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COHOMOLOGY OF QUASI-PROJECTIVE STIEFEL MANIFOLDS

A. DUANE RANDALL

1. **Introduction.** Let $V_{n,k}$ denote the Stiefel manifold of orthonormal k-frames in Euclidean n-space. The special orthogonal group $SO(2)$ acts freely on $V_{2n,k}$ via the diagonal embedding of S^1 in $U(n)$ and the standard embedding of $U(n)$ into $SO(2n)$ corresponding to realification $r: BU(n) \to BSO(2n)$. The quasi-projective Stiefel manifold $PV_{2n,k}$ is the quotient space of $V_{2n,k}$ under this action of $SO(2)$. The spaces $PV_{2n,k}$ are classifying spaces for sectioning multiples of a complex line bundle. If X is a finite complex and ξ a complex line bundle over X, then $n\xi$ has k linearly independent real sections if and only if there is a map $f: X \to PV_{2n,k}$ such that $f^*\eta_0 = \xi$ where η_0 is the complex line bundle over $PV_{2n,k}$ associated to the S^1-fibering $V_{2n,k} \to PV_{2n,k}$. In this paper we determine the cohomology algebras of the spaces $PV_{2n,k}$.

2. **Preliminaries.** We first establish some notation. Let $RE(x_i \mid i \in I)$ denote the exterior algebra over a ring R with generators x_i of degree i. Let $V(x_1, \cdots, x_m)$ denote the commutative associative algebra over Z_2 on generators $x_1, \cdots, x_m$ such that the monomials $x_1^{\epsilon_1} \cdots x_m^{\epsilon_m}$ with $\epsilon_i = 0$ or 1 form an additive basis. Let $\{{}_pE_r(X)\}$ denote the mod p Bockstein spectral sequence for X with ${}_pE_1(X) = H^*(X; Z_p)$. $C_{r,i}$ denotes the binomial coefficient $\binom{r}{i}$. Let ρ_p denote the universal coefficient map $H^*(\, ; Z) \to H^*(\, ; Z_p)$ for any prime p and let ρ_0 denote the map $H^*(\, ; Z) \to H^*(\, ; Q)$. Denote the image of an integral class x under the projection $H^*(X; Z) \to H^*(X; Z)/\text{Tors}$ by $\bar{x}$. Finally let $J_{n,k}$ represent the set of all integers j such that $[(2n-k)/2] < j < n$ where $0 < k < 2n$. We write $H^*(CP^\infty) = Z[\beta]$.

Recall from [5] the cellular structure of the Stiefel manifold $V_{2n,k}$ obtained from an embedding of real projective space RP^{2n-1} into $O(2n)$ composed with the projection map $O(2n) \to V_{2n,k}$. The image of RP^{2j} determines a class P^{2j} in $H^{2j}(V_{2n,k}; Z)$ of order 2 for every $j \in J_{n,k}$. *Set* $x_{2j} = \rho_2(P^{2j})$. RP^{2n-k} determines a free integral class y_{2n-k} for k even. Let $x_{2n-k} = \rho_2(y_{2n-k})$ for k even and let x_{2n-k} be the unique class such that $\mathrm{Sq}^1 x_{2n-k} = x_{2n-k+1}$ for k odd. By [2]

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there is a class $y_{2n-1} \in H^{2n-1}(V_{2n,k})$ such that $\tau(y_{2n-1}) = \chi_{2n}$ where τ denotes transgression in the spectral sequence for the fibration

$$V_{2n,k} \to BSO(2n-k) \to BSO(2n)$$

and χ_{2n} is the Euler class. Set $x_{2n-1} = \rho_2(y_{2n-1})$. By [1] and [5]

$$(2.1) \qquad \begin{aligned} H^*(V_{2n,k}; Z_2) &= V(x_{2n-k}, \cdots, x_{2n-1}) \quad \text{and} \\ \mathrm{Sq}^i x_j &= C_{j,i} x_{i+j}. \end{aligned}$$

For every $j \in J_{n,k}$, there is a class y_{4j-1} in $H^{4j-1}(V_{2n,k})$ such that $\tau(2y_{4j-1}) =$ the Pontryagin class p_j and $\rho_2(y_{4j-1}) = x_{2j}x_{2j-1} + x_{4j-1}$ from [2, 30.10]. By [1]

$$(2.2) \qquad H^*(V_{2n,k})/\text{Torsion} = ZE(\bar{y}_{2n-k}, \bar{y}_{2n-1}, \bar{y}_{4j-1} \mid j \in J_{n,k})$$

where $\bar{y}_{2n-k}$ is omitted for k odd. For integers $s, t \in J_{n,k}$ with $s < t$ let $u_{s,t}$ be the integral class of order 2 such that $\rho_2(u_{s,t}) = x_{2s}x_{2t-1} + x_{2t}x_{2s-1}$. We state the following known

Proposition 2.3. *The classes y_{2n-1}, y_{4j-1}, $u_{s,t}$, P^{2j} for $j, s, t \in J_{n,k}$ with $s < t$, y_{2n-k} for k even, and the unit generate the algebra $H^*(V_{2n,k})$.*

Consider the following commutative diagram of fibrations.

$$(2.4) \qquad \begin{array}{ccccccc} V_{2n,k} & = & V_{2n,k} & = & V_{2n,k} & = & V_{2n,k} \\ \downarrow & & \downarrow i & & \downarrow & & \downarrow \\ j^{\#}E_{2n,k} & \to & E_{2n,k} & \to & E' & \to & BSO(2n-k) \\ \downarrow & & \downarrow \pi & & \downarrow & & \downarrow \\ CP^l & \xrightarrow{j} & CP^\infty & \xrightarrow{n\eta} & BU(n) & \xrightarrow{r} & BSO(2n) \end{array}$$

Here $\pi : E_{2n,k} \to CP^\infty$ is the principal fibration induced from the fibration $BSO(2n-k) \to BSO(2n)$ by the map $r \circ n\eta : CP^\infty \to BSO(2n)$ classifying the n-fold sum of the Hopf bundle η over CP^∞ regarded as a real vector bundle. By construction $E_{2n,k}$ is the classifying space for finding k independent real sections to the n-fold Whitney sum of a complex line bundle over a finite complex. The method of proof of [3, Proposition 1.3] yields the following

Proposition 2.5. *The spaces $PV_{2n,k}$ and $E_{2n,k}$ have the same homotopy type.*

Consider the following homotopy commutative diagram of vertical fibrations with $m = n - k$.

(2.6)
$$\begin{array}{ccccc} S^{2m} & = & S^{2m} & = & S^{2m} \\ \downarrow & & \downarrow t & & \downarrow \\ V_{2n,2k} & \xrightarrow{i} & PV_{2n,2k} & \xrightarrow{s} & BSO(2m) \\ \downarrow & & \downarrow r & & \downarrow \\ V_{2n,2k-1} & \rightarrow & PV_{2n,2k-1} & \rightarrow & BSO(2m+1) \end{array}$$

The fibration $S^{2m} \to PV_{2n,2k} \to PV_{2n,2k-1}$ is totally nonhomologous to zero so we obtain

PROPOSITION 2.7. *$H^*(PV_{2n,2k}) = H^*(PV_{2n,2k-1}) \otimes H^*(S^{2m})$ as $H^*(PV_{2n,2k-1})$-modules.*

We select X uniquely in $H^{2m}(PV_{2n,2k})$ such that t^*X generates $\bar{H}^*(S^{2m})$ and

(2.8)
$$s^*\chi_{2m} = 2X + C_{n,k}\omega^m$$

where ω denotes $\pi^*\beta$ for $\pi : PV_{2n,2k} \to CP^\infty$ in (2.4). This selection is possible by [**6**, Theorem A] using the natural map $Y_{n,k} \to PV_{2n,2k}$ where $Y_{n,k}$ denotes the complex projective Stiefel manifold. Since $\chi_{2m}^2 = p_m$ in $H^*(BSO(2m))$, it follows that

(2.9)
$$4X^2 = C_{n,k}(1 - C_{n,k})\omega^{2m} - 4C_{n,k}X\omega^m.$$

Thus we consider the spaces $PV_{2n,k}$ primarily for k odd.

We remark that the problem of determining the geometric dimension of $n\eta$ based on CP^l is equivalent to finding the largest integer k such that $j^{\#}E_{2n,k} \to CP^l$ has a section where $j^{\#}E_{2n,k} \to CP^l$ is the fibration in (2.4) induced from π via the standard embedding $j : CP^l \to CP^\infty$. (See [7] for the case $l = n - 1$.) Note that $PV_{2n,1}$ is CP^{n-1}.

3. **Rational and mod p cohomology of $PV_{2n,k}$.** Let F denote Q or Z_p for an odd prime p. From (2.2) and (2.3) $H^*(V_{2n,k}; F) = FE(\tilde{y}_{2n-k}, \tilde{y}_{2n-1}, \tilde{y}_{4j-1} \mid j \in J_{n,k})$ where $\tilde{y}_{2n-k}$ is omitted for k odd, and $\tilde{\ }$ denotes the image under $\rho : H^*(\ ; Z) \to H^*(\ ; F)$.

THEOREM 3.1. *Let k be an odd integer with $k < n + 1$. Then $H^*(PV_{2n,k}; F) = F[\tilde{\omega}]/(\tilde{\omega}^n) \otimes FE(v_{4j-1} \mid j \in J_{n,k})$ where $i^*v_p = \tilde{y}_p$ and $\omega = \pi^*\beta$.*

PROOF. The Serre spectral sequence for the fibration $V_{2n,k} \xrightarrow{i} PV_{2n,k} \xrightarrow{\pi} CP^\infty$ in (2.4) with coefficients F has $E_2^{*,*} = F[\tilde{\beta}] \otimes FE(\tilde{y}_{2n-1}, \tilde{y}_{4j-1} \mid j \in J_{n,k})$. Since $\tau(2y_{4j-1}) = p_j(n\eta) = C_{n,j}\beta^{2j}$ in the integral spectral sequence for π, the fiber is transgressively generated over F. By dimensionality d_{2n} is the first nonzero differential and $d_{2n}(\tilde{y}_{2n-1}) = \tau(\tilde{y}_{2n-1}) = \tilde{\chi}(n\eta) = \tilde{\beta}^n$. Note that the image of the ideal

$(\tilde{\beta}^n)$ in $E_{2n+1}^{*,*}$ is 0 and $E_{2n+1}^{*,*} = E_{2n+1}^{*,0} \otimes E_{2n+1}^{0,*}$. All the following differentials are trivial so $E_\infty^{*,*} = E_{2n+1}^{*,*}$. The result follows from [1, Proposition 8.1].

Let l denote the smallest integer in $J_{n,k}$. Given an odd prime p, let $N(p)$ denote the smallest integer j in $J_{n,k}$ such that p does not divide $C_{n,j}$. If no such integer j exists, set $N(p) = \infty$.

THEOREM 3.2. *Let k be an odd integer with $k > n$. If $n \neq 2l$*

$$H^*(PV_{2n,k}; Q) = Q[\tilde{\omega}]/(\tilde{\omega}^{2l}) \otimes QE(v_{2n-1}, v_{4j-1} \mid l < j < n)$$

with $i^*v_s = \tilde{y}_s = \rho_0(y_s)$. If $n = 2l$,

$$H^*(PV_{2n,k}; Q) = Q[\tilde{\omega}]/(\tilde{\omega}^n) \otimes QE(v_{4j-1} \mid j \in J_{n,k})$$

where $i^*v_{4l-1} = 2\tilde{y}_{4l-1} - C_{n,l}\tilde{y}_{2n-1}$ and $i^*v_s = \tilde{y}_s$ otherwise. If $2N(p) < n$,

$$H^*(PV_{2n,k}; Z_p) = Z_p[\tilde{\omega}]/(\tilde{\omega}^{2N(p)}) \otimes Z_pE(v_{2n-1}, v_{4j-1} \mid j \in J_{n,k}, j \neq N(p))$$

with $i^*v_s = \tilde{y}_s = \rho_p(y_s)$. If $2N(p) > n$,

$$H^*(PV_{2n,k}; Z_p) = Z_p[\tilde{\omega}]/(\tilde{\omega}^n) \otimes Z_pE(v_{4j-1} \mid j \in J_{n,k})$$

with $i^*v_s = \tilde{y}_s$. If $2N(p) = n$,

$$H^*(PV_{2n,k}; Z_p) = Z_p[\tilde{\omega}]/(\tilde{\omega}^n) \otimes Z_pE(v_{4j-1} \mid j \in J_{n,k})$$

where $i^*v_{4N(p)-1} = 2\tilde{y}_{4N(p)-1} - C_{n,N(p)}\tilde{y}_{2n-1}$.

Theorem 3.2 follows similarly from the proof of (3.1). From (3.2) and (2.7) we obtain the following

COROLLARY 3.3. *$H^*(PV_{2n,k})$ has p-torsion for an odd prime p if and only if $k > n + 2$ for n even, $k > n + 1$ for n odd, and p divides $C_{n,l}$.*

The Z_2 cohomology algebra of $PV_{2n,k}$ and module structure over the Steenrod algebra A have essentially been determined up to a small indeterminacy by Gitler and Handel in [3]. Let N denote the smallest integer j with $C_{n,j}$ odd and $2n - k + 1 \leqq 2j \leqq 2n$. Applying the proof of [3, Theorem 2.8] gives the following

THEOREM 3.4. *As an algebra*

$$H^*(PV_{2n,k}; Z_2) = Z_2[\alpha]/(\alpha^N) \otimes V(z_{2n-k}, \cdots, z_{2N-2}, z_{2N}, \cdots, z_{2n-1})$$

where $i^*z_p = x_p$ *and* $\alpha = \rho_2(\omega)$. *If* $C_{2n,q}$ *is even,*

$$\mathrm{Sq}^i z_{q-1} = \sum_{k\in K} C_{q-1-2k,i-2k} w_{2k}(n\eta_0) z_{q+i-2k-1} + \lambda(q,i).$$

If $C_{2n,q}$ *is odd, then* $q = 2s$ *and*

$$\begin{aligned} Sq^i z_{q-1} = &\sum_{k\in K} C_{q-1-2k,i-2k} w_{2k}(n\eta_0) z_{q+i-2k-1} \\ &+ \sum_{j,k\in J} C_{2N-1-2k,j-2k}\,\mathrm{Sq}^{i-j}\alpha^{s-N} w_{2k}(n\eta_0) z_{2N+j-2k-1}. \end{aligned}$$

Here $\lambda(q,i) = 0$ *if* $q+i$ *is even, and* $\lambda(q,i) = \epsilon\alpha^r$ *if* $q+i-1 = 2r$ *where* $\epsilon = 0$ *or* 1. $K = \{k \mid 0 \leqq 2k \leqq i$ *and* $q+i-2k \neq 2N\}$ *and* $J = \{j,k \mid 0 \leqq 2k < j \leqq i\}$.

Suppose now that $k < n+2$ for n odd and $k < n+3$ for n even. We shall show that all torsion in $H^*(PV_{2n,k})$ has order 2. Note from (3.4) that $\mathrm{Sq}^1 z_{2j-1} = z_{2j} + \lambda(2j,1)$ where $j < N$ and $2n - k < 2j$. If $\lambda(2j,1) \neq 0$, we define z_{2j} to be $\mathrm{Sq}^1 z_{2j-1}$. Take $s \in J_{n,k}$ with $s \neq N$. If $s < N$, define

$$Z_{4s-1} = z_{2s-1}z_{2s} + \sum_{\substack{j=2s-n+1\\ j\neq 2s-N}}^{s} C_{n,j}\alpha^j z_{4s-2j-1} + \lambda_s \alpha^{2s-n} z_{2n-1}$$

where

$$\lambda_s = C_{n,2s-N} + \sum_{j=2s-n+1}^{2s-N-1} C_{n,j}C_{n,2s-j} \in Z_2.$$

If $s > N$, define

$$\begin{aligned} Z_{4s-1} = &\, z_{2s-1}z_{2s} + \sum_{j=2s-n+1}^{N-1} C_{n,j}\alpha^j z_{4s-2j-1} \\ &+ C_{n,s}\alpha^{s-N} z_{2N} z_{2s-1} \\ &+ C_{n,s} \sum_{l=2N-n+1}^{3N-2s-1} C_{n,l}\alpha^{2s-2n+l} z_{4N-2l-1}. \end{aligned}$$

Note from (3.4) that $Z_{4s-1} \in \ker \mathrm{Sq}^1$, and Z_{4s-1} is not in $\mathrm{im}\,\mathrm{Sq}^1$ since $i^*Z_{4s-1} = x_{2s-1}x_{2s}$. Clearly Z_{4s-1}^2 is in $\mathrm{im}\,\mathrm{Sq}^1$. Note also that $\mathrm{Sq}^1 z_{2n-1} = \alpha^{n-N} z_{2N}$ and, for $0 \leqq j < n - N$,

$$(\alpha^j z_{2N})^2 = \mathrm{Sq}^1\Big(\alpha^{2j} \sum_{l=2N-n+1}^{N-1} \alpha^l C_{n,l} z_{4N-2l-1}\Big).$$

Let T denote the graded algebra over Z_2 with trivial multiplication on generators z_{2N}, $z_{2N}z_{2n-1}$, and $\alpha^{2N-n}z_{2n-1}$. Similar computation using (3.4) yields the following for k odd and $k < n + 2$.

PROPOSITION 3.5.

$$_2E_2(PV_{2n,k}) = Z_2[\alpha]/(\alpha^N)$$
$$\otimes\, Z_2E(Z_{4s-1} \mid s \in J_{n,k}, s \neq N) \otimes T/I$$

where I is the ideal generated by $\alpha^{n-N} \otimes 1 \otimes z_{2N}$ and $\alpha^{n-N} \otimes 1 \otimes \alpha^{2N-n}z_{2n-1}$.

COROLLARY 3.6. *All torsion in $H^*(PV_{2n,k})$ has order 2 where $k < n + 2$ for n odd and $k < n + 3$ for n even.*

PROOF. Assume k is odd and $k < n + 2$. To show $_2E_2(PV_{2n,k}) = {_2E_\infty}(PV_{2n,k})$, it suffices to define an isomorphism $\varphi : {_2E_2}(PV_{2n,k}) \to H^*(PV_{2n,k}; Q)$ of graded vector spaces over Z_2. Define $\varphi(\alpha^s) = \tilde{\omega}^s$ for $s < N$, $\varphi(\alpha^s z_{2N}) = \tilde{\omega}^{N+s}$ and $\varphi(\alpha^{s+2N-n}z_{2n-1}) = \tilde{\omega}^s v_{4N-1}$ for $0 \leqq s < n - N$, $\varphi(\alpha^s z_{2N}z_{2n-1}) = \tilde{\omega}^{s+n-N}v_{4N-1}$ for $0 \leqq s < N$, and $\varphi(Z_{4j-1}) = v_{4j-1}$ for $j \in J_{n,k}$ and $j \neq N$. Extend φ to an isomorphism and apply (2.7) for k even.

4. **Integral cohomology.**

Case I. We assume in Case I that $k < n + 1$ with k odd. We determine the differentials and $E_\infty^{*,*}$ for the integral spectral sequence for the fibration $V_{2n,k} \xrightarrow{i} PV_{2n,k} \to CP^\infty$ in (2.4) and then use the Gysin sequence to specify generators for $H^*(PV_{2n,k})$. $E_2{}^{*,*} = Z[\beta] \otimes H^*(V_{2n,k})$. Since $\tau(P^{2j}) = \delta w_{2j}$ for the fibration $BSO(2n - k) \to BSO(2n)$ where δ denotes the integral Bockstein operator, P^{2j} for $j \in J_{n,k}$ survives in the integral spectral sequence for π. $\tau(y_{2n-1}) = \chi(n\eta) = \beta^n$ so $E_{2n+1}^{p,0} = 0$ for $p > 2n$. All differentials kill $2y_{4j-1}$ for $j \in J_{n,k}$ since $\tau(2y_{4j-1}) = p_j(n\eta)$. Note that d_{2N} is the first nontrivial differential in the integral spectral sequence for π since d_{2N} is the first nontrivial differential in the Z_2 spectral sequence by (3.4) and d_{2n} is the only nontrivial differential with F coefficients by (3.1). If $N = n$, clearly

$$E_\infty^{*,*} = E_{2n+1}^{*,*} = Z[\beta]/(\beta^n) \otimes H^*(V_{2n,k})/(y_{2n-1}).$$

Assume $N < n$. Now $d_{2N}(1 \otimes y_{4N-1}) = \beta^N \otimes P^{2N}$ since

$$d_{2N}(1 \otimes x_{2N-1}x_{2N}) = \beta^N \otimes x_{2N} = \rho_2(\beta^N \otimes P^{2N}).$$

Similarly, $d_{2N}(1 \otimes u) = \beta^N \otimes P^{2j}$ where $u = u_{j,N}$ if $j < N$, and $u = u_{N,j}$ if $N < j$. Since $d_{2j}(y_{4j-1})$ = image of $c_j \otimes P^{2j}$ in $E_{2j}^{*,*}$ in the integral spectral sequence for $E' \to BU(n)$ in (2.4), one checks that $d_{2j}(y_{4j-1})$ = image of $C_{n,j}\beta^j \otimes P^{2j}$ in $E_{2j}^{*,*}$ in the integral spectral sequence for π, and $d_{2r}(y_{4j-1}) = 0$ for $j < r < 2j$ if $d_{2j}(y_{4j-1}) = 0$. Note that for $s, t \in J_{n,k} - \{N\}$ with $s < t$,

$$u_{s,t} = i^* U_{s,t} \tag{4.1}$$

where $U_{s,t} = \delta(z_{2s-1}z_{2t-1})$ since

$$\begin{aligned} \mathrm{Sq}^1(z_{2s-1}z_{2t-1}) &= z_{2s}z_{2t-1} + z_{2s-1}z_{2t} \\ &\quad + C_{n,s}\alpha^{s-N}z_{2N}z_{2t-1} + C_{n,t}\alpha^{t-N}z_{2N}z_{2s-1} \end{aligned}$$

by (3.4). Thus $E_{2N+1}^{*,*} = Z[\beta] \otimes E_{2N+1}^{0,*}/K$ where K is the ideal generated by $\beta^N \otimes P^{2j}$ for $j \in J_{n,k}$. The differentials d_p are trivial for $p > 2N$ and $p \neq 2n$ so it follows that

$$E_\infty^{*,*} = E_{2n+1}^{*,*} = Z[\beta]/(\beta^n) \otimes E_{2n+1}^{0,*}/K \tag{4.2}$$

as graded algebras.

The only nontrivial extension from $E_\infty^{*,*}$ to $H^*(PV_{2n,k})$ is the nontrivial extension of Z by Z_2. Since $\tau(z_{2N-1}) = \rho_2(\beta^N)$, it follows from the universal example for division by 2 that

$$\omega^N = \pi^*\beta^N = 2e_{2N} \quad \text{with } i^*e_{2N} = P^{2N} \text{ and } \rho_2(e_{2N}) = z_{2N}. \tag{4.3}$$

Consider the following commutative diagram.

(4.4)

$$\begin{array}{ccccccccc}
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & H^{i+1}(V_{2n,k}; Z) & \xrightarrow{\theta} & H^i(PV_{2n,k}; Z) & \xrightarrow{\cdot\omega} & H^{i+2}(PV_{2n,k}; Z) & \xrightarrow{i^*} & H^{i+2}(V_{2n,k}; Z) & \to \\
 & \downarrow \times 2 & & \downarrow \times 2 & & \downarrow \times 2 & & \downarrow \times 2 & \\
\to & H^{i+1}(V_{2n,k}; Z) & \xrightarrow{\theta} & H^i(PV_{2n,k}; Z) & \xrightarrow{\cdot\omega} & H^{i+2}(PV_{2n,k}; Z) & \xrightarrow{i^*} & H^{i+2}(V_{2n,k}; Z) & \to \\
 & \downarrow \rho_2 & & \downarrow \rho_2 & & \downarrow \rho_2 & & \downarrow \rho_2 & \\
\to & H^{i+1}(V_{2n,k}; Z_2) & \xrightarrow{\theta} & H^i(PV_{2n,k}; Z_2) & \xrightarrow{\cdot\alpha} & H^{i+2}(PV_{2n,k}; Z_2) & \xrightarrow{i^*} & H^{i+2}(V_{2n,k}; Z_2) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

The above rows are the Gysin sequence for the fibration $S^1 \to V_{2n,k} \to PV_{2n,k}$.

For $s \in J_{n,k} - \{N\}$, we define

$$Y_{2s} = \delta(z_{2s-1}). \tag{4.5}$$

Note that $i^*Y_{2s} = P^{2s}$ and $\rho_2(Y_{2s}) = z_{2s}$. We define

$$(4.6) \qquad V_s = \delta(z_{2s-1}z_{2n-1}) \in H^{2n+2s-1}(PV_{2n,k}).$$

Clearly $2V_s = 0$ and $\rho_2(V_s) = z_{2s}z_{2n-1} + \alpha^{n-N}z_{2N}z_{2s-1} + C_{n,s}\alpha^{s-N}z_{2N}z_{2n-1}$. Note that $i^*V_s = P^{2s}y_{2n-1}$ since $i^*\rho_2(V_s) = x_{2s}x_{2n-1}$.

For any $j \in J_{n,k} - \{N\}$, we now show there exists

(4.7) $X_{4j-1} \in H^{4j-1}(PV_{2n,k})$ with $i^*X_{4j-1} = y_{4j-1}$ and $\rho_2(X_{4j-1}) = Z_{4j-1}$.

Let u be any class in $H^{4j-1}(PV_{2n,k})$ with $i^*u = y_{4j-1}$. Then $\rho_2(u) = Z_{4j-1} + \alpha z$ with $z \in H^{4j-3}(PV_{2n,k}; Z_2)$ by (4.4). $\mathrm{Sq}^1(\alpha z) = 0$ so $\alpha z = \rho_2(Z)$, and $i^*Z = 2V$ since $\rho_2(i^*Z) = i^*(\rho_2 Z) = i^*(\alpha z) = 0$. Select Z' such that $i^*Z' = V$ by (4.2). Then $Z - 2Z' = \omega Z''$ and $X_{4j-1} = u + \omega Z''$ satisfies (4.7).

Take any class $u \in H^{4N-1}(PV_{2n,k})$ with $i^*u = 2y_{4N-1}$. Then $\rho_2(u) \in \ker \mathrm{Sq}^1 \cap \ker i^*$ so $\rho_2(u) = \alpha w$ with $\mathrm{Sq}^1 w \neq 0$ by (4.4). It follows that $\rho_2(u) = \alpha^{2N-n}z_{2n-1} + \alpha\rho_2(V)$ for $V \in H^{4N-3}(PV_{2n,k})$. Define $X_{4N-1} = u + \omega V$ and note that

$$(4.8) \qquad i^*X_{4N-1} = 2y_{4N-1} \quad \text{and} \quad \rho_2(X_{4N-1}) = \alpha^{2N-n}z_{2n-1}.$$

Similarly, it follows from (4.4) and the fibration $V_{2N-1,k+2N-2n-1} \to PV_{2n,k} \to PV_{2n,2n-2N+1}$ that we can choose $Y \in H^{2n+2N-1}(PV_{2n,k})$ so that

$$(4.9) \qquad 2Y = \omega^{n-N}X_{4N-1}, \quad i^*Y = P^{2N}y_{2n-1}, \quad \text{and} \quad \rho_2(Y) = z_{2N}z_{2n-1}.$$

One checks using (4.4) that

$$(4.10) \qquad \omega^{2N-n}Y = e_{2N}X_{4N-1}.$$

Note that $H^*(PV_{2n,k}; Q) = Q[\tilde{\omega}]/(\tilde{\omega}) \otimes QE(\tilde{X}_{4j-1} \mid j \in J_{n,k})$. In summary we have the following

THEOREM 4.11. *Suppose that $k < n + 1$ with k odd. If $N = n$, $H^*(PV_{2n,k}) = H^*(CP^{n-1}) \otimes H^*(V_{2n,k})/(y_{2n-1})$ as algebras. If $N < n$, $H^*(PV_{2n,k})$ is generated by the classes ω, e_{2N}, Y_{2s}, V_s, X_{4j-1}, $U_{s,t}$ and Y where $j \in J_{n,k}$ and $s, t \in J_{n,k} - \{N\}$ with $s < t$. Relations among the generators and the product structure are determined by the rational and Z_2 cup products.*

REMARK. $H^*(PV_{2n,k})$ contains the subalgebra $Z[\omega]/(\omega^n) \otimes ZE(X_{4j-1} \mid j \in J_{n,k})$.

Case II. We assume that n is even and $k = n + 1$. Thus $n = 2l$. Set $d_l = \frac{1}{2}C_{n,l}$. Choose X_{4l-1} in $H^{2n-1}(PV_{2n,k})$ such that

$$(4.12) \qquad i^*X_{4l-1} = y_{4l-1} - d_l y_{2n-1} \quad \text{and} \quad \rho_2(X_{4l-1}) = Z_{4l-1}.$$

Then $H^*(PV_{2n,k})$ is again given by (4.11).

5. **Integral cohomology.**

Case III. Finally we assume $2l < n$ with k odd. Let d_j denote $\frac{1}{2}C_{n,j}$ for $l \leqq j < N$. Set $b_l = d_l$ and inductively define $b_i =$ G.C.D.(d_i, b_{i-1}) for $l < i < N$.

Set $b_N =$ G.C.D.$(b_{N-1}, C_{n,N})$. If $N = l$, set $b_N = C_{n,l}$. Define b_j inductively for $2N \leqq 2j < n$. Suppose $b_j > 1$. Set $b_{j+1} =$ G.C.D.(b_j, λ_j) where $\lambda_j \in Z_{b_j}$ is chosen uniquely such that $C_{n,j+1} = 2\lambda_j \bmod b_j$. If $b_j = 1$, set $b_i = 1$ for $2j < 2i < n$. The argument of [**6**, Proposition 5] shows

PROPOSITION 5.1. $\operatorname{Ker} \pi^* = [b_l\beta^{2l}, \cdots, b_j\beta^{2j}, \cdots, \beta^n]$ *for* $2l \leqq 2j < n$. $\operatorname{Ker} \bar{\pi}^* = [b_l\beta^{2l+1}, \cdots, b_j\beta^{2j+1}, \cdots, \beta^{n+1}]$ *where* $\bar{\pi} : (PV_{2n,k}, V_{2n,k}) \to (CP^\infty, *)$.

Set $a_i = b_{i-1}/b_i$ for $2l < 2i < n$. Set $y_{4N-1} = 2y_{4N-1}$. Recall $T^{q-1}/\operatorname{im} i^* = \operatorname{Ker}^q \pi^*/\operatorname{Ker}^q \bar{\pi}^*$ where $T^{q-1} \subseteq H^{q-1}(V_{2n,k})$ denotes the subgroup of transgressive elements. Thus we obtain from (5.1) the following

COROLLARY 5.2. $T^q/\operatorname{im} i^* = 0$ *for* $2n \leqq q$. $T^{4i-1}/\operatorname{im} i^* = Z_{a_i}$ *for* $2l < 2i < n$. $T^{4l-1}/\operatorname{im} i^* = Z$. $T^{2n-1}/\operatorname{im} i^* = Z_{b_{s-2}}$ *if* $n = 2s$ *and* Z_{b_s} *if* $n = 2s + 1$.

Thus there exist classes X_{4j-1} in $H^{4j-1}(PV_{2n,k})$ for $2l < 2j < n$ such that $i^*X_{4j-1} = a_jy_{4j-1}$. If $n = 2s + 1$, choose X_{2n-1} in $H^{2n-1}(PV_{2n,k})$ such that $i^*X_{2n-1} = b_sy_{2n-1}$. If $n = 2s$, choose X_{2n-1} such that $i^*X_{2n-1} = b_{s-1}y_{2n-1}$, and define X_{4s-1} so that $i^*X_{4s-1} = y_{4s-1} - d_sy_{2n-1}$ if $s < N$ and $i^*X_{4s-1} = y_{4s-1} - \lambda_sy_{2n-1}$ if $s > N$. Select X_{4j-1} so that $i^*X_{4j-1} = y_{4j-1}$ for $n < 2j < 2n$. Choose a fixed set of the above classes arbitrarily. Let p be a fixed odd prime and set $I_p = \{j \mid 2l < 2j < n,\ p \mid a_j\} \cup \{l\}$. For $j \in I_p$ with $j \neq N(p)$, set $\bar{v}_{4j-1} = v_{4j-1}$ from (3.2). Define $\bar{v}_{4j-1} = \rho_p(X_{4j-1})$ for $l < j < n$, $j \notin I_p$, $j \neq N(p)$. Set $\bar{v}_{2n-1} = \rho_p(X_{2n-1})$ if $p \not| \, b_{s-1}$. Then

$$\begin{aligned} H^*(PV_{2n,k}; Z_p) = {} & Z_p[\tilde{\omega}]/(\tilde{\omega}^{2N(p)}) \\ & \otimes Z_pE(\bar{v}_{4j-1}, \bar{v}_{2n-1} \mid j \in J_{n,k}, j \neq N(p)), \\ & \qquad \text{if } 2N(p) < n, \\ = {} & Z_p[\tilde{\omega}]/(\tilde{\omega}^n) \otimes Z_pE(\bar{v}_{4j-1} \mid j \in J_{n,k}), \\ & \qquad \text{if } n \leqq 2N(p). \end{aligned}$$

Note that

$$H^*(PV_{2n,k}; Q) = Q[\tilde{\omega}]/(\tilde{\omega}^{2l}) \otimes QE(\tilde{v}_{4j-1}, \tilde{v}_{2n-1} \mid l < j < n)$$

where $\tilde{v}_{4j-1} = \rho_0(X_{4j-1})$ and $\tilde{v}_{2n-1} = \rho_0(X_{2n-1})$. Arrange I_p so that $l = i(0) < i(1) < \cdots < i(j) < \cdots < i(t)$ and write $b_{i(j)} = p^{r(j)}e_j$ where $p \nmid e_j$. Then $r(j) > r(j+1)$ and $b_i = p^{r(j)}e_i$ for $i(j) \leqq i < i(j+1)$ where $p \nmid e_i$. The argument of [**6**, Lemmas 8, 10] determines the mod p Bockstein spectral sequence via the following

LEMMA 5.3. *The differential d_r for ${}_pE_r(PV_{2n,k})$ is trivial unless $r = r(j)$. $d_r(\bar{v}_{4i(j)-1}) = 0$ for $r < r(j)$. $d_{r(j)}(\bar{v}_{4i(j)-1}\tilde{\omega}^s) = k_j\tilde{\omega}^{2i(j)+s} \neq 0$ for $0 \leqq s < 2[i(j+1) - i(j)]$, $k_j \in Z_p$. If $n = 2s + 1 = 2N(p) + 1$, $s = i(t)$ and $d_{r(t)}(\bar{v}_{4s-1}) = k_s\tilde{\omega}^{2s}$ with $k_s \neq 0$ and $d_{r(t)}(\bar{v}_{4s-1}\tilde{\omega}) = 0$. If $n = 2s = 2N(p)$, then $i(t) = s - 1$ and $d_{r(t)}(\bar{v}_{4s-5}) = k_t\tilde{\omega}^{2(s-1)} \neq 0$ and $d_{r(t)}(\bar{v}_{4s-5}\tilde{\omega}^2) = 0$. If $2N(p) > n$, $d_{r(t)}(\bar{v}_{4i(t)-1}\tilde{\omega}) \neq 0$ for $s < n - 2i(t)$. If $2N(p) < n - 1$, $d_{r(t)}(\bar{v}_{4i(t)-1}) = 0$. Further, $H^*(PV_{2n,k})/$Tors. $\otimes Z_p = H^*(PV_{2n,k}; Q)$ as algebras over Z_p.*

We apply Poincaré duality to specify generators for $H^*(PV_{2n,k})$. Let U denote the fundamental cohomology class for the closed orientable manifold $PV_{2n,k}$ of dimension $\frac{1}{2}k(4n - k - 1) - 1$. Fix an arbitrary choice of generators for $H^*(PV_{2n,k}; Z_2)$ such that $z_{2s} = \mathrm{Sq}^1 z_{2s-1}$ for $s \in J_{n,k}$. Analogous to Case I, we define 2-torsion classes

(5.4) $$V_{2s} = \delta(z_{2s-1}) \quad \text{and} \quad U_{s,t} = \delta(z_{2s-1}z_{2t-1}) \quad \text{for } s, t \in J_{n,k} - \{N\} \text{ with } s < t.$$

Suppose $N = l$. Note that $\pi^*\beta^2 = 2e_{2l}$ where $\rho_2(e_{2l}) = z_{2l}$. Also $\rho_2(X_{4j-1}) = z_{2j-1}z_{2j} + z_{4j-1} + \gamma_j$ for some γ_j with $i^*\gamma_j = 0$. So

$$U = \omega^{l-1}e_{2l}X_{2n-1}\prod_{l<j<n} X_{4j-1}$$

since $\rho_p U \neq 0$ for all primes p. Thus we obtain

PROPOSITION 5.5. *For $2l < n$ with $N = l$ and k odd,*

$$H^*(PV_{2n,k})/\mathrm{Tor} = Z[\omega]/(\omega^l) \otimes ZE(e_{2l}, X_{2n-1}, X_{4j-1} \mid l < j < n).$$

If $2l \leqq N \leqq n$, then $N = 2^r$ for some integer r. Set $s = 2^{r-1}$ and note that $T^{4N-1}/\mathrm{im}\, i^*$ is generated by $\frac{1}{2}y_{4N-1} + P^{2N}y_{4s-1}$. Recall that y_{4N-1} was redefined to be twice the generator in (2.3). Let $I_2 = \{j \mid 2l < 2j < n, 2 \mid a_j\}U\{l\}$, and arrange I_2 so that

(5.6) $$l = i(0) < i(1) < \cdots < i(t).$$

Note that $i(t) = s$. Write $b_{i(j)} = 2^{r(j)}g_j$ where $2 \nmid g_j$. There exist classes $Z_{4j-1} = z_{2j-1}z_{2j} + z_{4j-1} + \gamma_j$ with $i^*\gamma_j = 0$ for $j \in J_{n,k} - \{N\}$ such that in the mod 2 Bockstein spectral sequence we have $d_{r(j)}(Z_{4i(j)-1}) = \alpha^{2i(j)} \neq 0$. Any choice of the classes X_{4j-1} for $l < j < n$ with j not in I_2 satisfies $\rho_2(X_{4j-1}) = Z_{4j-1} + \mu_j$ for some μ_j with $i^*\mu_j = 0$. Classes $X_{4i(j)-1}$ can be chosen so that

$$(5.7) \qquad \begin{aligned} i^*X_{4i(j)-1} &= a_{i(j)}y_{4i(j)-1}, \\ \rho_2(X_{4i(j)-1}) &= \alpha^{2i(j)-2i(j-1)}Z_{4i(j-1)-1}, \end{aligned} \qquad \text{for } 0 < j \leqq t.$$

Also $\rho_2(X_{2n-1}) = z_{2n-1} + \gamma$ and $\rho_2(X_{4N-1}) = z_{2N}Z_{4s-1} + z_{4N-1} + \mu$ with $i^*\gamma = i^*\mu = 0$ for $2l \leqq N < n$. If $N = n$, $\rho_2(X_{2n-1}) = z_{2s-1}z_{2s}$ for some choice of X_{2n-1}. Thus

$$U = \omega^{2l-1}X_{2n-1}\prod_{l<j<n} X_{4j-1}$$

since $\rho_p(U) \neq 0$ for all primes p, and we obtain the following

PROPOSITION 5.8.

$$H^*(PV_{2n,k})/\mathrm{Tors} = Z[\omega]/(\omega^{2l-1}) \otimes ZE(X_{2n-1}, X_{4j-1} \mid l < j < n)$$

for $2l \leqq N \leqq n$ *with* k *odd.*

Finally we consider the case $l < N < 2l < n$ where divisibility by 2 occurs among certain products in $H^*(PV_{2n,k})/\mathrm{Tors}$. Note that the free class $\pi^*\beta^N = 2e_{2N}$ with $i^*e_{2N} = P^{2N}$ and $\rho_2(e_{2N}) = z_{2N}$. Suppose $i(t) < N$ in (5.6). The higher order mod 2 Bocksteins are given by $d_{r(s)+1}(Z_{4i(s)-1}) = z_{2N}\alpha^{2i(s)-N}$ for $0 \leqq s \leqq t$. Thus $X_{4i(j)-1}$ for $1 \leqq j \leqq t$ can again be chosen to satisfy (5.7). For proper choices $\rho_2(X_{4N-1}) = \alpha^{2[N-i(t)]}Z_{4i(t)-1}$ and $\rho_2(X_{2n-1}) = z_{2n-1} + \mu$ for some μ with $i^*\mu = 0$. Now

$$U = e_{2N}X_{2n-1}\omega^{2l-N-1}\prod_{l<j<n} X_{4j-1}$$

since $\rho_p(U) \neq 0$ for all primes p. Note that $P^{2N}y_{4j-1}$ for $j \in J_{n,k} - \{N\}$ survives in the integral spectral sequence for π for $l < N < 2l < n$ so there exist classes Y_j in $H^*(PV_{2n,k})$ for $1 \leqq j \leqq t$ such that

$$(5.9) \qquad \begin{aligned} i^*Y_j &= P^{2N}y_{4i(j)-1} \quad \text{and} \\ Y_j\omega^{2[i(j)-i(j-1)]} &= e_{2N}X_{4i(j)-1} \quad \text{modulo torsion.} \end{aligned}$$

If $N = i(t)$, Y_t is not defined and X_{4N-1} can be chosen so that

$\rho_2(X_{4N-1}) = \alpha^{2[N-i(t-1)]} Z_{4i(t-1)-1}$. In summary, we have the following

PROPOSITION 5.10. *For* $l < N < 2l < n$ *and* k *odd,* $H^*(PV_{2n,k})/\mathrm{Tors}$ *is generated by* ω, e_{2N}, X_{2n-1}, X_{4j-1}, *and* Y_r *for* $l < j < n$ *and* $1 \leqq r \leqq t$ *with* Y_t *omitted if* $i(t) = N$.

THEOREM 5.11. *Suppose that* $2l < n$ *with* k *odd.* $H^*(PV_{2n,k})$ *is generated by the classes* ω, e_{2N}, X_{2n-1}, X_{4j-1}, Y_r, V_{2s}, *and* $U_{s,t}$ *for* $l < j < n$ *and* $s, t \in J_{n,k} - \{N\}$ *with* $s < t$. $H^*(PV_{2n,k})/\mathrm{Tors}$ *is given by* (5.5), (5.8), *and* (5.10).

REMARK. The known result that the real geometric dimension of $m\eta$ based on CP^j must be greater than $j - 2$ follows from the fact that $\pi^*\beta^{2l}$ is in Tor $H^*(PV_{2n,k})$.

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UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556