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THE BOUNDING PROPERTIES OF THE MULTIPOINT PADÉ APPROXIMANT TO A SERIES OF STIELTJES ON THE REAL LINE

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The present results have been proved by Barnsley [2] using a continued fractions approach, based on earlier work by Baker [1]. They have also been established variationally by Epstein and Barnsley [3].

Let $F(x)$ be representable by a series of Stieltjes with radius of convergence (at least) $R \geqq 0$. Then $F(x)$ is an analytic function for $x \in (-R, \infty)$, and can be written in the form

$$F(x) = \int_0^{1/R} \frac{d\phi(u)}{(1 + ux)},$$

where $\phi(u)$ is a bounded monotone non-decreasing function which attains infinitely many different values for $u \in [0, 1/R)$. Let $-R < x_1 < x_2 < \cdots < x_Q < \infty$ and $F^{(n)}(x_p)$ $(n = 0, 1, \cdots, N_p - 1; p = 1, 2, \cdots, Q)$ be given, with $N_p \geqq 1$ and $\sum_{p=1}^{Q} N_p = N$. Then we have N pieces of information about $F(x)$ associated with Q points. The corresponding *multipoint Padé approximant* to $F(x)$ is defined as the function $F_{N(Q)}(x) = A_N(x)/B_N(x)$ where $A_N(x)$ and $B_N(x)$ are the polynomials of degrees $[(N-1)/2]$ and $[N/2]$, respectively, which are uniquely specified by the requirements

$$F^{(n)}_{N(Q)}(x_p) = F^{(n)}(x_p),\ n = 0, 1, \cdots, N_p - 1;\ p = 1, 2, \cdots, Q,$$

together with $B_N(0) = 1$, say. Here $[R]$ denotes the integer part of the number R. The *existence* of $F_{N(Q)}(x)$ thus defined is assured because $F(x)$ is representable by a series of Stieltjes: for $F(x)$ an arbitrary function the corresponding multipoint Padé approximant must be defined in terms of the "modified" interpolation problem [4]. To actually obtain $F_{N(Q)}(x)$ one can either linearize the set of defining equations, or else use a continued fractions approach [2]. For the case $Q = N$, Wuytack [5] has presented applicable algorithms.

The bounding properties of $F_{N(Q)}(x)$ with respect to $F(x)$ are

$$F_{N(Q)}(x) < F(x), \text{ for } -R < x < x_1,$$

$$\operatorname{sgn}(F(x) - F_{N(Q)}(x)) = (-1)^{N_1 + N_2 + \cdots + N_p}, \quad \text{for } x_p < x < x_{p+1}\ (p = 1, 2, \cdots, Q-1),$$

$$\operatorname{sgn}(F(x) - F_{N(Q)}(x)) = (-1)^N, \text{ for } x_Q < x < \infty.$$

If, in addition to the original set of information, we are also given a lower bound to R, $R^L \leqq R$, then it is possible to construct the corresponding *complementary multipoint Padé approximant* $F^C_{N(Q)}(x) = C_N(x)/D_N(x)$ where $C_N(x)$ and $D_N(x)$ are the polynomials of degrees $[N/2]$ and $[(N+1)/2]$, respectively, which are uniquely specified by the requirements

$$(F^C_{N(Q)})^{(n)}(x_p) = F^{(n)}(x_p),\ n = 0, 1, \cdots, N_p - 1;\ p = 1, 2, \cdots, Q,$$

together with $D_N(-R^L) = 0$, and $D_N(0) = 1$, say. The bounds imposed by $F^C_{N(Q)}(x)$ on $F(x)$ are the precise complement to those imposed by $F_{N(Q)}(x)$. Together, $F_{N(Q)}(x)$ and $F^C_{N(Q)}(x)$ supply the best possible upper and lower bounds, (on the basis of the given information), to $F(x)$ for $x \in (-R^L, \infty)$.

In the special case that all of the given information is associated with the single point $x = 0$, the corresponding multipoint Padé approximant is seen to be the usual $[[(N-1)/2]/[N/2]]$ one point Padé approximant. The bounding properties of $F^C_{N(1)}(x)$ have been described by Baker [1].

If, further to the original set of information, one also has

$$F(x) \sim F_0 + \left(\frac{1}{x}\right)F_{-1} + \cdots + \left(\frac{1}{x}\right)^{J-1} F_{-J} + H$$

where H consists of all higher terms with unknown or divergent coefficients, then these additional J pieces of information can be incorporated into the construction of $F_{N(Q)}(x)$ and $F^C_{N(Q)}(x)$, yielding two new approximants $F_{N(Q)+J}(x)$ and $F^C_{N(Q)+J}(x)$. The directions of the bounds imposed by the new approximants are the same as if no additional information has been used; however, the bounds themselves, being best possible (on the basis now of a larger set of information), are tighter.

By working within the framework of the variational approach [3] it is possible to discuss those multipoint Padé approximants which reduce, in the case that all of the given information is associated with the single point $x = 0$, to the $[([(N-1)/2] + K)/([N/2] - K)]$ $(K = 1, 2, \cdots, [N/2])$ non-optimal one-point Padé approximants. Thus it is possible to discuss the Padé table for *multipoint* Padé approximants to functions representable by series of Stieltjes.

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