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A REMARK ON SIMPLE PATH FIELDS IN POLYHEDRA OF CHARACTERISTIC ZERO

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1. **Introduction.** A *path field* φ on a space X is a map $\varphi : X \to X^I$ such that $\varphi(x)(0) = x$, $x \in X$, and if $\varphi(x)(t) = x$ for $t > 0$ then $\varphi(x)$ is the constant path. φ is non-singular if $\varphi(x)$ is never the constant path. A non-singular path field φ is *simple* if $\varphi(x)$ is a simple arc for each x. Differentiable manifolds of (Euler) characteristic zero admit simple path fields while topological manifolds of characteristic zero are known to admit non-singular path fields [1]. The existence of simple path fields in the topological category is an open question. The purpose of this note is to observe that in the case of *triangulated* manifolds of characteristic zero it is easy to find a simple path field. In fact, every polyhedron K satisfying the so-called Wecken condition of characteristic zero admits a simple path field φ such that the track of $\varphi(x)$ is a broken line segment.

2. **Preliminaries.** Let K denote a finite polyhedron. We will not distinguish in the notation between K as a simplicial complex and K as the underlying space. If x is a point of K, then $\sigma(x)$ is the unique (open) simplex of K which contains x. Furthermore, if Δ represents the diagonal in $K \times K$, there is a special neighborhood of Δ given by

$$(1) \qquad \eta(\Delta) = \{(x, y) : \sigma(x) \text{ and } \sigma(y) \text{ have a common vertex}\}.$$

Each point $x \in K$ also has a special neighborhood defined by

$$(2) \qquad V(x) = \{y : (x, y) \in \eta(\Delta)\}.$$

Following R. F. Brown [2], we call a map $f : K \to K$ a *proximity map* if $f(x) \in V(x)$ for all $x \in K$.

If a, b are points of K in the same closed simplex, then $[a, b]$ will denote the segment from a to b. The following lemma is a somewhat stronger version of a lemma contained in [2] and [3]. The proof is the same except for additional observations.

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LEMMA 2.1. *There exists a map* $\alpha : \eta(\Delta) \to K^I$ *such that*

(1) $\alpha(x, y)$ *is a path from* x *to* y,

(2) $\alpha(x, x)$ *is the constant path at* x,

(3) *the track of* $\alpha(x, y)$ *has the form*
$[x, z] \cup [z, y]$,
where $z = z(x, y)$,

(4) $\alpha(y, x)$ *is the reverse of* $\alpha(x, y)$, i.e.,
$\alpha(y, x)(t) = \alpha(x, y)(1 - t)$, and

(5) *if* $x \neq y$, $\alpha(x, y)$ *is a simple path.*

PROOF. Following [2] and [3] we assume K is realized in some Euclidean space and each $x \in K$ is provided with coordinates, i.e.,

$$x = \Sigma \lambda_j v_j, \lambda_j \geqq 0, \Sigma \lambda_j = 1, \tag{3}$$

where $v_1, \cdots, v_n$ are the vertices of K. If $y = \Sigma \mu_j v_j$, set

$$\beta = \beta(x, y) = \Sigma (\lambda_j \mu_j)^{1/2} \tag{4}$$

and

$$z = z(x, y) = (1/\beta) \Sigma (\lambda_j \mu_j)^{1/2} v_j. \tag{5}$$

A simple argument shows that

$$[x, z] \cap [z, y] = z, (x, y) \in \eta(\Delta). \tag{6}$$

Assuming that K is simplicially imbedded in some Euclidean space with the usual metric d, define

$$\mathfrak{L}(w) = \begin{cases} d(x, w), & w \in [x, z], \\ d(x, z) + d(z, w), & w \in [z, y], \end{cases} \tag{7}$$

and parametrize $[x, z] \cup [z, y]$ with a map

$$\alpha(x, y) : I \to K, (x, y) \in \eta(\Delta) \tag{8}$$

characterized by

$$\mathfrak{L}(\alpha(x, y)(t)) = t\mathfrak{L}(\alpha(x, y)(1)); \tag{9}$$

then

$$\alpha : \eta(\Delta) \to K^I \tag{10}$$

is the required map. The fact that $\alpha(x, y)$ is a simple path when $x \neq y$ is an immediate consequence of (6). The symmetry (4) is proved as follows. Let $\mathcal{L}$ denote the length function (7) along $[x, z] \cup [z, y]$ and $\mathcal{L}'$ the length function along $[y, z] \cup [z, x]$. Then, $\alpha(x, y)(t)$ and $\alpha(y, x)(1 - t)$ are characterized by

(11) $$\mathcal{L}(\alpha(x, y)(t)) = t\mathcal{L}(\alpha(x, y)(1)) = t\mathcal{L}(y),$$

(12) $$\mathcal{L}'(\alpha(y, x)(1 - t) = (1 - t)\mathcal{L}'(\alpha(y, x)(1)) = (1 - t)\mathcal{L}'(x),$$

where $\mathcal{L}(y) = \mathcal{L}'(x)$. Since

(13) $$\mathcal{L}(\alpha(x, y)(t)) + \mathcal{L}'(\alpha(x, y)(t)) = \mathcal{L}'(x) = \mathcal{L}(y),$$

we have

(14) $$\mathcal{L}'(\alpha(x, y)(t)) = \mathcal{L}'(x) - t\mathcal{L}'(x) = (1 - t)\mathcal{L}'(x),$$

and, finally,

(15) $$\alpha(x, y)(t) = \alpha(y, x)(1 - t), 0 \leqq t \leqq 1.$$

3. **The Observation.** First we recall that a polyhedron K satisfied the *Wecken* conditions if every maximal simplex has dimension $\geqq 2$ and given maximal simplices σ and τ there is a sequence of maximal simplices $\sigma_1, \cdots, \sigma_k$ such that $\sigma_1 = \sigma$, $\sigma_k = \tau$ and $\sigma_i \cap \sigma_{i+1}$ has dimension $\geqq 1, i = 1, \cdots, k - 1$.

The following theorem is implicit in [2] or [3]. It is only necessary to observe that at each stage of the proof, one always obtains proximity maps.

THEOREM 3.1. *Let K denote a polyhedron which satisfies the Wecken condition. Then, there exists a proximity map $f : K \to K$ which has no fixed points if $\chi(K) = 0$ and exactly one fixed point if $\chi(K) \neq 0$.*

OBSERVATION 3.2. Let K denote a polyhedron which satisfies the Wecken condition. Then, if K has characteristic $\chi(K) = 0$, K admits a *simple* path field $\varphi : K \to K^I$ such that the track of $\varphi(x)$ is a broken segment for every $x \in K$. [If $\chi(K) \neq 0$, $\varphi(x)$ is simple for all but one point.]

PROOF. Let f denote the map in Theorem 3.1 and let $\varphi(x) = \alpha(x, f(x))$, for $x \in K$, where α is as in Lemma 2.1.

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