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# SOME $C^*$-ALGEBRAS WITH OUTER DERIVATIONS

GEORGE A. ELLIOTT

1. In [9], Sakai has given an example of a simple $C^*$-algebra without unit every derivation of which is inner. Theorem 2 below shows that such a $C^*$-algebra cannot be separable.

Theorem 3, the main result of this paper, gives a complete description of separable liminary $C^*$-algebras every derivation of which is inner.[1]

A consequence of Theorem 3 is that a separable liminary $C^*$-algebra every derivation of which is inner is the direct sum of a commutative algebra and an algebra with unit. Theorem 2 shows that this implication holds for a separable primitive $C^*$-algebra, and a modification of the proof of Theorem 3 (see 4.3) shows that it holds for a separable $C^*$-algebra whose primitive spectrum is separated.[2]

Another consequence of Theorem 3 is that if every derivation of a separable liminary $C^*$-algebra is inner then each quotient of this $C^*$-algebra has this property.

2. THEOREM. *Let $A$ be a separable $C^*$-algebra. If $A$ has a primitive quotient without unit then $A$ has outer derivations.*

PROOF. By [1] $A$ has a commutative approximate unit, contained, say, in a commutative sub-$C^*$-algebra $B$ of $A$. Let $t$ be a primitive ideal of $A$ such that $A/t$ does not have a unit. Then because $B$ is separable and $(B + t)/t$ does not have a unit, there is a bounded sequence $(x_n)$ of elements of $B$ whose images in $(B + t)/t$ have norm one, and whose supports in the spectrum of $B$ are compact, mutually disjoint, and, except for finitely many, disjoint from each fixed compact.

*Claim.* The inner derivation of $A$ defined by $\sum_{n=1}^{k} x_{2n}$, $k = 1, 2, \cdots$, converges simply to an outer derivation.

To show convergence on all of $A$ it is enough to show convergence on a dense subset of $A$. The set of $x \in A$ such that $yx = xy = x$ for some $y \in B$ of compact support in the spectrum of $B$ is dense in $A$, and for each such $x$, $x_nx = xx_n = 0$ for all but finitely many $n$.

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[1] *Added in proof*: It has been shown by Akemann, Elliott, Pedersen and Tomiyama (*Derivations and multipliers of $C^*$-algebras*, preprint) that in this theorem "liminary" can be replaced by "postliminary".

[2] *Added in proof*: This implication has now been established for an arbitrary separable $C^*$-algebra (op. cit.).

The limit is clearly a derivation; denote it by $\delta$. Since if $\delta$ is inner the derivation induced by $\delta$ in any quotient of $A$ is inner, to show that $\delta$ is outer it is sufficient to consider the case $t = 0$. In this case $A$ can be represented faithfully as an irreducible algebra of operators, and when this is done $\delta$ is implemented by the operator $\sum_{n=1}^{\infty} x_{2n}$. Moreover, any other operator implementing $\delta$ must differ from $\sum_{n=1}^{\infty} x_{2n}$ by a scalar. The properties of the $x_n$ together with the fact that $B$ contains an approximate unit for $A$ imply that no such operator can be an element of $A$.

3. Theorem. *Let $A$ be a separable liminary $C^*$-algebra.*[1] *Then the following two conditions are equivalent:*

(i) *every derivation of $A$ is inner;*

(ii) *$A$ is a finite direct sum of homogeneous $C^*$-algebras, each noncommutative homogeneous summand possessing a unit.*

Proof. (ii) $\Rightarrow$ (i). Let $\delta$ be a derivation of $A$. For each $t \in \operatorname{Prim} A$ let $\delta_t$ denote the derivation of $A/t$ induced by $\delta$. Since $A/t$ is isomorphic to a matrix algebra there exists an element of $A/t$ which determines $\delta_t$, and a unique such element $x_t$ of trace zero. To show that $\delta$ is inner it is enough to show that the field $t \mapsto x_t$ on $\operatorname{Prim} A$ is continuous with respect to the continuous field of $C^*$-algebras defined by $A$, and vanishes at infinity. In fact the field has compact support. Let $t_0$ be a point of $\operatorname{Prim} A$. By [8, Theorem 3.2], there is a compact neighbourhood $K$ of $t_0$ such that the restriction to $K$ of the continuous field of $C^*$-algebras defined by $A$ is trivial. The $C^*$-algebra defined by this restriction is then isomorphic to the tensor product of a matrix algebra and a commutative $C^*$-algebra with unit, and it is known that a derivation of such a $C^*$-algebra is inner (see e.g. [6, remark on page 254]). Hence the field $t \mapsto x_t$ differs on $K$ from a continuous field by the trace of this continuous field; since this trace is continuous the field $t \mapsto x_t$ is continuous at $t_0$.

(i) $\Rightarrow$ (ii). By Theorem 2, every primitive quotient of $A$ possesses a unit. Therefore the $C^*$-algebra $A^\sim$ obtained by adjoining a unit to $A$ is liminary. Since it is enough to prove (ii) with $A$ replaced by $A^\sim$, it is enough to prove (i) $\Rightarrow$ (ii) for the case that $A$ has a unit. In this case $\operatorname{Prim} A$ is compact, and (ii) follows once it is known that for each $r = 2, 3, \cdots$ the open set $\{t \in \operatorname{Prim} A \mid \dim A/t \geqq r^2\}$ is closed.

Suppose, then, that $A$ has a unit and that there exists a convergent sequence $t_n \to t$ in $\operatorname{Prim} A$ such that $\dim A/t_n \geqq r^2$, $n = 1, 2, \cdots$, and $\dim A/t < r^2$. The following argument shows that $A$ must have an outer derivation.

By [4, Proposition 2], there exists in $\operatorname{Prim} A$ a dense open set $S$

of separated points. The points $t_n$ may be assumed to be in S.

Since $A$ is separable, the space $\mathcal{C}(\text{Prim } A)$ of closed subsets of Prim $A$, with the Fell topology, is compact and metrizable ([7], and [5, Lemme 2]). Therefore the sequence $(\{t_n\})$ may be assumed to be convergent in $\mathcal{C}(\text{Prim } A)$, say to a closed subset $F$ of Prim $A$.

Choose an open neighbourhood $O_n$ of $\{t_n\}$ in $\mathcal{C}(\text{Prim } A)$, $n = 1, 2, \cdots$, in such a way that each neighbourhood of $F$ contains all but a finite number of $O_n$.

Let $\lambda$ denote the map $\text{Prim } A \ni s \mapsto \{s\} \in \mathcal{C}(\text{Prim } A)$. Fix $n = 1, 2, \cdots$. By [5, Theoreme 17], $\lambda \mid S$ is continuous, so $\lambda^{-1}(O_n) \cap S$ is open. Let $g_n$ be a continuous function on Prim $A$ satisfying $|g_n| \leqq 1$, $g_n(t_n) = 1$ and support $g_n \subset \lambda^{-1}(O_n) \cap S$.

The functions $g_n$ will be used to construct a sequence of elements of $A$, the derivations defined by finite sums of which converge to an outer derivation. The construction will be such that both the convergence (simple) of the derivations and the outer nature of the limit follow from the fact that for each element of $A$ the norm of the image of this element in the quotient of $A$ corresponding to a point of $\mathcal{C}(\text{Prim } A)$ depends continuously on this point ([8, Theorem 2.2]).

There are two cases to consider: $F = \{t\}$ and $F \neq \{t\}$.

If $F = \{t\}$, let $(e_{ij})$ be a complete system of matrix units for $A/t$. By [8, Theorem 3.1], there exists an array $(x_{ij})$ of elements of $A$ whose image in the quotient of $A$ corresponding to each point in a neighbourhood of $\{t\}$ in $\mathcal{C}(\text{Prim } A)$ is a system of matrix units, and whose image in $A/t$ is $(e_{ij})$. For all except finitely many $n = 1, 2, \cdots$, which will be ignored, the open set $O_n$ is a subset of this neighbourhood of $F$. For the moment, fix $n = 1, 2, \cdots$. There exists an element $y_n$ of $A$ whose image in $A/t_n$ is nonscalar but permutable with the image of each $x_{ij}$ in $A/t_n$. Set $\sum_i x_{i1}y_nx_{1i}$ equal to $z_n$. Then the element $g_nz_n$ of $A$ has the same image in $A/t_n$ as $y_n$ and moreover is permutable with each $x_{ij}$. Some scalar multiple of $g_nz_n$ defines a derivation of $A$ of norm one; denote this element by $x_n$. The inner derivation of $A$ defined by $\sum_{n=1}^{k} x_n$, $k = 1, 2, \cdots$, converges simply to an outer derivation. (Every element of $A$ is the sum of an element of $t$ and a linear combination of the $x_{ij}$. Since the $x_n$ are permutable with the $x_{ij}$, to show convergence it is enough to show that for each $y \in t$, $\sum_{n=k}^{m} x_ny - y\sum_{n=k}^{m} x_n \to 0$ $(k, m \to \infty)$. This holds because each $\sum_{n=k}^{m} x_n$ defines a derivation of norm one, and is zero for sufficiently large $k$ and $m$ in all primitive quotients of $A$ except those in which $y$ is small. If the limit derivation were inner, defined by $z \in A$, then $z + t$ would be zero in $A/t$, but in quotients of $A$ corresponding to points of $\mathcal{C}(\text{Prim } A)$ arbitrarily close to $\{t\}$ $z$

would define a derivation of norm one; this would contradict the above-mentioned continuity of the norm.)

If $F \neq \{t\}$, then $F$ must consist of at least two points. By [3, Proposition 12] (for example), the quotient of $A$ whose primitive spectrum is $F$ has a nonscalar central element; let $x$ be a representative in $A$ of such an element. The inner derivation of $A$ defined by $\sum_{n=1}^{k} g_{2n}x$, $k = 1, 2, \cdots$, converges simply to an outer derivation. (For any $y \in A$ the image of $xy - yx$ in a quotient corresponding to a point of $\mathcal{C}(\text{Prim } A)$ sufficiently close to $F$ is small; hence, for sufficiently large $k$ and $m$, $\sum_{n=k}^{m} g_{2n}xy - y\sum_{n=k}^{m} g_{2n}x$ is small. If the limit derivation were inner, defined by $z \in A$, then, for $r = 1, 2, \cdots$, on the one hand the image of $z$ in $A/t_{2n}$ would differ from the image of $x$ by a scalar, and on the other hand the image of $z$ in $A/t_{2n+1}$ would be a scalar. Since $\{t_n\}$ converges to $F$ in $\mathcal{C}(\text{Prim } A)$ it would follow by the above-mentioned continuity of the norm that the image of $x$ in the quotient of $A$ with primitive spectrum $F$ is scalar, contrary to the choice of $x$.)

4. Remarks. 4.1. Let $A$ be a noncommutative $C^*$-algebra and let $B$ be a $C^*$-algebra without unit. Then $A \otimes B$ has outer derivations. More precisely, if $x$ is a noncentral element of $A$ there exists a unique derivation $\delta_x$ of $A \otimes B$ such that $\delta_x(y \otimes z) = (xy - yx) \otimes z$, and $\delta_x$ is outer.

To show this it is enough to consider the case that $A$ is primitive, so that $A$ and $B$ can be realized as algebras of operators, $A$ acting irreducibly. If $\delta_x$ is inner then for every $\epsilon > 0$ there exists a finite sum $\sum y_i \otimes z_i$ in $A \otimes B$ such that the norm of the derivation of $A \otimes B$ defined by $x \otimes 1 - \sum y_i \otimes z_i$ is less than $\epsilon$. Since $A$ acts irreducibly, there exists a state $f$ of $A$ such that

$$\|(x \otimes 1 - \textstyle\sum y_i \otimes z_i) - (f(x) \otimes 1 - \textstyle\sum f(y_i) \otimes z_i)\| \leqq \epsilon.$$

Hence, if $g$ is any state of $A$,

$$\|(g(x) - f(x)) - \textstyle\sum (g(y_i) - f(y_i))z_i\| \leqq \epsilon.$$

Because $x$ is not a scalar there exists $\beta > 0$ such that for every state $f$ of $A$ there is a state $g$ of $A$ such that $|g(x) - f(x)| > \beta$. Since $\epsilon$ is arbitrary it follows that $B$ contains a nonzero scalar, a contradiction.

Corollaries 1 and 2 of [9] are special cases of this result.

A variation of the preceding argument shows that if a $C^*$-algebra has outer derivations then its tensor product with any nonzero $C^*$-algebra has outer derivations.

4.2 Let $A$ be an infinite-dimensional separable matroid $C^*$-algebra with unit, and let $B$ be an infinite-dimensional separable commutative $C^*$-algebra with unit. Then $A \otimes B$ has outer derivations.

To see this it is enough to consider the case that $B$ is the $C^*$-algebra of convergent sequences of complex numbers; then $A \otimes B$ is isomorphic to the $C^*$-algebra of convergent sequences of elements of $A$. If $(x_n)$ is a sequence of elements of $A$ each of which defines a derivation of norm one, such that $x_n x - x x_n \to 0$ for each $x \in A$, then the sequence $(x_n)$, although not a multiplier of $A \otimes B$, defines a derivation. This derivation is not inner because for no choice of scalars $\lambda_n$ does $(x_n + \lambda_n)$ converge.

4.3. Let $A$ be a separable $C^*$-algebra. Suppose that every derivation of $A$ is inner. Then by Theorem 2 each primitive quotient of $A$ has a unit. If Prim $A$ is separated then it is not difficult to show that the set of $t \in \operatorname{Prim} A$ with $\dim A/t > 1$ is compact. This shows that $A$ is the direct sum of a commutative algebra and an algebra with compact primitive spectrum. In fact, a summand with compact primitive spectrum must have a unit, for arguments similar to those in the proof of Theorem 3 show that the extra point added to the primitive spectrum upon adjunction of a unit is isolated. (The existence of a dense open set of separated points, needed in the proof, follows from [3, Proposition 11].)

4.4. Consideration of the $C^*$-algebra of convergent sequences of complex $2 \times 2$ matrices shows that it is not always possible to extend a derivation from an ideal. Consideration of the $C^*$-algebra of convergent sequences of complex numbers shows that it is not always possible to lift an automorphism from a quotient. An example due to Kadison and Ringrose (unpublished) shows that it is not always possible to lift a derivation from a quotient. This question is open, however, in the separable case.

Knowledge that derivations could be lifted from quotients of a separable $C^*$-algebra would make possible some simplification in the proof of Theorem 3.

## References

1. J. F. Aarnes and R. V. Kadison, *Pure states and approximate identities*, Proc. Amer. Math. Soc. **21** (1969), 749–752. MR 39 #1980.

2. C. A. Akemann, *Approximate units and maximal abelian $C^*$-subalgebras*, Pacific J. Math. **33** (1970), 543–550. MR 41 #9001.

3. C. Delaroche, *Sur les centres des $C^*$-algèbres*, II, Bull. Sci. Math. (2) **92** (1968), 111–128. MR 38 #2612.

4. J. Dixmier, *Points séparés dans le spectre d'une $C^*$-algèbre*, Acta Sci. Math. (Szeged) **22** (1961), 115–128. MR 23 #A4030.

5. ——, *Sur les espaces localement quasi-compacts,* Canad. J. Math. **20** (1968), 1093–1100. MR 38 #5171.

6. G. A. Elliott, *Derivations of matroid C*-algebras,* Invent. Math. **9** (1969/70), 253–269. MR 42 #857.

7. J. M. G. Fell, *A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space,* Proc. Amer. Math. Soc. **13** (1962), 472–476. MR 25 #2573.

8. ——, *The structure of algebras of operator fields,* Acta Math. **106** (1961), 233–280. MR 29 #1547.

9. S. Sakai, *Derivations of simple C*-algebras,* III, Tôhoku Math. J. **23** (1971), 559–564.

QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA

INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

*Current address*: Mathematics Institute, Universitetsparken 5, 2100 Copenhagen Ø, Denmark