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UNIQUENESS AND CONTINUOUS DEPENDENCE CRITERIA FOR THE NAVIER-STOKES EQUATIONS[1]

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I. **Introduction.** This paper is mainly of expository nature and deals with certain computational or practical questions regarding the Navier-Stokes equations. It is rather surprising that although the questions of existence and regularity have been pretty well answered by the important works of Leray [12] and others (see e.g. papers cited in the works of Finn [5], [6], Ladyzhenskaya [11], and Serrin [22]), usable criteria for stability and uniqueness were almost non-existent until very recently. These latter questions are in some ways much more difficult than those of existence and regularity, and, in fact, only the first rather crude steps in the determination of explicit uniqueness and stability criteria have been made (see e.g. Payne [17], [18]).

It would of course be very useful to have some a priori criterion involving the viscosity coefficient, the geometry of the domain, and the prescribed data, which would guarantee uniqueness of stationary solutions or stability of unsteady flows. It has long been known that if the viscosity coefficient is large enough, the domain is small enough and the data are small enough, then there do exist unique, stable solutions, but it was not known how small (large) was small (large) enough or just what the right measure of smallness (largeness) ought to be.

My primary interest in the Navier-Stokes equations has, therefore, been directed at the following questions:

1. The determination of explicit criteria for uniqueness in the steady state problem.
2. The determination of explicit criteria for convergence to steady state.
3. The determination of explicit growth criteria for solutions of the time dependent problem.
4. Solutions of the Navier-Stokes equations backward in time.

The discussion in this paper will be concerned primarily with these

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four questions and with related eigenvalue problems. For simplicity we confine our remarks to classical solutions of the Navier-Stokes equations although in many cases it will be possible to interpret the results directly in a more general sense. We treat in particular:

A. *Steady state problem.* Let D denote a bounded domain in three-space with smooth boundary ∂D. We are interested in the solution u_i of problem $\mathcal{S}$, i.e.,

$$\mathcal{S} \qquad \left.\begin{aligned} \nu\,\Delta u_i &= u_j u_{i,j} + p_{,i} \\ u_{j,j} &= 0 \end{aligned}\right\} \text{ in } D, \qquad u_i = f_i \quad \text{on } \partial D.$$

Here Δ denotes the Laplace operator, $\nu > 0$ is the coefficient of kinematic viscosity, and p the unknown pressure. We have also used the comma to denote differentiation and have adopted the summation convention.

B. *Dynamic problem.* We denote the three dimensional flow domain at $t = 0$ by D. Since the flow history during the time interval $0 \leqq t \leqq \tau$ is to be considered we denote by $\Omega(\tau)$ the cartesian product domain $D \times (0, \tau)$. We shall be concerned with integrals over D at time t and shall express them as follows: $\int_{D_t} f(x, t)dx$. The symbol ∂D_t will be used for the boundary of D_t and $S(\tau) \equiv \partial D \times [0, \tau]$. We seek a vector function $u_i(x, t)$ which satisfies (problem $\mathcal{D}$).

$$\mathcal{D} \qquad \left.\begin{aligned} \nu\,\Delta u_i &= p_{,i} + u_j u_{i,j} + \frac{\partial u_i}{\partial t} \\ u_{j,j} &= 0 \end{aligned}\right\} \text{ in } \Omega(\tau), \qquad u_i = f_i \quad \text{on } S(\tau), \qquad u_i(x, 0) = g_i(x).$$

Thus question 1 deals with the uniqueness of problem $\mathcal{S}$, while question 2 deals with the convergence of solutions of $\mathcal{D}$ to those of $\mathcal{S}$ as $t \to \infty$ (τ assumed infinite) if f_i is independent of time. Question 3 is concerned with the growth of solutions of $\mathcal{D}$ as $t \to \infty$, and the final question involves a nonstandard problem for $\Omega(\tau)$.

Let us remark on the fact that in $\mathcal{S}$ and $\mathcal{D}$ we deal with the homogeneous system of equations and inhomogeneous boundary data. We could handle the inhomogeneous system with little additioanl difficulty, but this would merely complicate the ideas we wish to present. On the other hand it is quite easy to deal with the problem of inhomogeneous system with homogeneous boundary data, and there has been a tendency to emphasize this particular problem in the literature. We have restricted consideration here to three dimensional problems. Some simplifications can be realized in two dimensions.

II. Uniqueness in problem $\mathcal{S}$. We start by giving a type of uniqueness theorem which does appear in the literature. In the usual manner we assume the existence of two solutions u_i^1 and u_i^2, set $w_i = u_i^1 - u_i^2$, and form:

$$E(w) \equiv \int_D w_{i,j}w_{i,j}dx = - \int_D w_i \,\Delta w_i dx$$

$$(2.1) \qquad = -\frac{1}{\nu}\int_D w_i\{(p_1 - p_2)_{,i} + w_j u^1_{i,j} + u_j^2 w_{i,j}\}\, dx$$

$$= -\frac{1}{\nu}\int_D w_i w_j u^1_{i,j}dx = \frac{1}{\nu}\int_D w_{i,j}w_j u_i^1 dx.$$

It follows then by Schwarz's inequality that

$$(2.1a) \qquad E^2(w) \leqq \frac{1}{\nu^2} E(w) \int_D w_j w_j u_i^1 u_i^1 dx.$$

Now suppose we know an upper bound for $u_i^1 u_i^1$ in $\bar{D}$, i.e.

$$(2.2) \qquad \sup_{x \in \bar{D}} u_i^1 u_i^1 \leqq M^2.$$

This leads to (assuming $E(w) \neq 0$)

$$(2.3) \qquad E(w) \leqq \frac{M^2}{\nu^2}\int_D w_j w_j dx \leqq \frac{M^2}{\nu^2 \lambda} E(w)$$

where

$$(2.4) \qquad \lambda = \inf_{\varphi_i = 0 \text{ on } \partial D;\, \varphi_{j,j} = 0 \text{ in } D} \frac{E(\varphi)}{\int_D \varphi_i \varphi_i dx}, \quad \varphi_i \in PC^1 \text{ in } D.$$

Now if we have any computable lower bound for λ, say

$$\lambda \geqq \hat{\lambda},$$

it follows that

$$(2.5) \qquad E(w)(1 - M^2/\nu^2\hat{\lambda}) \leqq 0.$$

Thus if

$$(2.6) \qquad M^2 < \nu^2 \hat{\lambda}$$

the inequality (2.5) implies $E(w) = 0$ which in turn implies that w_i must be a constant vector. The boundary conditions then imply

$w_i \equiv 0$. The criterion would therefore be explicit if we knew both M and $\hat{\lambda}$. In fact we would have the following theorem.

THEOREM I. *If there exists one classical solution of problem* $\mathcal{S}$ *satisfying* (2.2) *then there is no other solution which satisfies the same data provided* $M^2 < \nu^2\hat{\lambda}$.

Serrin [**21**], and later Velte [**24**], as well as Weinberger and the author [**19**], have given methods for determining $\hat{\lambda}$ in general, and improved methods for particular geometries. Note that by removing constraints and applying the Faber-Krahn inequality one would always have

$$\lambda \geqq \tilde{\lambda}$$

where $\tilde{\lambda}$ is the first eigenvalue of the problem $\mathcal{C}$,

$$\mathcal{C} \qquad \begin{aligned} \Delta u + \tilde{\lambda} u &= 0 \quad \text{in } K, \\ u &= 0 \quad \text{on } \partial K, \end{aligned}$$

where K denotes the sphere of the same volume as D. For obvious reasons we call $\mathcal{C}$ the fixed membrane problem for K. This problem is easily solvable. Weinberger and I showed, in fact, that

$$\lambda \geqq \tilde{\tilde{\lambda}}_2$$

where $\tilde{\tilde{\lambda}}_2$ is the second eigenvalue in problem $\mathcal{C}$, but in this case K denotes the smallest sphere which circumscribes D. It has been conjectured that

$$\lambda_2 \geqq \lambda_2{}'$$

where $\lambda_2{}'$ is the second fixed membrane eigenvalue for D. Velte has shown that λ is not less than the second membrane eigenvalue for any circumscribing parallelepiped, but the conjecture is as yet unproved.

Velte [**24**] has shown further that for a two dimensional simply connected domain, λ is just the first eigenvalue in the buckling problem for the elastic plate defined on D, i.e.,

$$\begin{aligned} \Delta^2\varphi + \lambda\, \Delta\varphi &= 0 \quad \text{in } D, \\ \varphi, \frac{\partial \varphi}{\partial n} &= 0 \quad \text{on } \partial D. \end{aligned} \tag{2.7}$$

It is worth noting that the minimum problem for λ defined by (2.4) has as its Euler equation

$$\left.\begin{aligned} \Delta\tilde{v}_i + \lambda\tilde{v}_i &= p_{,i} \\ \tilde{v}_{j,j} &= 0 \end{aligned}\right\} \text{ in } D, \tag{2.8}$$
$$\tilde{v}_i = 0 \quad \text{on } \partial D.$$

This equation for $\tilde{v}_i$ also describes the displacement of a vibrating incompressible elastic body occupying D.

We have thus seen that the determination of an explicit $\hat{\lambda}$ is not difficult. However, the computation of an explicit M essentially requires a knowledge of the solution u_i^1 and hence can hardly be considered as an a priori criterion. What makes this computation particularly impractical is the fact that very few explicit solutions of the Navier-Stokes equations are known.

Another type of explicit criteria was given by Hopf [7]. One denotes by $\mu_i(x)$ the three eigenvalues of the matrix $u^1_{i,j}(x)$. If we assume $\mu_1 \leqq \mu_2 \leqq \mu_3$ and set

$$-m = \inf_{x \in D} \mu_1(x) \tag{2.9}$$

then

$$-\frac{1}{\nu}\int_D w_i w_j u^1_{i,j} dx \leqq \frac{m}{\nu}\int_D w_i w_i dx \leqq \frac{m}{\nu\hat{\lambda}} E(w). \tag{2.10}$$

Hopf's criterion then is that, provided

$$m < \nu\hat{\lambda}, \tag{2.11}$$

problem $\mathcal{S}$ has at most one solution. Again m is not likely to be known unless $u_i^1(x)$ itself is known, and again we are left with a condition that is of little practical utility.

Let me say at this point that it would be possible using techniques indicated in Finn's survey papers [5], [6] to find explicit criteria involving ν, f_i and the geometry of D, which would be sufficient to guarantee uniqueness. Such a criterion has actually not been determined explicitly, but it appears that it would involve upper bounds for the absolute value of the boundary data together with their first and second derivatives. Also the boundary might well be required to have differentiable curvature at every boundary point. The criterion which I shall indicate will require somewhat less smoothness.

As we have remarked the condition $u_i^1 u_i^1 \leqq M^2$ is not an a priori condition. To apply it one must essentially know the base flow. Let us, therefore, return to our original expression for $E(w)$ and see whether we can extract more useful information. Instead of using a sup measure for u_i^1 let us use Schwarz's inequality and obtain

$$(2.12)\qquad \begin{aligned} E^2(w) &= \frac{1}{\nu^2}\left[\int_D w_{i,j}w_j u_i{}^1 dx\right]^2 \leqq \frac{1}{\nu^2} E(w)\int_D w_j w_j u_i{}^1 u_i{}^1 dx \\ &\leqq \frac{1}{\nu^2} E(w)\|w\|_4^2\|u^1\|_4^2 . \end{aligned}$$

Here we have used the notation

$$\|u\|_{2p} = \left[\int_D (u_i u_i)^p dx\right]^{1/2p}.$$

If we could now compute a lower bound for the first eigenvalue Ω_1 in the problem

$$(2.13)\qquad \Omega_1 = \inf_{\varphi_i = 0 \text{ on } \partial D;\ \varphi_{j,j} = 0 \text{ in } D} [E(\varphi)]^2/\|\varphi\|_4^4$$

for piecewise continuously differentiable functions φ_i, we would obtain

$$(2.14)\qquad E^4(w) \leqq \frac{1}{\Omega_1 \nu^4}\|u^1\|_4^4 E^4(w).$$

It would then follow that a sufficient condition for uniqueness in problem $\mathcal{S}$ would be

$$(2.15)\qquad \|u^1\|_4^4 < \nu^4 \Omega_1.$$

Even if a lower bound for Ω_1 were known, (2.15) would still not be an explicit criterion as it stands. We require, in addition, an upper bound for $\|u^1\|_4^4$ in terms of the data.

Before pursuing this question, let us first look, however, at the problem of obtaining a bound for Ω_1. Such bounds have been derived by Serrin [**22**] who showed that

$$(2.16)\qquad \begin{aligned} &\Omega_1 \geqq 2\lambda, && n = 2, \\ &\Omega_1 \geqq 3\sqrt{3\lambda}, && n = 3, \end{aligned}$$

where λ is the eigenvalue defined previously. For $n = 3$, Crooke [4] has replaced Serrin's result by $8\pi\sqrt{\lambda}/3$.

Making use of symmetrization arguments and a reduction to the Emden equation, Crook has succeeded in obtaining a bound for Ω_1 which is usually sharper than those given above, i.e., for $n = 3$,

$$(2.17)\qquad \Omega_1 \geqq 35\pi/R$$

where R is the radius of the sphere of the same volume as that of D. We have thus proved the following theorem:

THEOREM II. *If there exists one classical solution $u_i{}^1$ of $\mathcal{S}$ which* *satisfies*

$$\|u^1\|_4^4 < 35\pi\nu^4/R, \tag{2.18}$$

then it is the only classical solution satisfying the given boundary data. Here R is the radius of the sphere having the same volume as that of D.

These bounds for Ω_1 are undoubtedly very crude and may be off by a factor of 10 or even 100. In computing the bounds the divergence-free condition could not effectively be taken into account. It would be extremely helpful if bounds of a more or less isoperimetric nature could be obtained.

We mention in passing that a bound for Ω_1 could easily be obtained from γ_1 defined as follows:

$$\gamma_1 = \inf_{\varphi_i = 0 \text{ on } \partial D;\ \varphi_{j,j} = 0 \text{ in } D} \|\varphi\|_2^{4-n}[E(\varphi)]^{n/2}/\|\varphi\|_4^4$$

for n (the number of dimensions) not greater than 4. One interesting property of γ_1 is that it is dimensionless. It is also a monotone function of domain and hence γ_1 should take the same value for every domain. Crooke [4] has studied this eigenvalue problem and has given arguments which indicate that no smooth minimizing function exists.

Returning now to the criterion (2.15) we see that we have been able to find a (perhaps crude) lower bound for Ω_1. We now require an upper bound for $\|u^1\|_4^4$.

We first note that if v_i is the solution to the corresponding linear problem, i.e.,

$$\left.\begin{aligned} \nu\,\Delta v_i &= q_{,i} \\ v_{j,j} &= 0 \end{aligned}\right\} \text{ in } D, \qquad v_i = f_i \quad \text{on } \partial D, \tag{2.19}$$

then

$$\|u^1 - v\|_4^4 \leqq [E(u^1 - v)]^2/\Omega_1. \tag{2.20}$$

But

$$\begin{aligned} E(u^1 - v) &= \frac{1}{\nu}\int_D (u_i{}^1 - v_i)u_j{}^1 u_{i,j}^1 dx \\ &= -\frac{1}{\nu}\int_D (u_i{}^1 - v_i)_{,j} u_j{}^1 u_i{}^1 dx \\ &= -\frac{1}{\nu}\int_D (u_i{}^1 - v_i)_{,j} u_j{}^1 v_i dx. \end{aligned} \tag{2.21}$$

Using Schwarz's inequality we have

$$[E(u^1 - v)]^2 \leqq E(u^1 - v)\|u^1\|_4^2\|v\|_4^2/\nu^2 \tag{2.22}$$

or

$$E(u^1 - v) \leqq \|u^1\|_4^2\|v\|_4^2/\nu^2 \ . \tag{2.23}$$

The triangle inequality now yields

$$\begin{aligned} \|u^1\|_4 &\leqq \|v\|_4 + \|u^1 - v\|_4 \\ &\leqq \|v\|_4 + \|u^1\|_4\|v\|_4/\Omega_1^{1/4}\nu. \end{aligned} \tag{2.24}$$

This leads to

$$[1 - \|v\|_4/\Omega_1^{1/4}\nu] \ \|u^1\|_4 \leqq \|v\|_4. \tag{2.25}$$

Thus if

$$\|v\|_4^4 < \nu^4\Omega_1 \tag{2.26}$$

we can bound $\|u^1\|_4$ in terms of $\|v\|_4$. In fact if

$$\|v\|_4^4 < \nu^4\Omega_1/16 \tag{2.27}$$

this will imply

$$\|u^1\|_4^4 < \nu^4\Omega_1. \tag{2.28}$$

Thus (2.27) is also a sufficient condition for uniqueness. If the solution of the corresponding linear problem is known then (2.27) can easily be checked, but again (2.27) is not truly of a priori nature. It is likely that for a smooth boundary ∂D, there exists a constant k such that

$$\|v\|_4^4 \leqq k \int_{\partial D}(v_iv_i)^2 ds, \tag{2.29}$$

but an explicit constant k has to my knowledge not yet been determined. If one knew such a constant then he would have the following a priori criterion for uniqueness:

If

$$\int_{\partial D} [f_if_i]^2 ds \leqq \nu^4\Omega_1/16k \tag{2.30}$$

problem $\mathcal{S}$ has a unique solution.

On the other hand if we let H_i denote the harmonic function which takes on the boundary value f_i then

$$\begin{aligned} \|v\|_4 &\leqq \|H\|_4 + \|v - H\|_4 \\ &\leqq \|H\|_4 + [E(v - H)]^{1/2}/\Omega_1{}^{1/4}. \end{aligned} \tag{2.31}$$

Since (see Bramble and Payne [3])

$$\int_D [H_iH_i]^2 dx \leqq \sigma \int_{\partial D} [H_iH_i]^2 ds \tag{2.32}$$

where, for smooth boundaries, σ is the first eigenvalue in the problem

$$\begin{aligned} \Delta^2\chi &= 0 \quad \text{in } D, \\ \left.\begin{aligned} \chi &= 0 \\ \Delta\chi - \sigma^{-1}\partial\chi/\partial n &= 0 \end{aligned}\right\} & \quad \text{on } \partial D, \end{aligned} \tag{2.33}$$

it follows that

$$\|v\|_4 \leqq \left[\sigma \oint_{\partial D} (f_if_i)^2 ds\right]^{1/4} + [E(v-H)]^{1/2}/\Omega_1^{1/4} \tag{2.34}$$

Now

$$\begin{aligned} E(v-H) &= E(v) - E(H) \\ &= -(1/\nu) \int_D (v_i - H_i)q_{,i}\, dx \\ &= -(1/\nu) \int_D qH_{i,i}dx. \end{aligned} \tag{2.35}$$

By Schwarz's inequality and the fact that

$$\int_D H_{i,i}^2 dx = \int_D (H_{i,j} - V_{i,j})(H_{j,i} - V_{j,i})\, dx$$

we have

$$E(v-H) \leqq \frac{1}{\nu^2} \int_D q^2 dx, \tag{2.36}$$

which leads to

$$\|v\|_4 \leqq \left[\sigma \oint_{\partial D} (f_if_i)^2 ds\right]^{1/4} + \frac{1}{\nu\Omega_1^{1/4}} \left[\int_D q^2 ds\right]^{1/2}. \tag{2.37}$$

In an earlier paper, Payne [17] showed how to compute an explicit inequality for $\int_D q^2 dx$ of the following form:

$$\int_D q^2 dx \leqq k_1 \oint_{\partial D} f_if_i ds + k_2 \oint_{\partial D} \operatorname{grad}_s f_i \cdot \operatorname{grad}_s f_i ds. \tag{2.38}$$

Here grad_s denotes the tangential projection of the gradient on ∂D. We thus have

THEOREM III. *If*

$$\left\{\sigma \int_{\partial D} (f_i f_i)^2 ds\right\}^{1/4}$$

$$(2.39) \qquad + \frac{R^{1/4}}{\nu(35\pi)^{1/4}} \left\{k_1 \oint_{\partial D} f_i f_i ds + k_2 \oint_{\partial D} \text{grad}_s \cdot \text{grad}_s f_i \, ds\right\}^{1/2}$$

$$< \frac{\nu}{2}\left(\frac{35\pi}{R}\right)^{1/4},$$

where the symbols are as defined in the previous equations, then there exists at most one classical solution of $\mathcal{S}$.

Note that since σ, k_1, and k_2 are computable this criterion is truly an a priori sufficient condition for uniqueness. The criterion derived in [17] was similar to (2.39).

We consider the following example: Let D be the interior of an ellipsoid of semi-axes a, b and c. Let u_i satisfy the Navier-Stokes equations in D subject to the boundary conditions

$$f_1 = \omega x_2, \qquad f_2 = -\omega x_1, \qquad f_3 = 0 \quad \text{on } D.$$

This problem has a solution given by

$$u_1 \equiv \omega x_2, \qquad u_2 \equiv -\omega x_1, \qquad u_3 \equiv 0.$$

Then

$$u_i u_i = \omega^2[x^2 + y^2]$$

and

$$\int_D (u_i u_i)^2 dx = \frac{4\pi\omega^4 abc}{105}\{3a^4 + 2b^2a^2 + 3b^4\}.$$

The criterion for uniqueness then (using (2.17)) is

$$\frac{4}{105}\,\omega^4 abc\{3a^4 + 2b^2a^2 + 3b^4\} < \frac{35\nu^4}{(abc)^{1/3}}.$$

Setting $b = \alpha a$, $c = \beta a$ this inequality becomes

$$\frac{4}{105}\,\omega^4(\alpha\beta)^{4/3}\{3\alpha^4 + 2\alpha^2 + 3\} < 35\nu^4.$$

Thus in particular if α is less than unity we are assured of uniqueness provided

$$\left(1 + \frac{\alpha^2}{6} + \frac{\alpha^4}{4}\right)\frac{\omega a^2(\alpha\beta)^{1/3}}{\nu} < 4.12.$$

Inequality (2.6) with the Faber-Krahn inequality gives the criterion

$$\omega a^2(\alpha\beta)^{1/3}/\nu < \pi,$$

which is weaker for $\alpha < 1/2$.

III. Convergence to steady state and stability. We now look at the question of convergence of the solution of $\mathcal{S}$ to that of $\mathfrak{D}$. Here of course we take τ to be $+\infty$ and consider the problem to be defined over $\Omega(\infty)$. It is not difficult to show that if in problem $\mathfrak{D}$, f_i is independent of t then the same criterion which implies uniqueness in the steady state problem $\mathcal{S}$ implies convergence to steady state (in L_2) as $t \to \infty$.

We merely set $w_i(x, t) = u_i(x, t) - \tilde{u}_i(x)$ where $u_i(x, t)$ is the solution of problem $\mathfrak{D}$ and $\tilde{u}_i(x)$ the solution of problem $\mathcal{S}$ for the same boundary data. Then if we let

$$I(t) = \|w\|_2^2, \tag{3.1}$$

it follows that

$$\begin{aligned} dI/dt &= 2\int_{D_t} w_i \frac{\partial w_i}{\partial t} dx \\ &= -2\nu E(w) + 2\int_{D_t} w_{i,j} w_j \tilde{u}_i dx \\ &\leqq -2\nu E(w)[\nu - \|\tilde{u}\|_4/\Omega_1{}^{1/4}]. \end{aligned} \tag{3.2}$$

Thus if

$$\|\tilde{u}\|_4 \leqq \Omega_1{}^{1/4}(\nu - \epsilon) \tag{3.3}$$

for some positive ϵ then

$$dI/dt \leqq -2\nu\epsilon I \tag{3.4}$$

which implies

$$I(t) \leqq I(0)e^{-2\nu\epsilon t}. \tag{3.5}$$

Clearly $I(t) \to 0$ as $t \to \infty$ and convergence in L_2 is established.

Thus we have the same criterion as that for uniqueness, i.e.,

THEOREM IV. *If there exists a solution $u_i{}^1$ of $\mathcal{S}$ satisfying* (2.15) *then the solution of problem $\mathfrak{D}$ with the same boundary data as $\mathcal{S}$ and arbitrary L_2 initial data must converge to $u_i{}^1$ in L_2 as $t \to \infty$.*

Or as an analog to Theorem III we establish

THEOREM V. *If the boundary data of problem $\mathcal{S}$ satisfies* (2.39), *then the solution of problem $\mathfrak{D}$ with the same boundary data as $\mathcal{S}$ and arbitrary L_2 initial data must converge to steady state in L_2 as $t \to \infty$.*

Note that we have tacitly assumed the uniqueness of solution of $\mathfrak{D}$, a fact easily established (see e.g., Serrin [20]).

The question of asymptotic stability or that of growth of solutions at infinity is more difficult. Explicit a priori stability criteria (with one exception) do not appear in the literature. The only exception involves a criterion on the boundary data which essentially insures that the solution for arbitrary initial data converges to the solution of the linearized problem at infinity. This clearly implies asymptotic stability under perturbations of the initial data since if u_i^1 and u_i^2 are solutions of problem $\mathfrak{D}$ for different initial data and v_i^1 and v_i^2 the solutions of the corresponding linearized Stokes problems then

$$(3.6) \qquad \|u_i^1 - u_i^2\| \leqq \|u_i^1 - v_i^1\| + \|v_i^1 - v_i^2\| + \|v_i^2 - u_i^2\|.$$

If then all quantities on the right tend to zero as $t \to \infty$ for given coefficients and boundary data, the solution of problem $\mathfrak{D}$ will be asymptotically stable under perturbations of the initial data.

We do not here go into these computations, which are rather complicated and indirect. One finds that if a certain combination of the boundary data and its tangential derivatives over $\partial D \times (0, T)$ decays fast enough as $T \to \infty$, then the solution of problem $\mathfrak{D}$ is stable under perturbations of the initial data. The criterion itself is very complicated and requires a lot of notational explanation so we merely refer to the author's paper [18]. It should be mentioned in passing that if one is perturbing off a base flow which is known then one can compute an M such that $u_i^1 u_i^1 \leqq M^2$. Provided $M < \lambda\nu$, the base flow, therefore, is stable under perturbations of the initial data. Thus clearly perturbations from rest always decay exponentially (in L_2) as $t \to \infty$. These results date back 25 or 30 years to T. Y. Thomas [23], E. Hopf [7], and J. Kampé de Fériet [9].

IV. **Uniqueness of solution of $\mathcal{S}$ in exterior regions.** If the flow domain lies in the region D^* exterior to a closed boundary D in three-space, there are, to my knowledge, no known a priori criteria which will guarantee uniqueness of steady state flow or stability or convergence to a steady state for dynamical problems. We recall that the a priori criteria for interior problems involved certain eigenvalues. These, unfortunately, vanish for regions exterior to closed bounded surfaces. One can, however, obtain a Thomas-type result for problem $\mathcal{S}$ if one restricts attention to flows with finite energy and further imposes the restrictions that $u_i(x) \to c_i$ as $|x| \to \infty$ and

$$(4.1) \qquad \sup_{x \in D^*} \left[r^2(u_i - c_i)(u_i - c_i) \right] \leqq M_1^2,$$

where r measures the distance from some origin inside D. In this case one obtains for the steady state exterior problem, with $w_i = u_i^1 - u_i^2$,

$$
\begin{aligned}
E^2(w) &= \frac{1}{\nu^2}\left[\int_{D^*} w_{i,j}w_j(u_i^1 - c_i)dx\right]^2 \\
&\leqq \frac{M_1^2}{\nu^2}E(w)\int_{D^*}\frac{w_iw_i}{r^2}dx.
\end{aligned} \tag{4.2}
$$

But

$$
0 = \oint_{\partial D^*}\frac{x_kn_kw_iw_ids}{r^2} = \int_{D^*}\frac{w_iw_idx}{r^2} + 2\int_{D^*}\frac{x_kw_iw_{i,k}}{r^2}dx,
$$

which implies that

$$
\int_{D^*}\frac{w_iw_i}{r^2}dx \leqq 4E(w).
$$

Thus if

$$
4M_1^2 < \nu^2 \tag{4.3}
$$

the exterior steady state problem has at most one solution. Of course, (4.3) is not an a priori condition.

V. **Logarithmic convexity and stability of solution of the dynamical problem backward in time.** It is well known that the solution of the dynamical problem with homogeneous Dirichlet data depends continuously on the initial data forward in time. However, unless the class of admissible solutions is suitably restricted the solution backward in time will not depend continuously on the data. We now indicate how Knops and Payne [**10**] have made use of convexity arguments to determine such a class of admissible solutions. We first illustrate how the arguments go for linear operators.

Let M and N be two linear operators defined on the same dense subdomain of a Hilbert space H. In general M and N might depend on a parameter t (which we could generally associate with time) but for simplicity let us suppose that this is not the case. Let us assume further that M is real, symmetric and positive definite, and that N is real and symmetric. Suppose further that

$$
Mu_t = Nu, \qquad 0 \leqq t < T. \tag{5.1}
$$

With $(\cdot,\cdot)$ denoting the inner product in H, we define

$$
F(t) = (u, Mu). \tag{5.2}
$$

Then

$$(5.3)\qquad \begin{aligned} F'(t) &= 2(u, Mu_t), \\ F''(t) &= 2(u_t, Mu_t) + 2(u, Mu_{tt}) \\ &= 2(u_t, Mu_t) + 2(u, Nu_t) \\ &= 2(u_t, Mu_t) + 2(u_t, N_u) = 4(u_t, Mu_t). \end{aligned}$$

Since

$$(5.4)\qquad FF'' - (F')^2 = 4[(u, Mu)(u_t, Mu_t) - (u, Mu_t)^2] \geqq 0,$$

it can be shown that either $F(t) \equiv 0$ for $0 \leqq t \leqq \tau$ or $(\log F)'' \geqq 0$.
Using Jensen's inequality and finite Taylor series expansion it follows that

$$(5.5)\qquad F(t) \geqq [F(\tau)]^{t/\tau}[F(0)]^{1-t/\tau},$$

and

$$(5.6)\qquad F(t) \geqq F(0) \exp(F'(0)t/F(0)).$$

We note the following consequences of (5.5) and (5.6):

(a) *Uniqueness.* If $F(0) = 0$, $F(\tau) < \infty$ then $F(t) \equiv 0$, $0 \leqq t \leqq \tau$, which implies $u \equiv 0$.

(b) *Stability.* We say $\varphi \in \mathcal{M}$ if $F(\tau) = (u(\tau), Mu(\tau)) \leqq K^2$ for some prescribed K. Thus a solution $u \in \mathcal{M}$ is Lyapunov stable on $0 \leqq t \leqq t_1 < \tau$ under perturbation $\varphi \in \mathcal{M}$.

(c) *Nonexistence.* For $\tau = \infty$, if $F'(0) > 0$, $F(0) \neq 0$ every solution must grow exponentially (in the sense of $F(t)$) as $t \to \infty$.

These techniques have been used, for considerably more general operators by Agmon [1] and Agmon and Nirenberg [2] to study questions of growth of solutions and nonexistence. They have also been used by Ogawa [15], H. Levine [13], and others for similar purposes. Somewhat independently such methods have been used by Lavrentiev [14] and others (see [16] and references therein), to study various types of improperly posed problems, e.g.,

(a) Cauchy problems for elliptic equations (a priori bounds).

(b) Improperly posed transonic flow problems.

(c) Singular perturbations in improperly posed problems.

(d) Uniqueness and stability for dynamical problems of anisotropic elasticity.

We now return to the problem at hand, i.e., to problem $\mathfrak{D}$ but with $\Omega(\tau)$ replaced by $\Omega(-t_0) = D \times (-t_0, 0)$. The data is still given at $t = 0$ and along ∂D.

We shall say that a vector $w_i(x, t) \in \mathcal{M}$ if

$$\sup_{x,t \in \Omega(-t_0)} w_i w_i \leqq M^2 \tag{5.7}$$

for some prescribed constant M. A solution u_i will be said to belong to the set $\mathcal{N}$ if, for some prescribed N,

$$\sup_{x,t \in \Omega(-t_0)} \left[u_i u_i + (u_{i,j} - u_{j,i})(u_{i,j} - u_{j,i}) + \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} \right] \leqq N^2. \tag{5.8}$$

Let $u_i^1(x, t)$ and $u_i^2(x, t)$ be solutions of problem $\mathfrak{D}$ for $\Omega(-t_0)$, corresponding to final data $f_i^1(x)$ and $f_i^2(x)$ respectively, and set $v_i(x, t) = u_i^1(x, t) - u_i^2(x, t)$. For classical solutions of problem $\mathfrak{D}$ we have:

THEOREM 1. *If $u_i^1 \in \mathcal{M}$ and $u_i^2 \in \mathcal{N}$ then $\exists$ a constant K and a $\delta(t)$ $(0 < \delta < 1)$, independent of u_i^1 and u_i^2, such that, for every t and t_1 satisfying $-t_0 < -t_1 \leqq t \leqq 0$,*

$$\int_{D_t} v_i(x, t) v_i(x, t) \leqq K\epsilon^{\delta} \tag{5.9}$$

where

$$\epsilon = \int_{D_0} (f_i^1 - f_i^2)(f_i^1 - f_i^2) dx. \tag{5.10}$$

To prove this theorem we set

$$F(t) = \int_{D_t} v_i v_i dx$$

and form

$$\begin{aligned} F'(t) &= 2 \int_{D_t} v_i \frac{\partial v_i}{\partial t} dx, \\ F''(t) &= 2 \int_{D_t} [\dot{v}_i \dot{v}_i + v_i \ddot{v}_i]\, dx \\ &= 4 \int_{D_t} \dot{v}_i \dot{v}_i dx + 4 \int_{D_t} v_{i,t} v_{i,j} u_j^1 dx \\ &\quad + 2 \int_{D_t} v_{i,t} v_j (u^2_{i,j} - u^2_{j,i}) dx + 2 \int_{D_t} v_{i,j} v_j \frac{\partial u_i^2}{\partial t} dx. \end{aligned} \tag{5.11}$$

We observe that

$$FF'' - (F')^2 \geqq -k_1 FF' - k_2 F^2 \tag{5.12}$$

where k_1, k_2 depend on M, N, and ν. Serrin [22] has shown that if $F(t_2) = 0$ for some t_2 in $-t_0 < t_2 \leqq 0$ then $F(t)$ must vanish identically in $\Omega(-t_0)$. Thus without loss we may assume $F(t) \neq 0$ for any t in this interval, divide by F and write

$$(5.13)\qquad (e^{-k_1 t} F'/F)' \geqq -k_2 e^{-k_1 t}.$$

We now set

$$(5.14)\qquad \sigma = e^{k_1 t}$$

and observe that

$$(5.15)\qquad \begin{gathered} \frac{d^2}{d\sigma^2}(\ln F) \geqq -\frac{k_2}{k_1{}^2\sigma^2} = \frac{d^2}{d\sigma^2}[\ln \sigma^{k_2/k_1{}^2}], \quad \text{or} \\ \frac{d^2}{d\sigma^2}[\ln (F\sigma^{-k_2/k_1{}^2})] \geqq 0. \end{gathered}$$

Jensen's inequality then implies

$$(5.16)\qquad F(t) \leqq \exp\,[k_2[t + t_0(1-\delta)]]\;[F(-t_0)]^{1-\delta}\epsilon^{\delta}$$

where

$$(5.17)\qquad \delta = [e^{k_1 t} - e^{-k_1 t_0}]/[1 - e^{-k_1 t_0}].$$

But

$$(5.18)\qquad F(-t_0) \leqq 2[M^2 + N^2]\,V$$

where V is the volume of D. We thus have

$$(5.19)\qquad \|V\|_2^2 \leqq \exp\,[k_2/k_1(t + t_0[1-\delta]))]\;\{2(M^2+N^2)V\}^{1-\delta}\epsilon^{\delta}$$

which was to be proved.

One can weaken the hypotheses somewhat (see [**10**]), but we shall not pursue the question here.

VI. **Extensions.** Some of the stability studies discussed here have been extended to more general systems. Ladyzhenskaya [**11**] and others have studied certain Navier-Stokes-like systems, D. Joseph [**8**] has carried out stability studies on the Boussinesq convective flow equations and Crooke [**4**] has investigated the Saffman model of the "dusty" gas equations. These latter equations are a coupled system of the form

$$(6.1)\qquad \left.\begin{aligned} &\rho\left(\frac{\partial u_i}{\partial t} + u_j u_{i,j}\right) = u\,\Delta u_i + p_{,i} - kN(u_i - v^i)u_{j,j} = 0 \\ &mN\left\{\frac{\partial v_i}{\partial t} + v_j v_{i,j}\right\} = kN(u_i - v_i) \\ &\frac{\partial N}{\partial t} + \frac{\partial}{\partial x_j}(Nv_j) = 0;\; N \geqq 0 \end{aligned}\right\} \text{ in } D.$$

Here u_i denotes the ith component of the fluid velocity, v_i the ith component of the dust particle velocity, N ($\geqq 0$) the number density of particles and m the mass of the dust particles. Even for homogeneous boundary conditions and prescribed initial conditions for u_i, v_i, and N, very little of an explicit nature can be said. First of all it is not clear what the right boundary conditions ought to be. It would appear to be physically reasonable to prescribe the quantities u_i and Nv_jn_j on ∂D. However, the last condition seems questionable. Crooke [4] has shown that for smooth solutions this latter condition has some unexpected implications (pointed out below). Of course, it may well happen that, in general, smooth solutions to this system do not exist.

Crooke [4] has shown that if ∂D is convex and the flow is smooth then the condition $Nv_jn_j = 0$ on ∂D actually implies that $Nv_j = 0$ on ∂D. This fact permits one to determine $(Nv_j)_{,j}$ and N itself on ∂D. Crooke also showed that if at some point $x_0 \in \partial D$ and time t_1 the condition $N(x_0, t_1) = 0$ holds, then this implies $N(x_0, t) = 0$ for all t in the interval $0 \leqq t \leqq \tau$. He showed in addition, for smooth flows, that at every point $x_p \in \partial D$ at which $N(x_p, t) > 0$ the following relation holds:

$$\frac{1}{N(x_p, t)} = \frac{1}{N(x_p, 0)} - \frac{m}{k}\,\frac{\partial N(x_p, 0)/\partial t}{N^2(x_p, 0)} \tag{6.2}$$

which automatically imposes the restriction that

$$N(x_p, 0) > \frac{m}{k}\,\frac{\partial N(x_p, 0)}{\partial t}. \tag{6.3}$$

These results were established only for domains with boundaries of positive Gaussian curvature.

If we denote by $J(t)$ the energy

$$J(t) = \tfrac{1}{2}\int_{D_t} \{\rho u_i u_i + mNv_iv_i\}dx, \tag{6.4}$$

then it is easily shown that

$$\begin{aligned} J(t) &+ \mu \int_0^t \int_{D_2} u_{i,j}u_{i,j}dx d\eta \\ &+ k\int_0^t \int_{D_\eta} N(u_i - v_i)(u_i - v_i)dx d\eta = J(0). \end{aligned} \tag{6.5}$$

This fact implies directly that if $\tau = \infty$ then provided the indicated limits exist

(a) $\int_{D_t} u_{i,j}u_{i,j}dx \to 0$ as $t \to \infty$,
(b) $\int_{D_t} N(u_i - v_i)(u_i - v_i)dx \to 0$ as $t \to \infty$.
But (a) implies that $\int_{D_t} u_iu_idx \to 0$ as $t \to \infty$. It follows then that if u_i is sufficiently smooth,

$$\int_{D_t} Nu_iu_idx = u_i(\xi, t)u_i(\xi, t) \int_{D_t} N(x, t)dx$$
$$= u_i(\xi, t)u_i(\xi, t) \int_{D_0} N(x, 0)dx$$

and
(c) $\int_{D_t} Nu_iu_idx \to 0$ as $t \to \infty$.
Then (b) and (c) together imply that
(d) $\int_{D_t} Nv_iv_idx \to 0$ as $t \to \infty$.

One could, of course, compute more precisely a rate of decay, but what we really expect is exponential decay of energy. In order to establish this type of decay one would seem to need information about $N(x, t)$ which has not as yet been established. One would need to know that as the flow evolves in time the dust particles do not pile up, i.e., that some inequality of the type

(6.6) $$\sup_{x,t \in \Omega(\infty)} N(x, t) \leqq M < \infty$$

or

(6.7) $$\sup_{0 \leqq t \leqq \infty} \int_{D_t} N^2(x, t)dx \leqq M_1^2 < \infty$$

holds. Crooke [4] showed how either of these conditions would lead to exponential decay of the energy. He further showed that under certain additional assumptions the energy cannot decay to zero in finite time.

Concluding remarks. In this paper we have restricted our attention to the problem of deriving explicit criteria sufficient to guarantee uniqueness, stability and convergence to a steady state. Although the criteria which we have indicated are explicit, they are unfortunately not very sharp. Hopefully substantial improvements will be forthcoming.

We have not dealt here with the questions of existence and regularity of solutions of the Navier-Stokes equations. There is by now a vast literature on these topics, much of which is referred to in [5], [6], [11], and [22]. We have ignored also questions of unique continuation, lower bounds on growth of solutions, and a number of other topics. Several papers which deal with these questions have appeared in the literature during the past eight or ten years.

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