [4] has shown that if we take a free group F of finite rank and add a new predicate $E(x, y)$ to our language whose interpretation in F is to be that x and y have the same length (in some arbitrary but fixed set of free generators for F.), then the elementary theory of F is undecidable. She also showed that the elementary theory of all non-abelian free groups (in this extended vocabulary) is undecidable, and that the free groups of finite rank are not elementarily equivalent. Her argument proceeds by first observing that the following predicates are definable in terms of $E(x, y)$

$P(x)$	"x is a positive power of some fixed element a"
$S(x, y, z)$	"x, y, and z are positive powers of a and $z = x \cdot y$"
$D(x, y)$	"x and y are positive powers of a and there is an $n > 0$ such that $x^n = y$."

From this information, she concludes that one of the essentially undecidable subsystems of arithmetic of Tarski, Mostowski and Robinson [32] is interpretable in the elementary theory of F. This gives the undecidability of this theory. The elementary inequivalence follows from the fact that the predicate $E(x, y)$ can be used to define a free set of generators for F.

III. **The Universal Sentences of the Theory of Free Groups.** Since all of the countable non-abelian free groups are subgroups of one another, they all have the same universal theory. For universal sentences ϕ involving at most two variables, there is a simple algorithm to determine whether or not ϕ holds in a non-abelian free group; however, for sentences which are even slightly more complicated, no algorithm is known.

Much of the interest in this problem stems from the fact (observed by Sanov [29]) that an undecidability result would give a class of very simple polynomials whose solvability in the integers is undecidable. Since the converse does not hold, Matiyasevich's results give no answer to this problem. A related problem, posed by Rabin [23], asks whether or not one can decide which finitely presented groups have non-abelian free quotients. Miller [21] showed that this problem is in fact a special case of Question III, and the paper cited gives several other open unsolved problems.

IIIA. **Algorithms to Solve Equations Over Free Groups.** Let F be a free group, and let X be the group freely generated by the distinct elements $x_1, x_2, \cdots$. An *equation over* F is an expression of the form $W = W'$ where W and W' are elements of $F * X$. A *solution to* $W = W'$ *in* F is a sequence $u_1, u_2, \cdots, u_i, \cdots$ of elements of F, such that if U and U' result from W and W' by replacing each occurrence of each letter x_i by the corresponding element u_i of F, then $U = U'$ in F. One can readily see that the problem of solving equations over F is a special case of Question III.

The only general solution obtained to this problem is for the case in which only one letter x_i appears in W and W'. Lyndon's solution [**16**, **17**], which was sharpened by Appel [1] and Lorenc [**13**, **14**, **15**], showed that there was a finite list of parametric words P which involved the generators of F and integer-valued parameters as exponents, such that the solutions to $W = W'$ in F are precisely the sequences in which u_i is an instance of a word in P.

For the case of equations involving more than one variable, the situation is much more difficult. Appel [2] showed that Lyndon's method could not be generalized. The positive results are all fragmentary. For example, Lyndon and Schutzeberger [**18**], showed that the equation $x_1{}^m x_2{}^n = x_3{}^p$ (where all of $|m|$, $|n|$, and $|p| > 1$) has as its only solutions sequences in which u_1, u_2, and u_3 are powers of a common element. Paul Schupp [**30**], Malcolm Wicks [**36**], and Z. M. Asel'derov [3] have given an algorithm to solve equations $W = W'$ in which $W \in F$, and $W' = (x_1, x_2)$, the commutator of x_1 and x_2. These results have been extended by C. Edmunds [5]. Further results in this direction were obtained by Ju. I. Hmelevskii [9] who gave an algorithm to solve systems which consist of finitely many equations of equations of the form $W = W'$ just described or of the form $W''(x_1) = W'''(x_2)$.

IV. **Properties Preserved under Free Product.** Let G and H be non-trivial groups, and let ϕ be a sentence of the elemtntary language of group theory which is true of both G and H. Under what circumstances is ϕ true of the free product $G * H$? H. J. Keisler [**11**] has shown that if ϕ satisfies the following property S, then the truth of ϕ is preserved to retractable embeddings:

(S): If x is a variable which occurs in ϕ in the scope of an even number of negation symbols then each occurrence of $\exists x$ is in the scope of an even number of negation symbols and each occurrence of $\forall x$ is in the scope of an odd number of negation symbols.

In particular the embeddings of both G and H into their free product $G * H$ can be retracted onto the factors, by simply factoring the free product by the normal subgroup generated by the other factor.

I have given a related result [**25**], by showing that if ϕ is a positive sentence and if one of the non-trivial groups differs from the group Z_2 with two elements, then ϕ is *never* preserved unless it is a consequence of the axioms of group theory.

V. **Properties of all Free Products of Groups.** I know of only two general results on this problem. The first [25] asserts that any positive or negative sentence which holds in a non-trivial free product of groups other than $Z_2 * Z_2$ holds of all groups. The second, an unpublished result, asserts than an analogous result holds for most generalized free products of groups, provided that the amalgam is not of index two in both of the factors.

VI. **Free Product and Elementary Equivalence.** Question VI was first posed by Feferman and Vaught [7].

For structures other than groups there are a great many results now available. Olin [22] has shown that Question VI is true if one considers structures in which there is one binary total multiplication, and false if one considers free products of semi-groups. Olin and Jónsson [10] have also answered this question negatively for all non-trivial varieties of lattices. For groups Olin has recently announced that this equivalence is not preserved under V-free products for certain varieties V of groups. The question for the variety of all groups remains open.

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