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MORE ON COMPLEMENTS OF MINIMAL SPANNING SURFACES

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ABSTRACT. W. R. Alford in volume 91 of the Annals of Mathematics has shown the existence of a knot which has two minimal spanning surfaces whose complements in S^3 are not homeomorphic. The trefoil knot is a companion to the knot. This paper shows that any nontrivial knot k is a companion to a knot K which has at least two minimal spanning surfaces.

Introduction. In [1], W. R. Alford exhibited a knot k and two minimal spanning surfaces S_1 and S_2 for k such that $S^3 - S_i$ are not homeomorphic. The knot was formed by sending the torus T containing the knot l in Fig. 1 faithfully to a regular neighborhood of the trefoil knot.

In a later paper [2], Alford and C. B. Schaufele constructed knots with 2^m really distinct minimal spanning surfaces; the surfaces do not have homeomorphic complements. The examples were constructed by sending the torus T containing the knot l in Fig. 1 faithfully to a regular neighborhood of the sum of m "nice" knots. The selection of the knots was strongly influenced by their algebraic properties.

The purpose of this paper is to show that any nontrivial knot is a companion to a knot K which has at least two minimal surfaces.

The knot K is the image of the knot l in T in Fig. 1 under a faithful homeomorphism of the solid torus T to a regular neighborhood V of the knot l.

The Alexander polynomial of K is $(2 - t) \cdot (2t - 1)$ [4] for any nontrivial k used. Thus K had genus at least one. The spanning surfaces for K have genus one, so K has genus one.

The surfaces. The surfaces for K are constructed as in [2]. The knot l is spanned by a singular disk in T as shown in Fig. 2. Only one side is shown; the singularities are in heavy lines.

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The image of the disk in V is as in Fig. 3 with the band portion tied in the knot k (with twists in the band).

The two singularities are cut out and a tube is attached to the boundaries of the excised disks as indicated in Fig. 4.

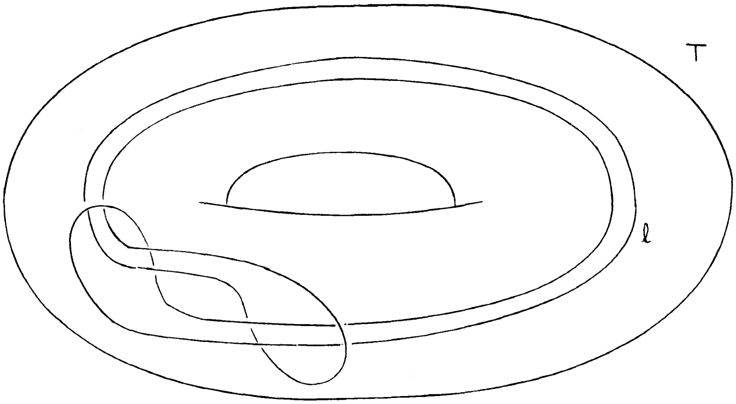

Figure 1

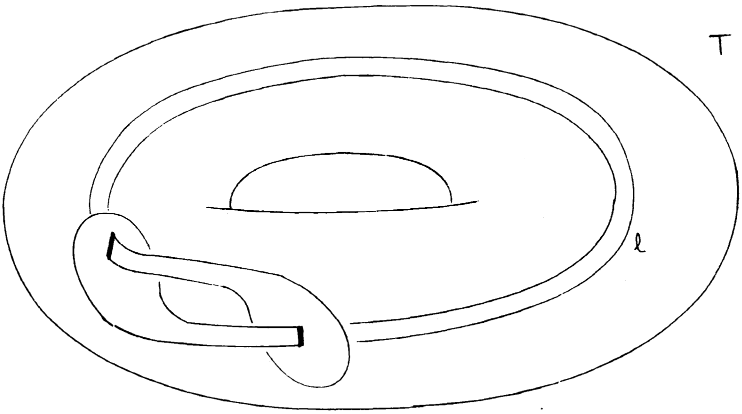

Figure 2

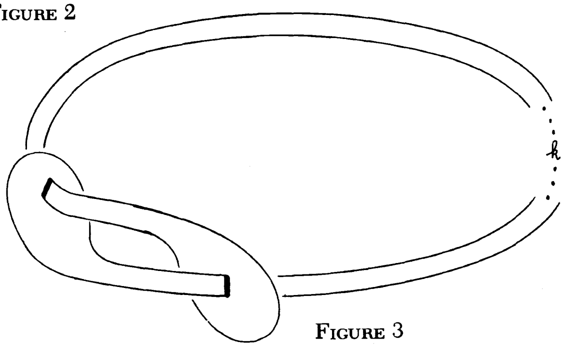

Figure 3

There are two possibilities for the tube surrounding the knot k in the band as indicated in Figs. 5 and 6 for the figure eight knot.

S_1 will be the surface when the tube does not go "through" the knotted band; S_2 when the tube does go "through" the knotted band. A spine for S_1 is shown in Fig. 7. Fig. 8 has the same knot type as Fig. 7. Thus $\pi_1(S^3 - S_1)$ is the free product of the integers with the knot group of k.

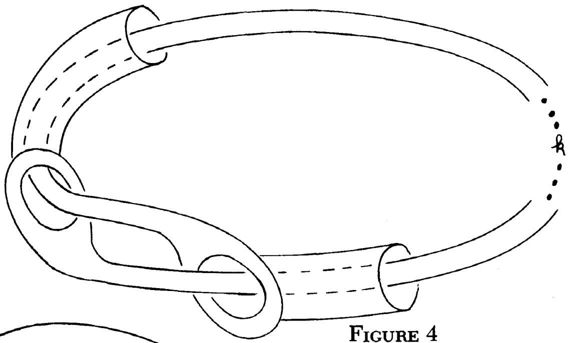

FIGURE 4

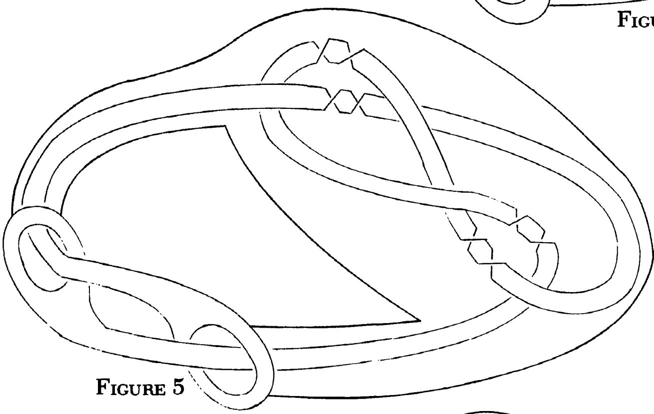

FIGURE 5

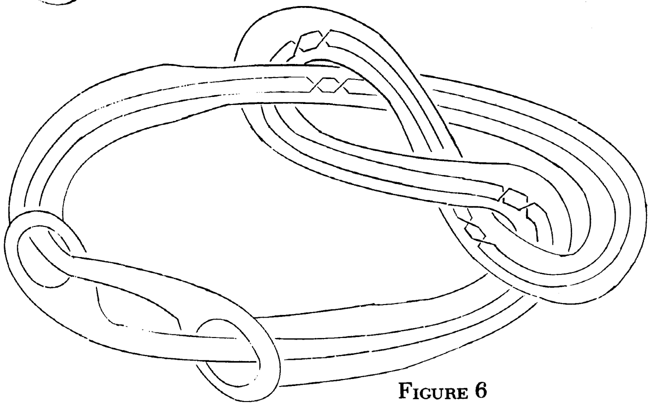

FIGURE 6

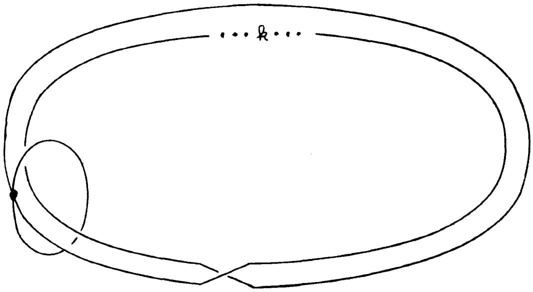

FIGURE 7

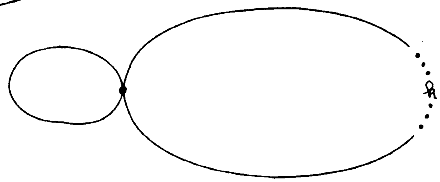

FIGURE 8

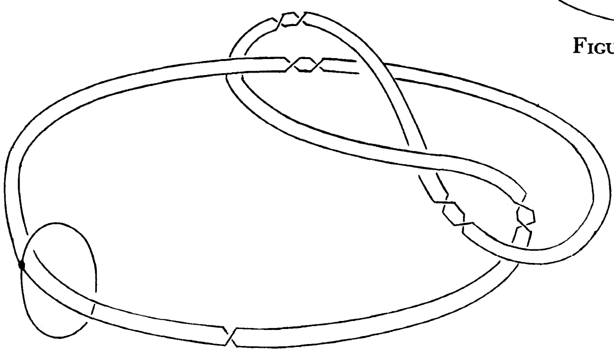

FIGURE 9

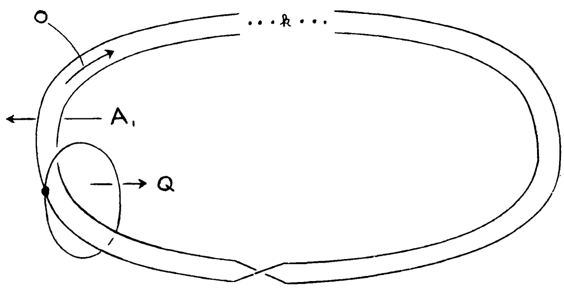

FIGURE 10

A spine C for S_2 will be taken so that the part of the spine which lies on the tube is "homologous" to zero in the complement of the knot k in the band. An example is shown in Fig. 9.

Let $G = (A_1, \cdots, A_n : R_1, \cdots, R_n)$ be the group of k obtained from an over presentation with A_1 as in Fig. 10.

Let C_1 be the part of the spine C bounding a disk in S^3 and C_2 the part bounding a Möbius band M. Let T be the boundary of a relative regular neighborhood of M in the complement of $C_1 - C_1 \cap C_2$. Let O be the center line of M. If $S^3 - C$ is decomposed into the part not inside T and the part not outside T, then a simple application of Van Kampen's Theorem [3] gives $\pi_1(S^3 - S_2) = (O, Q, A_1, \cdots, A_n : R_1, \cdots, R_n, O^2 = QWA^{-1}QW)$ where W is the word in G obtained from twisting the spine on the tube around the band. W is a generator of the first homology of the boundary of a small regular neighborhood of k in S^3.

Let $G_1 = \pi_1(S^3 - S_1) = Z * H$ and $G_2 = \pi_1(S^3 - S_2)$ where H is isomorphic to G. The theorem that will be proved is

THEOREM 1. *G_1 and G_2 are not isomorphic.*

Preliminaries. Suppose $\varphi : G_2 \to G_1$ is an isomorphism. G_2 contains a copy of G, the knot group of k. G is not a free product since k is not trivial [8]. Therefore, since rank $(G) > 1$, $\varphi(G)$ is conjugate to a subgroup of the free factor H in G_1 by the Kurosh Subgroup Theorem [5]. It can then be assumed that the isomorphism φ also sends G to a subgroup of H.

Let z be a generator of Z and let $\varphi(O) = v$, $\varphi(Q) = u$, $\varphi(A_1) = x$, $\varphi(WA_1^{-1}) = t$, $\varphi(W) = t' = t \cdot x$. x, t, and t' are in H. The following lemma will be needed later.

LEMMA 2. *v, u^2, x, v^2, v^2x^{-1}, t, t', $t' \cdot t^{-1}$ are each nontrivial words in G_1.*

PROOF OF LEMMA 2. The first five are nontrivial because G_2 is a free product with amalgamation containing as a subgroup the free group generated by Q free product with G. A_1 and W generate $Z \oplus Z$ [7, p. 57] as a subgroup of G since k is nontrivial. Because φ is an isomorphism, t, t' and $t't^{-1}$ cannot be trivial.

The relation $O^2 = QWA_1^{-1}QW$ in G_2 gives $v^2 = utut'$ in G_1. The strategy will be to show there is no $u \neq 1$ which satisfies the relation.

v has one of the following as its reduced form (b_i's belong to H):

Form 1: $v = b_1 z^{\alpha(1)} \cdots b_n z^{\alpha(n)}$.
Form 2: $v = b_1 z^{\alpha(1)} \cdots z^{\alpha(n-1)} b_n$.
Form 3: $v = z^{\alpha(1)} b_1 \cdots z^{\alpha(n)} b_n$.
Form 4: $v = z^{\alpha(1)} b_1 \cdots b_{n-1} z^{\alpha(n)}$.

Conjugation by an element of H in $Z * H$ sends H to itself. Thus conjugating Form 3 by b_n^{-1} and Form 4 by x gives rise to new isomorphisms of G_2 to G_1, sending G to a subgroup of H and giving the new v's Form 1 and Form 2 respectively. Form 2 can be changed to Form 1 when $b_n \cdot b_1 \neq 1$. Thus the existence of φ depends on v's ability to assume Form 1 or Form 2 with $b_n \cdot b_1 = 1$.

There are three cases to consider according to the reduced form of v^2.

Case 1. v has Form 1, $v^2 = b_1 z^{\alpha(1)} \cdots b_n z^{\alpha(n)} b_1 z^{\alpha(1)} \cdots b_n z^{\alpha(n)}$ is already reduced.

Case 2. v has Form 2 with $b_n \cdot b_1 = 1$, v^2 has reduced form $v^2 = b_1 z^{\alpha(1)} \cdots b_{q-1} z^{\alpha(q-1)}(b_q \cdot b_{n-q+1})\ z^{\alpha(n-q+1)} \cdots z^{\alpha(n-1)} b_n$ for $1 \leqq q < n$.

Case 3. v has form 2 with $b_n \cdot b_1 = 1$, v^2 has reduced form $v^2 = b_1 z^{\alpha(1)} \cdots b_q(z^{\alpha(q)+\alpha(n-q)}) b_{n-q+1} \cdots z^{\alpha(n-1)} b_n$ for $1 \leqq q < n$.

Therefore to prove Theorem 1, it need only be shown that Cases 1–3 cannot occur.

Proof of Theorem 1. The following lemma will contribute greatly to the demise of Case 1 and Case 2.

Lemma 3. *Let g_i's be elements of H and let $\beta(i)$'s be integers. Suppose the following two lists of equations hold for integers r, k, and p with $1 \leqq k - p \leqq p - 1$:*

$$(3.1)\qquad \begin{cases} g_{k-r+1} \cdot g_r = 1 & \text{for } 2 \leqq r \leqq k-p, \\ \beta(k-r) + \beta(r) = 0 & \text{for } 1 \leqq r \leqq k-p, \end{cases}$$

$$(3.2)\qquad \begin{cases} g_r = g_{k-p+r} & \text{for } 2 \leqq r \leqq p-1, \\ \beta(r) = \beta(k-p+r) & \text{for } 1 \leqq r \leqq p-1. \end{cases}$$

Then either there is an r, $2 \leqq r \leqq k-p$, so that $g_r = 1$ or there is an r, $1 \leqq r \leqq k-p$, so that $\beta(r) = 0$.

Proof of Lemma 3. The differences $A = \{k - 2r + 1 : 2 \leqq r \leqq k - p\}$ and $B = \{k - 2r : 1 \leqq r \leqq k - p\}$ of indices in (3.1) give $2(k-p) - 1$ consecutive integers and hence all equivalence classes modulo $(k-p)$ since $k - p \geqq 1$. If $0 \bmod (k-p)$ appears in A then there is an r, $2 \leqq r \leqq k-p$ so that $g_{k-r+1} \cdot g_r = 1$ and $(k - 2 + 1) \equiv 0 \bmod (k-p)$. Using the latter fact and $k - r \leqq p - 1$, one can deduce from (3.2) that $g_r = g_{k-r+1}$. Thus $g_r^2 = 1$. H has no torsion so $g_r = 1$. The alternate conclusion is reached in a similar manner if $0 \bmod (k-p)$ appears in B.

Case 1. Note length $(v^2) = 4n > 0$, v^2 begins with $b_1 \neq 1$ from H and ends with $z^{\alpha(n)} \neq 1$.

A. If $l(u) = 2k \geqq 2$ then either

$$u = a_1 z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)} \quad \text{or} \quad u = z^{\epsilon(1)} a_1 \cdots z^{\epsilon(k)} a_k.$$

If $u = a_1 z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)}$ then

$$v^2 = a_1 z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)} (ta_1) z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)} t'.$$

Spelling forces cancellation. Because of length, $v^2 = 1$ is the only possibility, contradicting Lemma 2.

If $u = z^{\epsilon(1)} a_1 \cdots z^{\epsilon(k)} a_k$ then

$$v^2 = z^{\epsilon(1)} a_1 \cdots z^{\epsilon(k)} (a_k t) z^{\epsilon(1)} a_1 \cdots z^{\epsilon(k)} (a_k t').$$

Because of spelling, $a_k t' = 1$ and $a_k t = 1$. Hence $t = t'$, a contradiction to Lemma 2.

B. If $l(u) = 2k - 1 \geqq 1$ then either

$$u = z^{\epsilon(1)} a_1 \cdots a_{k-1} z^{\epsilon(k)} \quad \text{or} \quad u = a_1 z^{\epsilon(1)} \cdots z^{\epsilon(k-1)} a_k.$$

The second is the only possible choice because no cancellation is possible in computing v^2 by $v^2 = utut'$ and there is a contradiction because of spelling.

If $u = a_1 z^{\epsilon(1)} \cdots z^{\epsilon(k-1)} a_k$ then

$$v^2 = a_1 z^{\epsilon(1)} \cdots z^{\epsilon(k-1)} (a_k t a_1) z^{\epsilon(1)} \cdots z^{\epsilon(k-1)} (a_k t').$$

Length and spelling force $a_k t' = 1$ and

$$v^2 = a_1 z^{\epsilon(1)} \cdots z^{\epsilon(k-1)} (a_k t a_1) z^{\epsilon(1)} \cdots z^{\epsilon(k-1)}.$$

If $a_k t a_1 \neq 1$ then using the two reduced forms for v^2 we have $a_1 = b_1 = a_k t a_1$ or $a_k t = 1$. Since $a_k t' = 1$ already, we have a contradiction to Lemma 2. Thus $a_k t a_1 = 1$ and cancellation will continue until the reduced form is either

$$v^2 = a_1 z^{\epsilon(1)} \cdots a_p (z^{\epsilon(p) + \epsilon(k-p)}) a_{k-p+1} \cdots a_{k-1} z^{\epsilon(k-1)}$$

or

$$v^2 = a_1 z^{\epsilon(1)} \cdots z^{\epsilon(p-1)} (a_p \cdot a_{k-p+1}) z^{\epsilon(k-p+1)} \cdots a_{k-1} z^{\epsilon(k-1)}$$

for $1 \leqq p \leqq k - 1$. The first can be eliminated because its length is $4p - 2$, which is not 0 mod 4. If the reduced form for v^2 is the second then $l(v^2) = 4(p - 1)$, so $n = p - 1$. Because of cancellations we have

$$a_{k-r+1} \cdot a_r = 1 \qquad \text{for } 2 \leqq r \leqq k - p,$$

$$\epsilon(k - r) + \epsilon(r) = 0 \qquad \text{for } 1 \leqq r \leqq k - p.$$

Using the two reduced forms for v^2 we have

$$a_r = a_{k-p+r} \qquad \text{for } 2 \leqq r \leqq p-1,$$
$$\epsilon(r) = \epsilon(k-p+r) \quad \text{for } 1 \leqq r \leqq p-1.$$

To apply Lemma 3 to obtain a contradiction that u is reduced we need only show

LEMMA 4. $1 \leqq k - p \leqq p - 1$.

PROOF OF LEMMA 4. $1 \leqq k-p$ follows from $1 \leqq p \leqq k-1$. If $k > 2p-1$ then in the cancellation to obtain the reduced form for v^2 using $v^2 = utut'$, the kth letter of u must be cancelled. The kth letter of u is $z^{\epsilon(k/2)}$ if k is even, $a_{(k+1)/2}$ if k is odd. Since the sum of the indices on the ϵ's must be k and on the a's must be $k+1$, either $2\epsilon(k/2) = 0$ or $a^2_{(k+1)/2} = 1$. So either $\epsilon(k/2) = 0$ or $a_{(k+1)/2} = 1$, a contradiction to u being reduced.

Lemma 3 can be applied to obtain that u is not reduced, a contradiction. Thus $l(u) \neq 2k - 1 \geqq 1$.

This completes the proof that Case 1 cannot occur.

Case 2. We note that $l(u^2) = 4q - 3 > 0$. v^2 begins with $b_1 \neq 1$ from H, v^2 ends with $b_n \neq 1$ from H. Because of cancellation to obtain the reduced form we have

$$b_{n-r+1} \cdot b_r = 1, \qquad 2 \leqq r \leqq n-q,$$
$$\alpha(n-r) + \alpha(r) = 0, \qquad 1 \leqq r \leqq n-q,$$

half of the equations needed to apply Lemma 3.

LEMMA 5. $1 \leqq n - q \leqq q - 1$.

The proof is exactly as in Lemma 4 using v instead of u.

A. If $l(u) = 2k \geqq 2$ then $u = z^{\epsilon(1)}a_1 \cdots z^{\epsilon(k)}a_k$ or $u = a_1 z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)}$.

If $u = z^{\epsilon(1)}a_1 \cdots z^{\epsilon(k)}a_k$ then

$$v^2 = z^{\epsilon(1)}a_1 \cdots z^{\epsilon(k)}(a_k t) z^{\epsilon(1)}a_1 \cdots z^{\epsilon(k)}(a_k t').$$

Because of spelling, $z^{\epsilon(1)}a_1 \cdots (a_k t) \cdots z^{\epsilon(k)} = 1$; in particular, $a_k t = 1$. Thus $v^2 = a_k t' = t^{-1}t' = x$, a contradiction to Lemma 2.

If $u = a_1 z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)}$ then

$$v^2 = a_1 z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)}(ta_1) z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)} t'.$$

If $ta_1 \neq 1$ then v^2 does not reduce further; $l(v^2) = 4k + 1$, so $q = k + 1$. Using the two reduced forms for v^2 we obtain

$$b_r = b_{n-q+r} \qquad \text{for } 2 \leqq r \leqq q-1,$$
$$\alpha(r) = \alpha(n-q+r) \quad \text{for } 1 \leqq r \leqq q-1.$$

Lemma 3 applied here gives a contradiction to v being reduced. Thus $ta_1 = 1$.

Observe that $l(v^2) = 1$ is impossible for then $ta_1 = 1$ and $v^2 = a_1 t'$ imply $v^2 = x$ which is impossible by Lemma 2. Thus if v^2 is allowed to reduce using $v^2 = utut'$ and if it is compared to the reduced form from v, we will always have the relation $a_1 = b_1$ and $t' = b_n$. Since $b_n \cdot b_1 = 1$ then $t' \cdot a_1 = 1 = ta_1$, a contradiction.

Hence $l(u) \neq 2k \geqq 2$.

B. If $l(u) = 2k - 1 \geqq 1$ then $u = z^{\epsilon(u)}a_1 \cdots a_k z^{\epsilon(k)}$ or $u = a_1 z^{\epsilon(1)} \cdots z^{\epsilon(k-1)}a_k$.

The former cannot occur because no cancellation is possible for v^2 from $v^2 = utut'$ and v^2 is spelled incorrectly.

If $u = a_1 z^{\epsilon(1)} \cdots z^{\epsilon(k-1)} a_k$ then

$$v^2 = a_1 z^{\epsilon(1)} \cdots z^{\epsilon(k-1)}(a_k t a_1) z^{\epsilon(1)} \cdots a_{k-1} z^{\epsilon(k-1)}(a_k t').$$

If $a_k t' = 1$, spelling of v^2 forces $v^2 = a_1$ and $a_k t a_1 = 1$. Since $t' = tx$, $a_1 = x$. Thus $v^2 = x$ contradicting Lemma 2. Therefore $a_k t' \neq 1$ and from this point the proof of the second part of A of this case can be imitated to deduce that $u = a_1 z^{\epsilon(1)} \cdots z^{\epsilon(k-1)} \cdots z^{\epsilon(k-1)} a_k$ is impossible. Hence $l(u) \neq 2k - 1 \geqq 1$.

This completes the proof that Case 2 cannot occur.

Case 3. We note that $l(v^2) = 4q - 1 > 0$, v^2 begins with $b_1 \neq 1$ from H and ends with $b_n \neq 1$ from H.

A. If $l(u) = 2k \geqq 2$ then $u = z^{\epsilon(1)}a_1 \cdots z^{\epsilon(k)}a_k$ or $u = a_1 z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)}$. The former is easily shown to be impossible by spelling and length arguments.

If $u = a_1 z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)}$ then

$$v^2 = a_1 z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)}(ta_1) z^{\epsilon(1)} \cdots a_k z^{\epsilon(k)} t'.$$

Because of length, $ta_1 = 1$ and since $l(v^2) \neq 1$ then using the two reduced forms for v^2 we always obtain the relations $a_1 = b_1$ and $t' = b_n$. Since $b_n \cdot b_1 = 1$, $t' \cdot a_1 = 1 = ta_1$, a contradiction. Thus $l(u) \neq 2k \geqq 2$.

B. The proof that $l(u) \neq 2k - 1 \geqq 1$ is very much like part A of this case.

Thus Case 3 cannot occur and Theorem 1 is proved.

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