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MELLIN CONVOLUTIONS AND H-FUNCTION TRANSFORMATIONS

H. M. SRIVASTAVA [1] AND R. G. BUSCHMAN [2]

ABSTRACT. An elegant expression is obtained for an H-function transform of the Mellin convolution of two functions in terms of the Mellin convolution of H-function transforms of the functions. This main result leads to several new and interesting relations involving, for instance, Fourier, Hankel, Laplace, Stieltjes, Whittaker, and K-transforms. Some of the special cases discussed provide useful additions to known tables of these integral transforms.

1. **Introduction.** While exploring convolutions for integral transformations we came upon an interesting relation involving a convolution of the Mellin type in connection with the H-function transformation. Thus, an H-function transform of the Mellin convolution of two functions can also be expressed as the Mellin convolution of H-function transforms of the functions. Since the H-function transformation includes G-function kernels among its special cases, further specialization leads to relations involving, for instance, Laplace, Stieltjes, Whittaker, and K-transforms. Many of these special cases are of interest in themselves, and they do not seem to be given in the literature. After the introductory definitions and a discussion of some simple properties of the H-function transformation we obtain the basic convolution relation and, as a corollary, an important special case which allows us to specialize all three H-functions in a similar manner. In the latter sections we obtain the explicit forms for a number of the special cases.

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In regard to notation, we shall use the symbol $*$ to denote the Mellin convolution taken in the form

$$(1)\qquad (k*f)(y) = \int_0^\infty x^{-1}k(y/x)f(x)\,dx.$$

We shall have use for the following two properties of the Mellin convolution, both of which follow from the definition (1) in a straightforward manner. To avoid some notational difficulties we let $\odot$ denote composition, that is, $(k \odot g)(x) = k(g(x))$, and we let Ω_α denote the function such that $\Omega_\alpha(x) = x^\alpha$ for $x \geqq 0$. Then we readily have

$$(2)\qquad \Omega_c((\Omega_a k) * (\Omega_b f)) = (\Omega_{a+c}k) * (\Omega_{b+c}f),$$

$$(3)\qquad (k \odot b\Omega_\alpha) * (f \odot c\Omega_\alpha) = \alpha^{-1}(k*f) \odot bc\Omega_\alpha,$$

it being understood that $(\Omega_a f)(x) = x^a f(x)$, $(\Omega_a k)(x) = x^a k(x)$, etc., and $(f \odot \Omega_\alpha)(x) = f(\Omega_\alpha(x)) = f(x^\alpha)$.

We take the definition of the H-function in the form

$$(4)\qquad \begin{aligned} H_{p,q}^{m,n}[z] &= H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_1, A_1), \cdots, (a_p, A_p)\\ (b_1, B_1), \cdots, (b_q, B_q)\end{matrix}\right] \\ &= \frac{1}{2\pi i}\int_L \theta(\zeta) z^\zeta\,d\zeta, \end{aligned}$$

where

$$(5)\qquad \theta(\zeta) = \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j\zeta)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j\zeta)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + B_j\zeta)\prod_{j=n+1}^{p}\Gamma(a_j - A_j\zeta)}.$$

Here an empty product is interpreted as 1, the integers m, n, p, q satisfy $0 \leqq m \leqq q$ and $0 \leqq n \leqq p$, the A_j and B_j are all positive, the parameters are such that no poles of the integrand coincide, and the contour $\mathrm{Re}(\zeta) = \zeta_0$ separates the poles of one product from those of the other. If we let

$$(6)\qquad \lambda = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \sum_{j=n+1}^{p} A_j - \sum_{j=m+1}^{q} B_j,$$

then for $\lambda > 0$ the integral is absolutely convergent and defines the H-function, analytic in the sector $|\arg(z)| < \lambda\pi/2$. If $\lambda = 0$ and z is real, then other conditions must be imposed; except where it is mentioned to the contrary, we asume $\lambda > 0$. The shorter notations will be used only where confusion is not likely to arise.

We shall have need for the Mellin convolution of two H-functions. This was obtained by K. C. Gupta and U. C. Jain [4]; we use it here in the slightly altered form

$$\int_0^\infty x^{-1} H_{p,q}^{m,n}\left[a/x \left| \begin{matrix} (a_1, A_1), \cdots, (a_p, A_p) \\ (b_1, B_1), \cdots, (b_q, B_q) \end{matrix} \right.\right]$$

$$\cdot H_{v,w}^{t,u}\left[bx \left| \begin{matrix} (c_1, C_1), \cdots, (c_v, C_v) \\ (d_1, D_1), \cdots, (d_w, D_w) \end{matrix} \right.\right] dx \tag{7}$$

$$= H_{p+v,q+w}^{m+t,n+u}\left[ab \left| \begin{matrix} (e_1, E_1), \cdots, (e_{p+v}, E_{p+v}) \\ (f_1, F_1), \cdots, (f_{q+w}, F_{q+w}) \end{matrix} \right.\right],$$

in which the sets of parameters $\{(e_j, E_j)\}$ and $\{(f_j, F_j)\}$ are given, respectively, by

$$\begin{cases} (a_1, A_1), \cdots, (a_n, A_n), (c_1, C_1), \cdots, (c_v, C_v), \\ \qquad (a_{n+1}, A_{n+1}), \cdots, (a_p, A_p), \\ (b_1, B_1), \cdots, (b_m, B_m), (d_1, D_1), \cdots, (d_w, D_w), \\ \qquad (b_{m+1}, B_{m+1}), \cdots, (b_q, B_q). \end{cases} \tag{8}$$

2. **The H-function Transformation.** We take the definition of the H-function transformation in a form which differs slightly from that used by K. C. Gupta and P. K. Mittal [3]. The notation $\hat{f}$ and various abbreviated forms will be used where suitable to denote the H-transform of a function f. Let

$$\hat{f}(y) = \mathcal{H}_{p,q}^{m,n}\{f(x); y\} = \left(\mathcal{H}_{p,q,\{b_j,B_j\}}^{m,n,\{a_j,A_j\}}\{f(x)\} \right)(y)$$

$$= \int_0^\infty x^{-1} H_{p,q}^{m,n}\left[xy \left| \begin{matrix} (a_1, A_1), \cdots, (a_p, A_p) \\ (b_1, B_1), \cdots, (b_q, B_q) \end{matrix} \right.\right] f(x)\, dx, \tag{9}$$

provided that the integral exists.

Some general operational formulas for the H-transformation will be needed; they follow fairly easily by changes of variables and by using the corresponding basic relations for the H-function (cf., e.g., [4]).

$$\mathcal{H}_{p,q}^{m,n}\{f(ax); y\} = \mathcal{H}_{p,q}^{m,n}\{f(x); y/a\}. \tag{10}$$

$$\mathcal{H}_{p,q,\{b_1,B_1\}}^{m,n,\{a_1,A_1\}}\{f(x^{-1}); y\} = \mathcal{H}_{q,p,\{1-a_1,A_1\}}^{n,m,\{1-b_1,B_1\}}\{f(x); y^{-1}\}. \tag{11}$$

$$\mathcal{H}_{p,q,\{b_1,B_1\}}^{m,n,\{a_1,A_1\}}\{x^c f(x); y\} = y^{-c}\, \mathcal{H}_{p,q,\{b_1+cB_1,B_1\}}^{m,n,\{a_1+cA_1,A_1\}}\{f(x); y\}. \tag{12}$$

$$\mathcal{H}_{p,q,\{b_1,B_1\}}^{m,n,\{a_1,A_1\}}\{f(x^c); y\} = \mathcal{H}_{p,q,\{b_1,cB_1\}}^{m,n,\{a_1,cA_1\}}\{f(x); y^c\}, \quad c > 0. \tag{13}$$

To obtain (11), (12), and (13) we have used those relations which allow $H[x^{-1}]$, $x^c H[x]$ and $H[x^c]$, respectively, to be expressed in terms of $H[x]$ with altered parameters; these basic relations could be obtained directly from manipulations in our formulas (4) and (5). The multiplication formula for the Γ-function can be applied to (4) and (5) to express $H_{p,q}^{m,n}$ in terms of $H_{Np,Nq}^{Nm,Nn}$ as in [4]. We shall have use for the case $N = 2$ written in the form

(14)
$$H_{2p,2q}^{2m,2n}\left[z \,\middle|\, \begin{matrix}(a_1, A_1), (a_1 + 1/2, A_1), \cdots, (a_p, A_p), (a_p + 1/2, A_p)\\(b_1, B_1), (b_1 + 1/2, B_1), \cdots, (b_q, B_q), (b_q + 1/2, B_q)\end{matrix}\right]$$
$$= \pi^{\delta/2} 2^{\phi - 1} H_{p,q}^{m,n}\left[2^{-\theta} z^{1/2} \,\middle|\, \begin{matrix}(2a_1, A_1), \cdots, (2a_p, A_p)\\(2b_1, B_1), \cdots, (2b_q, B_q)\end{matrix}\right],$$

where

(15)
$$\begin{cases}\delta = 2(m + n) - (p + q),\ \theta = \sum_{j=1}^{p} A_j - \sum_{j=1}^{q} B_j,\\[2mm] \phi = m + n - p + 2 \sum_{j=1}^{p} a_j - 2 \sum_{j=1}^{q} b_j.\end{cases}$$

From this we can write

(16)
$$\mathcal{H}_{2p,2q,\{(b_j,B_j),(b_j+1/2,B_j)\}}^{2m,2n,\{(a_j,A_j),(a_j+1/2,A_j)\}}\{f(x); y\}$$
$$= \pi^{\delta/2}\, 2^{\phi} \mathcal{H}_{p,q,\{2b_j,B_j\}}^{m,n,\{2a_j,A_j\}}\{f(x^2); 2^{-\theta} y^{1/2}\},$$

in which δ, ϕ, and θ are given by (15). Combining formulas (13) and (16) leads to an alternative result which appears simpler in some respects and may be of some use; however, (16) is the appropriate form when specializing to the G-function kernels, in which case $\theta = p - q$.

3. **The Convolution Property.** The derivation is straightforward under suitable restrictions on the parameters such that the interchange of order of integration can be justified by absolute convergence. We write down the Mellin convolution of the two H-function transforms and note that we can manipulate so as to use (7). For convenience, let

(17)
$$\hat{k}(y) = \mathcal{H}_{p,q}^{m,n}\{k(x); y\} \quad \text{and} \quad \hat{f}(y) = \mathcal{H}_{v,w}^{t,u}\{f(x); y\};$$

then we shall establish our main result given by

(18)
$$(\hat{k} * \hat{f})(y) = \int_0^\infty z^{-1} H_{p+v,q+w}^{m+t,n+u}[yz] \int_0^\infty x^{-1} k(z/x) f(x)\, dx\, dz,$$

provided that each side exists. We first state the following

THEOREM. *If each of the following five integrals*

$$\int_0^\infty \int_0^\infty x^{-1}y^{s-1}k(y/x)f(x)\,dx\,dy, \tag{19}$$

$$\int_0^\infty \int_0^\infty x^{-1}y^{s-1}H_{p,q}^{m,n}\,[xy]\;k(x)\,dx\,dy, \tag{20}$$

$$\int_0^\infty \int_0^\infty x^{-1}y^{s-1}H_{v,w}^{t,u}[xy]\;f(x)\,dx\,dy, \tag{21}$$

$$\int_0^\infty \int_0^\infty x^{-1}y^{s-1}\hat{k}(y/x)\hat{f}(x)\,dx\,dy, \tag{22}$$

and

$$\int_0^\infty \int_0^\infty x^{-1}y^{s-1}H_{p+v,q+w}^{m+t,n+u}\;[xy]\;(k*f)(x)\,dx\,dy, \tag{23}$$

be assumed to be an absolutely convergent double integral, then equation (18) *holds.*

REMARK. The assumptions about the absolute convergence of the double integrals (19) through (23) would enable us to compute the Mellin transforms of both sides of (18), and, as we shall observe below, our theorem then follows easily by using the well-known Fubini's theorem.

PROOF OF THE THEOREM. If we denote the Mellin transform of $\phi(x)$ by $\Phi(s)$, that is,

$$\Phi(s) = \mathcal{M}\{\phi(x) : s\} = \int_0^\infty x^{s-1}\phi(x)\,dx, \tag{24}$$

and

$$K(s) = \mathcal{M}\{k(x) : s\},\ F(s) = \mathcal{M}\{f(x) : s\}, \text{ etc.}, \tag{25}$$

from the definition (1) we have

$$\mathcal{M}\{(k*f)(y) : s\} = \int_0^\infty y^{s-1}\left\{\int_0^\infty x^{-1}k(y/x)f(x)\,dx\right\}dy. \tag{26}$$

Since the double integral (19) is assumed to be absolutely convergent, we can use Fubini's theorem to change the order of integration on the right-hand side of (26) to get

$$\begin{aligned}\mathcal{M}\{(k*f)(y):s\} &= \int_0^\infty x^{-1}f(x)\left\{\int_0^\infty y^{s-1}k(y/x)\,dy\right\}dx \\ &= \mathcal{M}\{k(y):s\}\,\mathcal{M}\{f(x):s\} = K(s)\,F(s),\end{aligned} \tag{27}$$

where the notations in (25) are used.

By absolute convergence of the double integrals (20) and (21), and by using Fubini's theorem again, equations (17) yield

$$\begin{cases}\mathcal{M}\{\hat{k}(y):s\} = \mathcal{M}\{H_{p,q}^{m,n}[y]\,:s\}K(-s), \\ \mathcal{M}\{\hat{f}(y):s\} = \mathcal{M}\{H_{v,w}^{t,u}[y]\,:s\}F(-s).\end{cases} \tag{28}$$

Now using (27) in conjunction with (28), we obtain

$$\begin{aligned}&\mathcal{M}\{(\hat{k}*\hat{f})(y):s\} = \mathcal{M}\{\hat{k}(y):s\}\mathcal{M}\{\hat{f}(x):s\} \\ &= \mathcal{M}\{H_{p,q}^{m,n}[y]\,:s\}\mathcal{M}\{H_{v,w}^{t,u}[x]\,:s\}K(-s)\,F(-s),\end{aligned} \tag{29}$$

since the double integral (22) is absolutely convergent. Equation (29) expresses the Mellin transform of the first member of (18).

On the other hand, the Mellin transform of the second member of (18) is

$$I = \int_0^\infty y^{s-1}\left\{\int_0^\infty z^{-1}\,H_{p+v,q+w}^{m+t,n+u}[yz]\,(k*f)(z)\,dz\right\}dy. \tag{30}$$

Assuming the double integral (23) {and hence (30)} to be absolutely convergent, and inverting the order of integration in (30) by appealing to Fubini's theorem once again, we find that

$$I = \mathcal{M}\{H_{p+v,\,q+w}^{m+t,n+u}[\xi]\,:s\}\int_0^\infty z^{-s-1}(k*f)(z)\,dz, \tag{31}$$

where we have set $yz = \xi$. Applying (7) and (27) to the second member of this last equation (31), we observe that

$$I = \mathcal{M}\{H_{p,q}^{m,n}[\xi]\,:s\}\mathcal{M}\{H_{v,w}^{t,u}[\eta]\,:s\}\,K(-s)\,F(-s). \tag{32}$$

Equations (29) and (32), together, exhibit the fact that the Mellin transforms of the two sides of (18) are equal, and so, the two sides of (18) must also be equal.

This evidently completes the proof of our formula (18) under the various hypotheses contained in the theorem.

Application of Formula (18). At the outset we write the formula (18) in the more convenient form

$$(33)\quad \begin{aligned}&(\mathcal{H}^{m,n,\{a_j,A_j\}}_{p,q,\{b_j,B_j\}}\{k(x)\} * \mathcal{H}^{t,u,\{c_j,C_j\}}_{v,w,\{d_j,D_j\}}\{f(x)\})(y)\\ &= (\mathcal{H}^{m+t,n+u,\{e_j,E_j\}}_{p+v,q+w,\{f_j,F_j\}}\{(k*f)(x)\})(y),\end{aligned}$$

where the parameters e_j, E_j, f_j, F_j are described in (8). If we set $t = m$, $u = n$, $v = p$, $w = q$, and $c_j = a_j + 1/2$, $C_j = A_j$ for $1 \leqq j \leqq p$, and $d_j = b_j + 1/2$, $D_j = B_j$ for $1 \leqq j \leqq q$, and then rearrange the parameters we obtain, by using (16),

$$(34)\quad \begin{aligned}&(\mathcal{H}^{m,n,\{a_j,A_j\}}_{p,q,\{b_j,B_j\}}\{k(x)\} * \mathcal{H}^{m,n,\{a_j+1/2,A_j\}}_{p,q,\{b_j+1/2,B_j\}}\{f(x)\})(y)\\ &= (\pi^{\delta/2}2^{\phi}\mathcal{H}^{m,n,\{2a_j,A_j\}}_{p,q,\{2b_j,B_j\}}\{(k*f)(x^2)\})(2^{-\theta}y^{1/2}),\end{aligned}$$

where δ, ϕ, and θ are given in (15).

4. **The Laplace and Stieltjes Transformations.** If we let $m = q = 1$ and $n = p = 0$, then from the relation

$$(35)\quad H^{1,0}_{0,1}\left[x \,\middle|\, \begin{matrix}-\\(b,1)\end{matrix}\right] = G^{1,0}_{0,1}\left[x \,\middle|\, \begin{matrix}-\\b\end{matrix}\right] = x^b e^{-x},$$

we have

$$(36)\quad \begin{aligned}\mathcal{H}^{1,0}_{0,1,(b,1)}\{f(x);\, y\} &= y^b\mathcal{L}\{x^{b-1}f(x);\, y\}\\ &= (\Omega_b\mathcal{L}\{x^{b-1}f(x)\})(y),\end{aligned}$$

in the standard notation for the Laplace transform (cf. [2])

$$(37)\quad \mathcal{L}\{f(x);\, y\} = \int_0^\infty e^{-xy}f(x)\,dx.$$

First we consider the Mellin convolution of two Laplace transforms. From (33) we obtain

$$(38)\quad \begin{aligned}&((\Omega_b\mathcal{L}\{x^{b-1}k(x)\}) * (\Omega_d\mathcal{L}\{x^{d-1}f(x)\}))(y)\\ &= \int_0^\infty x^{-1}G^{2,0}_{0,2}\left[xy \,\middle|\, \begin{matrix}-\\b,d\end{matrix}\right](k*f)(x)\,dx\\ &= 2y^{(b+d)/2}\int_0^\infty K_{b-d}(2(xy)^{1/2})x^{(b+d-2)/2}(k*f)(x)\,dx,\end{aligned}$$

where we have used the relationship (4), p. 216 in [1] to express $G^{2,0}_{0,2}$ in terms of the modified Bessel function of the third kind of order $b - d$. Although the K-transform is sometimes defined with this kernel, in the tables [2, Vol. II, p. 125 et seq.] it appears in the form

$$\mathcal{K}_\nu\{f(x);\, y\} = \int_0^\infty (xy)^{1/2} K_\nu(xy) f(x)\, dx. \tag{39}$$

After a change in variables in the last part of (38) we have

$$\begin{aligned}((\Omega_b \mathcal{L}\{x^{b-1}k(x)\}) * (\Omega_d \mathcal{L}\{x^{d-1}f(x)\}))(y) \\ &= 2^{3/2} y^{(b+d-1/2)/2} \mathcal{K}_{b-d}\{x^{b+d-3/2}(k*f)(x^2);\, 2y^{1/2}\} \\ &= 2^{2-b-d}(\Omega_{b+d-1/2}\mathcal{K}_{b-d}\{x^{b+d-3/2}(k*f)(x^2)\})(2y^{1/2}).\end{aligned} \tag{40}$$

If $d = b + 1/2$ then from (34) or (40) we can obtain the formula

$$\begin{aligned}((\Omega_b \mathcal{L}\{x^{b-1}k(x)\}) * (\Omega_{b+1/2}\mathcal{L}\{x^{b-1/2}f(x)\}))(y) \\ = 2\pi^{1/2} y^b \mathcal{L}\{x^{2b-1}(k*f)(x^2);\, 2y^{1/2}\},\end{aligned} \tag{41}$$

involving only the Laplace transforms. The convolution property (2) can be used to simplify (41) to the form

$$\begin{aligned}((\mathcal{L}\{x^{b-1}k(x)\}) * (\Omega_{1/2}\mathcal{L}\{x^{b-1/2}f(x)\}))(y) \\ = 2\pi^{1/2} \mathcal{L}\{x^{2b-1}(k*f)(x^2);\, 2y^{1/2}\}.\end{aligned} \tag{42}$$

From (2) also it can be seen that in (41) and (42) there is actually no restriction to choose a particular value for b. For $b = 1/2$ we have the especially simple form

$$\begin{aligned}((\mathcal{L}\{x^{-1/2}k(x)\}) * (\Omega_{1/2}\mathcal{L}\{f(x)\}))(y) \\ = 2\pi^{1/2} \mathcal{L}\{(k*f)(x^2);\, 2y^{1/2}\},\end{aligned} \tag{43}$$

which can be interpreted as the Laplace transform of a slightly modified Mellin convolution, since it can be rewritten in the form

$$\mathcal{L}\{(k*f)(x^2);\, y\} = 2^{-1}\pi^{-1/2}(\mathcal{L}\{x^{-1/2}k(x)\} * \Omega_{1/2}\mathcal{L}\{f(x)\})(y^2/4). \tag{44}$$

The generalized Stieltjes transformation is taken in the form [2, Vol. II, p. 233]

$$\mathcal{S}_\rho\{f(x);\, y\} = \int_0^\infty (x+y)^{-\rho} f(x)\, dx, \tag{45}$$

which reduces to the ordinary case [op. cit., p. 215] for $\rho = 1$. From the relation

$$H_{1,1}^{1,1}\left[x \,\middle|\, \begin{matrix}(c,1)\\(b,1)\end{matrix}\right] = G_{1,1}^{1,1}\left[x \,\middle|\, \begin{matrix}c\\b\end{matrix}\right] = \Gamma(b+1-c)x^b(1+x)^{c-b-1}, \tag{46}$$

we have

$$\mathcal{H}^{1,1,(c,1)}_{1,1,(b,1)}\ \{f(x); y\} = \Gamma(b+1-c)y^{c-1}\mathcal{S}_{b+1-c}\ \{x^{b-1}f(x); y^{-1}\}$$
$$(47) \qquad = \Gamma(b+1-c)y^{b}\mathcal{S}_{b+1-c}\{x^{-c}f(x^{-1}); y\}$$
$$= \Gamma(b+1-c)(\Omega_b\mathcal{S}_{b+1-c}\{x^{-c}f(x^{-1})\})(y),$$

where the two forms are related by equation (4), p. 233 in [2, Vol. II]. From the convolution property (34) after simplification, including the use of (2), we have

$$(48) \qquad \begin{aligned} &((\mathcal{S}_\rho\{x^{-c}k(x^{-1})\}) * (\Omega_{1/2}\mathcal{S}_\rho\{x^{-c-1/2}f(x^{-1})\}))(y) \\ &\quad = (2\pi^{1/2}\Gamma(\rho-1/2)/\Gamma(\rho))\mathcal{S}_{2\rho-1}\{x^{-2c}(k*f)(x^{-2}); y^{1/2}\}, \end{aligned}$$

or in the special case $\rho = 1, c = 0$,

$$(49) \qquad \begin{aligned} &((\mathcal{S}\{k(x^{-1})\}) * (\Omega_{1/2}\mathcal{S}\{x^{-1/2}f(x^{-1})\}))(y) \\ &\quad = 2\pi\mathcal{S}\{(k*f)(x^{-2}); y^{1/2}\}. \end{aligned}$$

Since we also have

$$(50) \qquad H^{0,1}_{1,0}\left[\, x \;\middle|\; \begin{matrix}(c,1)\\ -\end{matrix}\right] = x^{c-1}e^{-1/x},$$

we can write

$$(51) \qquad \mathcal{H}^{0,1,(c,1)}_{1,0}\ \{f(x); y\} = y^{c-1}\mathcal{L}\{x^{-c}f(x^{-1}); y^{-1}\}.$$

If we now use the convolution formula (33) and the relations (36), (51), and (47) we obtain a formula relating the Laplace and the Stieltjes transforms. A little simplification gives us

$$(52) \qquad \begin{aligned} &((\mathcal{L}\{x^{b-1}k(x)\}) * (\Omega_{-\rho}\mathcal{L}\{x^{\rho-b-1}f(x^{-1}); \Omega_{-1}\}))(y) \\ &\quad = \Gamma(\rho)\mathcal{S}_\rho\{x^{\rho-b-1}(k*f)(x^{-1}); y\}; \end{aligned}$$

or in the special case $\rho = 1, b = 0$,

$$(53) \qquad \begin{aligned} &((\mathcal{L}\{x^{-1}k(x)\}) * (\Omega_{-1}\mathcal{L}\{f(x^{-1}); \Omega_{-1}\}))(y) \\ &\quad = \mathcal{S}\{(k*f)(x^{-1}); y\}. \end{aligned}$$

5. **The Whittaker Transformation.** The generalized Whittaker transform was defined by H. M. Srivastava [5] in the form

$$(54) \qquad \mathcal{S}^{(\rho,\sigma)}_{q,k,m}\ \{f(x); y\} = \int_0^\infty (xy)^{\sigma-1/2}e^{-qxy/2}W_{k,m}(\rho xy)f(x)\,dx.$$

(Notice that the special case $\sigma = m$ and $\rho = q = 1$ of this transform was introduced, several years ago, by R. S. Varma [6].) We consider here the case $\rho = q = 1$, for which we shall condense the symbol to the form $\mathcal{S}^{\sigma}_{k,m}$. Inasmuch as the formula (6), p. 216 in [1] gives us

$$(55)\quad \begin{aligned} x^{-1}G^{2,0}_{1,2}\left[x\,\middle|\,\begin{matrix}\sigma+3/2-k\\ \sigma+m+1,\sigma-m+1\end{matrix}\right] &= G^{2,0}_{1,2}\left[x\,\middle|\,\begin{matrix}\sigma+1/2-k\\ \sigma+m,\sigma-m\end{matrix}\right]\\ &= x^{\sigma-1/2}e^{-x/2}W_{k,m}(x),\end{aligned}$$

we can thus relate this transform to the H-function transform by

$$(56)\quad \mathcal{H}^{2,0,(\sigma+3/2-k,1)}_{1,2,(\sigma+m+1,1),(\sigma-m+1,1)}\{f(x);y\} = (\Omega_1\mathcal{S}^{\sigma}_{k,m}\{f(x)\})(y).$$

From our convolution theorem (33) we can obtain

$$(57)\quad \begin{aligned} &((\mathcal{S}^{\sigma}_{h,m}\{k(x)\}) * (\mathcal{S}^{\rho}_{\lambda,\mu}\{f(x)\}))(y)\\ &= \int_0^{\infty} G^{4,0}_{2,4}\left[xy\,\middle|\,\begin{matrix}\rho+1/2-\lambda,\sigma+1/2-h\\ \sigma+m,\sigma-m,\rho+\mu,\rho-\mu\end{matrix}\right](k*f)(x)\,dx.\end{aligned}$$

Either by specializing parameters in (57), or directly from the convolution relation (34), we have

$$(58)\quad \begin{aligned} &((\mathcal{S}^{\sigma}_{\kappa+1/4,\mu}\{k(x)\}) * (\mathcal{S}^{\sigma+1/2}_{\kappa+1/4,\mu}\{f(x)\}))(y)\\ &= \pi^{1/2}2^{-2\kappa-2\sigma+1/2}y^{-1/2}\mathcal{S}^{2\sigma+1}_{2\kappa,2\mu}\{(k*f)(x^2);2y^{1/2}\}.\end{aligned}$$

The convolution of a Whittaker transform and a Laplace transform leads to a kernel involving $G^{3,0}_{1,3}$; a Whittaker transform and a Stieltjes transform, to $G^{3,1}_{2,3}$; and a Whittaker transform and a K-transform to $G^{4,0}_{1,4}$.

The K-transform, defined by (39), does not follow as a special case of the Whittaker transform considered in this section, since $\mathcal{S}^{\sigma}_{0,\nu}$ has an exponential function in the kernel along with the Bessel function K_{ν}. We can obtain a convolution formula for the K-transform of (39) by use of (34), but it seems messy in comparison with the result (40).

6. **Hankel and Fourier Transforms.** The Bessel function J_{λ} can be represented in terms of the H-function by use of [1, p. 216 (3)] as

$$(59)\quad H^{1,0}_{0,2}\left[x\,\middle|\,\begin{matrix}\overline{}\\ ((\mu+\lambda)/2,1),((\mu-\lambda)/2,1)\end{matrix}\right] = x^{\mu/2}J_{\lambda}(2x^{1/2}).$$

In order to match the tables [2] we choose the Hankel transform defined in the form

$$(60)\quad \mathcal{H}_{\lambda}\{f(x);y\} = \int_0^{\infty}(xy)^{1/2}J_{\lambda}(xy)f(x)\,dx;$$

hence for $\lambda = \pm 1/2$ we obtain $\sqrt{(2/\pi)}$ times the Fourier sine and cosine transforms as defined in the tables [2, Vol. I]. These kernels are examples for the case where the expression in equation (6) vanishes, and we need the additional condition in regard to the contour involved in the definition of the H-function that $\zeta_0 > \mathrm{Re}(\mu/2 - 1/4)$. The relation between the Hankel and H-function transforms is given by

$$\begin{aligned}(61)\qquad &\mathcal{H}^{1,0}_{0,2,((\mu+\lambda)/2,1),\,((\mu-\lambda)/2,1)}\{f(x); y\}\\ &= 2^{1/2}y^{\mu/2-1/4}\mathcal{H}_\lambda\{x^{\mu-3/2}f(x^2); 2y^{1/2}\}\\ &= 2^{1-\mu}(\Omega_{\mu-1/2}\mathcal{H}_\lambda\{x^{\mu-3/2}f(x^2)\})(2y^{1/2}).\end{aligned}$$

Starting with the convolution formula (33), if we use (3) and set $h(x) = k(x^2)$ and $g(x) = f(x^2)$, we shall obtain

$$\begin{aligned}(62)\qquad &((\Omega_{\mu-1/2}\mathcal{H}_\lambda\{x^{\mu-3/2}h(x)\}) * (\Omega_{\sigma-1/2}\mathcal{H}_\rho\{x^{\sigma-3/2}g(x)\}))(y)\\ &= 2^{\mu+\sigma-1}\int_0^\infty x^{-1}G^{2,0}_{0,4}\left(x^2y^2/16 \,\middle|\, \overline{(\mu+\lambda)/2, (\sigma+\rho)/2, (\sigma-\rho)/2, (\mu-\lambda)/2}\right)\\ &\qquad\cdot (h * g)(x)\,dx.\end{aligned}$$

The special case of interest in which $\rho = \lambda$ and $\sigma = \mu + 1$ can be obtained from (34) by employing (2) and (3) or from (62) with the use of [1, p. 216 (11)] to obtain the desired kernel. If we further set $\tau = \mu - 3/4$, we obtain the nice form

$$\begin{aligned}(63)\qquad &((\Omega_{-1/4}\mathcal{H}_\lambda\{x^{\tau-3/4}h(x)\}) * (\Omega_{3/4}\mathcal{H}_\lambda\{x^{\tau+1/4}g(x)\}))(y)\\ &= \sqrt{2}\mathcal{H}_{2\lambda}\{x^{2\tau}(h * g)(x^2); 2y^{1/2}\}.\end{aligned}$$

From (63) we write down the following special cases for $\lambda = \pm 1/4$, which display the Fourier sine and cosine transforms of the Mellin convolution.

$$\begin{aligned}(64)\qquad &\mathcal{F}_s\{x^{2\tau}(h * g)(x^2); y\}\\ &= (\sqrt{\pi}/2)((\Omega_{-1/4}\mathcal{H}_{1/4}\{x^{\tau-3/4}h(x)\}) * (\Omega_{3/4}\mathcal{H}_{1/4}\{x^{\tau+1/4}g(x)\}))(y^2/4),\end{aligned}$$

$$\begin{aligned}(65)\qquad &\mathcal{F}_c\{x^{2\tau}(h * g)(x^2); y\}\\ &= (\sqrt{\pi}/2)((\Omega_{-1/4}\mathcal{H}_{-1/4}\{x^{\tau-3/4}h(x)\}) * (\Omega_{3/4}\mathcal{H}_{-1/4}\{x^{\tau+1/4}g(x)\}))(y^2/4).\end{aligned}$$

If we let $\lambda = \pm 1/2$, we obtain convolutions of Fourier transforms. In order to present these in a convenient form we use (2) and let $\tau = \kappa + 3/4$ to simplify a bit in order to obtain

$$\begin{aligned}(66)\qquad &((\mathcal{F}_s\{x^\kappa h(x)\}) * (\Omega_1\mathcal{F}_s\{x^{\kappa+1}g(x)\}))(y)\\ &= (\pi/2)(\Omega_{1/2}\mathcal{H}_1\{x^{2\kappa+3/2}(h * g)(x^2)\})(2y^{1/2}),\end{aligned}$$

$$
\begin{aligned}
&((\mathcal{F}_c\{x^{\kappa}h(x)\}) * (\Omega_1 \mathcal{F}_c\{x^{\kappa+1}g(x)\}))(y) \\
&= -(\pi/2)(\Omega_{1/2}\mathcal{H}_1\{x^{2\kappa+3/2}(h * g)(x^2)\})(2y^{1/2}).
\end{aligned} \tag{67}
$$

If we return to (62), set $\lambda = -\rho = 1/2$, $\sigma = \mu = \kappa + 3/2$, and simplify the resulting equation, we obtain

$$
\begin{aligned}
&(\mathcal{F}_s\{x^{\kappa}h(x)\} * \mathcal{F}_c\{x^{\kappa}g(x)\}) \\
&= \pi(\Omega_{-1/2}\mathcal{H}_0\{x^{2\kappa+1/2}(h * g)(x^2)\})(2y^{1/2}).
\end{aligned} \tag{68}
$$

It should be noted that in view of (2) there is no restriction to take special values for τ and κ, such as $\tau = 0$ or $\kappa = 0$, in equations (63) through (68), since these powers can be absorbed by writing $x^{\kappa}h(x) = h_1(x)$ and $x^{\kappa}g(x) = g_1(x)$.

7. **Related Formulas.** It may be of interest to note the consequences of choosing one of the functions which is to be transformed as the generalized function $\delta(x - 1)$, the shifted "Dirac delta function". By the sifting property of $\delta(x - 1)$ we readily have

$$
f(x) = x^{\alpha}\delta(x - 1) \Longrightarrow (k * f)(x^2) = k(x^2) \tag{69}
$$

As a first example, we choose $x^{b-1/2}f(x) = \delta(x - 1)$ in formula (42) and replace y by $y^2/4$ so that we have

$$
\int_0^{\infty} u^{-1/2}e^{-u}\hat{k}_1(y^2/4u)\,du = 2\pi^{1/2}\mathcal{L}\{x^{2b-1}k(x^2); y\}, \tag{70}
$$

where $\hat{k}_1(y) = \mathcal{L}\{x^{b-1}k(x)\}$. For $b = 1$, for example, after a change of variable we obtain the known result [2, Vol. I, p. 131 (23)] with $n = 1$.

If we choose $\lambda = 0$, $\tau = 3/4$, and $xg(x) = \delta(x - 1)$ in (63) we obtain an analogous result

$$
y^{-1/2}\int_0^{\infty} u^{1/2}J_0(u)\hat{h}(y^2/4u)\,du = \mathcal{H}_0\{x^{3/2}h(x^2); y\}, \tag{71}
$$

which does not appear in the tables [2].

In fact, we have an entire family of these formulas arising from (34), if we set $f(x) = \delta(x - 1)$. Writing this out we see that in general the transform of $k(x^2)$ is connected with chains of two transformations with closely related kernels.

If we take $k(x) = U(x - 1)$, the shifted step function, then from (44) and the known result [2, Vol. I, p. 135 (15)] we shall have

$$(72)\qquad \mathcal{L}\left\{\int_0^{x^2} u^{-1}f(u)\,du;\,y\right\}=y^{-1}\int_0^{\infty}\operatorname{Erfc}(v^{-1/2}y/2)\hat{f}(v)\,dv.$$

We conclude by remarking that various types of formulas can be obtained for other integral transforms by different choices for one of the functions of the convolution.

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University of Victoria, Victoria, British Columbia, Canada V8W 2Y2
University of Wyoming, Laramie, Wyoming 82071