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INVARIANT MEANS ON SUBSEMIGROUPS OF LOCALLY COMPACT GROUPS

ANTHONY TO-MING LAU [1]

1. **Introduction.** Throughout this paper G will denote a locally compact group and locally null subsets of G are defined with respect to a fixed left Haar measure λ of G.

Recently J. W. Jenkins [5] shows that if S is an open subsemigroup of G and G is left amenable, then S is left amenable if and only if S has finite intersection property for open right ideals. In this paper, we shall prove an analogue result for any nonlocally null Borel measurable subsemigroups S of G, generalising a result of Frey [2] (see also [8, Theorem 3.5]) for discrete left amenable groups.

2. **Preliminaries and some notations.** For any subset A of a topological space Y, $\bar{A}$ will denote the closure of A in Y and 1_A will be the characteristic one function on A. The class of Borel sets in Y is the smallest σ-algebra of sets containing all open subsets of Y.

Let S be a *topological semigroup*, i.e., S is a semigroup with a Hausdorff topology such that, for each $a \in S$, the two mappings from S into S defined by $s \to as$ and $s \to sa$ for all $s \in S$ are continuous. Let $MB(S)$ be the space of bounded Borel measurable real valued functions on S equipped with the sup norm topology. For each $a \in S$, define two operators, r_a and l_a, from $MB(S)$ into $MB(S)$ by $r_af(s) = f(sa)$ and $l_af(s) = f(as)$ for all $s \in S$, $f \in MB(S)$. Let X be a closed subspace of $MB(S)$ containing 1_S. An element ϕ in X^*, the conjugate space of X, is a *mean* if $\phi(1_S) = \|\phi\| = 1$. Furthermore, the restriction of any element in the convex hull of $\{p_s; s \in S\} \subseteq MB(S)^*$ to X is called a *finite mean* on X, where $p_s(f) = f(s)$. As known [1] the set of finite means on X is weak* dense in the set of means on X. If X is invariant under l_a for each $a \in S$, then a mean ϕ on X is a *left invariant mean* (LIM) if $\phi(l_af) = \phi(f)$ for all $a \in S$ $f \in X$. S is *left amenable* if $MB(S)$ has a LIM.

A bounded continuous real valued function f on S is *uniformly*

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continuous if the two mappings from S in $MB(S)$ defined by $s \to l_s f$ and $s \to r_s f$ are continuous when $MB(S)$ has the sup norm topology. Then as known, $UC(S)$, the space of uniformly continuous functions on S, is a closed subspace of $MB(S)$ containing 1_S. Furthermore, $UC(S)$ is invariant under l_a and r_a for each $a \in S$.

A jointly continuous action of S on a topological space Y is a continuous mapping from $S \times Y$ (with the product topology) into Y, denoted by $(x, y) \to s \cdot y$, such that $s_1 \cdot (s_2 \cdot y) = (s_1 s_2) \cdot y$ for all $s, s_1, s_2 \in S$ and $y \in Y$.

3. **Technical lemmas.** In preparation of our main results, we shall prove in this section a series of lemmas.

Lemma 1. *Let S be a topological semigroup with the finite intersection property for closed right ideals. Then, for any jointly continuous action of S on a compact Hausdorff space Y the set $K = \bigcap \{s \cdot X; s \in S\}$ is nonempty and $s \cdot K = K$ for all $s \in S$.*

Proof. We shall show that the family $\{aX; a \in S\}$ has the finite intersection property, which will imply K is nonempty by compactness of X. For any finite subset σ of S, choose $c \in \bigcap \{\overline{aS}; a \in \sigma\}$, then $cX \subseteq \bigcap \{aX; a \in \sigma\}$. In fact, if $a \in \sigma$ and $x \in X$, there is a net $\{s_\alpha\}$ in S such that $\lim_\alpha a s_\alpha = c$. By compactness of X (and passing to a subnet if necessary), we may assume $\lim_\alpha s_\alpha x = y$ for some $y \in X$. Therefore

$$ay = \lim_\alpha a \cdot (s_\alpha \cdot x) = \lim_\alpha (a s_\alpha) x = cx$$

and hence $cx \in aX$. Since x is arbitrary, it follows that $cX \subseteq \{\overline{aX}; a \in \sigma\}$.

To see that $aK = K$ for all $a \in S$, let $y \in K$ and $a \in S$ be arbitrary but fixed. If $s \in S$, choose $u \in \overline{aS} \cap \overline{sS}$ and nets $\{s_\alpha\}$, $\{t_\beta\}$ in S such that $\lim_\alpha a s_\alpha = \lim_\beta s t_\beta = u$. Since $y \in K$, we can choose for each α an element x_α in X such that $s_\alpha \cdot x_\alpha = y$. We may assume (by compactness of X and passing to subnets if necessary) that $\lim_\alpha x_\alpha = x_0$ and $\lim_\beta t_\beta x_0 = x_1$ for some x_0, $x_1 \in X$. We have, by virtue of the continuity of the mapping $S \times X \to X$, that

$$\begin{aligned} a \cdot y &= \lim_\alpha a \cdot (s_\alpha \cdot x_\alpha) = \lim_\alpha (a s_\alpha) \cdot x_\alpha = u \cdot x_0 \\ &= \lim_\beta (s t_\beta) \cdot x_0 = \lim_\beta s \cdot (t_\beta \cdot x_0) = s x_1. \end{aligned}$$

Hence $ay \in sX$. Since y and s are arbitrary, it follows that $aK \subseteq K$.

To obtain the other inclusion, let σ be a finite subset of S and

$c \in \bigcap \{\overline{sS}; a \in \sigma\}$. Then $cX \subseteq \bigcap \{sX; s \in \sigma\}$ as shown in the earlier part of the proof.

Hence

$$a^{-1}\{y\} \cap (\bigcap \{sX; s \in \sigma\}) \subseteq a^{-1}\{y\} \cap cX \neq \emptyset$$

where $a^{-1}\{y\} = \{x \in X; ax = y\}$. Consequently $a^{-1}\{y\} \cap K \neq \emptyset$ is nonempty by compactness of X and $aK \subseteq K$.

The next lemmas were proved by Day in [1, Theorem 2] for discrete semigroups.

LEMMA 2. *Let S be a Borel measurable subsemigroup of a topological semigroup H. If there is a LIM μ on $MB(H)$ such that $\mu(1_S) > 0$, then S is left amenable.*

This lemma can be proved by repeating "mutatis mutandis," the argument used in [1, Theorem 2]. We omit the details.

LEMMA 3. *Let S be a nonlocally null Borel measurable subsemigroup of G. If there is a mean μ on $MB(G)$ such that $\mu(1_S) = 1$ and the restriction of μ on $UC(G)$ is a LIM, then there is a LIM ψ on $MB(G)$ such that $\psi(1_S) = 1$.*

PROOF. Let E be a compact subset of G such that $E \subseteq S$ and $\lambda(E) > 0$ (see [4, p. 127]); let Φ_E and $\Phi_{E^{-1}}$ be the normalised characteristic functions on E and E^{-1} respectively. For each f in $MB(G)$, define two bounded continuous real valued functions $\Phi_{E^{-1}} * f$ and $f * \tilde{\Phi}_{E^{-1}}$ on G by

$$(\Phi_{E^{-1}} * f)(g) = \int f(t^{-1}g)\Phi_{E^{-1}}(t)\, dt,$$
$$(f * \tilde{\Phi}_E)(g) = \int f(t)\tilde{\Phi}_E(t^{-1}g)\, dt,$$

where $\tilde{\Phi}_E(g) = \Phi_E(g^{-1})$ for all $g \in G$ and the integration is taken with respect to the left Haar measure λ on G (see [4, 20.14 and 20.16]). Since $\Phi_{E^{-1}} * f * \tilde{\Phi}_E$ is in $UC(G)$ (see [3, Lemma 2.1.2]), we may define a mean ψ on $MB(G)$ by $\psi(f) = \psi(\Phi_{E^{-1}} * f * \tilde{\Phi}_E)$ for all $f \in MB(G)$. Furthermore, an argument similar to that given in the proof of [3, Theorem 2.2.1, (5)$\Rightarrow$(1)] and [3, Proposition 2.1.3] will show that ψ is even a LIM on $MB(G)$. Finally if $a \in S$, then

$$\begin{aligned}\Phi_{E^{-1}} * 1_S(a) &= \int \Phi_{E^{-1}}(t)1_S(t^{-1}a)\, dt \\ &= \int \Phi_{E^{-1}}(at)1_S(t^{-1})\, dt \\ &= \lambda(S^{-1} \cap a^{-1}E^{-1})/\lambda(E^{-1}) = 1.\end{aligned}$$

It follows that

$$((\Phi_{E^{-1}} * 1_S) * \tilde{\Phi}_E)(a) = 1_S * \tilde{\Phi}_E(a)$$
$$= \lambda(S \cap aE)/\lambda(E) = 1$$

for all $a \in S$. Hence $\psi(1_S) = 1$.

4. **Main results.** We are now ready to prove our main results.

THEOREM 1. *Let S be a nonlocally null Borel measurable subsemigroup of G.*

If G is left amenable, then each of the following conditions are equivalent:

(a) *S is left amenable.*

(b) *S has the finite intersection property for right ideals.*

(c) *S has the finite intersection property for closed right ideals.*

PROOF. If ψ is LIM on $MB(S)$ and $a \in S$, then $1_{aS} \in MB(S)$ and $\psi(1_{aS}) = \psi({}_a 1_{aS}) \geqq \psi(1_S) = 1$. Hence, for any finite subset $\sigma \subseteq S$, $\bigcap \{aS; a \in \sigma\}$ is nonempty and (b) follows.

That (b) implies (c) is trivial.

Finally if (c) holds, let H be the closed subgroup of G generated by S (note that S is also Borel measurable in H and it is not locally null with respect to any left Haar measure on H). For each $g \in H$, let l_g^* denote the adjoint of the operator l_g from $UC(H)$ into $UC(H)$. Let K be the collection of all mean ϕ on $UC(H)$ which has an extension to a mean $\tilde{\phi}$ on $MB(H)$ with the property $\tilde{\phi}(1_S) = 1$. Certainly K is a nonempty weak* compact convex subset of $UC(H)^*$ and $l_s^*(K) \subseteq K$ for all $s \in S$. Since the mapping $(s, \phi) \to l_s^*\phi$, $s \in S$, $\phi \in K$, defines a jointly continuous action of S on K (with the weak* topology), it follows from Lemma 1 that the set $K_0 = \bigcap \{l_s^*(K); s \in S\}$ is nonempty and $l_s^* K_0 = K_0$ for all $s \in S$. Furthermore, if $s \in S$ and $\phi \in K_0$, then $l^*_{s^{-1}}\phi = l^*_{s^{-1}}(l_s^*\psi) = \psi$ for some $\psi \in K_0$. It follows that $l_g^*(K_0) \subseteq K_0$ for all $g \in H$. Since H is left amenable [3, Theorem 2.3.2], it follows from a fixed point theorem of Rickert [7, Theorem 4.2] that the jointly continuous affine action of H on the weak* compact convex set K_0 defined by $(g, \phi) \to l_g^*\phi$, $g \in H$ and $\phi \in H$ and $\phi \in K_0$, must have a common fixed point ψ in K_0 for H. Our result now follows from Lemma 2 and Lemma 3.

The next result is due to Greenleaf [3, Theorem 2.2.1] for the case when $S = G$.

THEOREM 2. *For any nonlocally null Borel measurable subsemigroup S of G, S is left amenable if and only if UC(S) has a LIM.*

PROOF. One direction is trivial; conversely, if ψ is a LIM on $UC(S)$ and H is the closed subgroup of G generated by S, let $\{\psi_\alpha\}$ be a net of finite means on $MB(S)$ such that $\lim_\alpha \psi_\alpha(f) = \psi(f)$ for all $f \in UC(S)$. Define a net of finite means $\{\bar{\psi}_\alpha\}$ on $MB(H)$ by $\bar{\psi}_\alpha(h) = \psi_\alpha(\Pi h)$ where $\Pi h(s) = h(s)$ for all $s \in S$, $h \in MB(H)$. Let $\bar{\psi}$ be a weak* cluster point of the net $\{\bar{\psi}_\alpha\}$ in $MB(H)^*$.

Clearly $\bar{\psi}(1_S) = 1$. Hence if we can show that the restriction of $\bar{\psi}$ to $UC(H)$ is a LIM, then it follows from Lemma 3 that $MB(S)$ has a LIM. Indeed, if $a \in S$, then $l_a{}^*\bar{\psi} = \bar{\psi}$ and $l^*_{a^{-1}}\bar{\psi} = \bar{\psi}$, where $l_a{}^*$ is the conjugate of the operator l_a from $UC(H)$ into $UC(H)$. Since the mapping from $H \times K$ into K, $(g, \phi) \to l_g{}^*\phi$, where K is the set of means on $UC(H)$, $g \in H$ and $\phi \in K$, is continuous when K has the weak* topology, it follows that $\bar{\psi}$ is a LIM on $UC(H)$.

COROLLARY (JENKINS). *If S is an open subsemigroup of G and G is left amenable, then S is left amenable if and only if S has the finite intersection property for open right ideals.*

PROOF. Note that any ideal I of S contains an open ideal aS for some $a \in S$. Use Theorem 1.

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UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA