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CONVOLUTION IN $K\{M_p\}$ SPACES

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In this note we establish a characterization of convolution operators on certain $K\{M_p\}$ spaces. In particular our results contain the characterization of the space $\mathcal{O}_c{}'$ of L. Schwartz [**5**, Chapter VII, §5, Theorem IX] and the characterization of convolutes on the space of distributions of exponential order as given in [**3**], [**6**] and [**7**]. We also obtain a characterization of convolutes on the $W_{M,a}$ spaces introduced by Gelfand and Shilov [**2**].

Throughout this note we use the terminology and notation of [**1**]. We recall the definition of $K\{M_p\}$ spaces. Let $\{M_p\}$ be a sequence of real-valued continuous functions defined on R^n which satisfy $1 \leqq M_1(x) \leqq M_2(x) \leqq \cdots$ for all $x \in R^n$. (In [**1**, Chapter II, 1.2], a slightly more general definition is given.) The space $K\{M_p\}$ consists of all infinitely differentiable functions ϕ such that $M_p D^\alpha \phi$ is bounded for every positive integer p and $|\alpha| \leqq p$. The vector space $K\{M_p\}$ is then given a locally convex Hausdorff topology by means of the norms

$$\|\phi\|_p = \sup \{M_p(x)|D^\alpha\phi(x)| : x \in R^n, |\alpha| \leqq p\} \qquad (p = 1, 2, \cdots).$$

We will only consider $K\{M_p\}$ spaces which satisfy the conditions (M) and (N) as introduced in [**1**, Chapter II, 4.2]. The sequence $\{M_p\}$ satisfies conditions (M) or (N) if:

(M) the functions M_p are quasi-monotonic in each coordinate, i.e., if $x_j' x_j'' \geqq 0$ and $|x_j'| \leqq |x_j''|$ then $M_p(x_1, \cdots, x_j', \cdots, x_n) \leqq M_p(x_1, \cdots, x_j'', \cdots, x_n)$ for each fixed point $(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$,

(N) for each p there is $p' > p$ such that the ratio $M_p(x)/M_{p'}(x) = m_{pp'}(x)$ tends to zero as $|x| \to \infty$ and is a summable function on R^n.

In [**1**, Chapter II, 4.2], it is shown that if $\{M_p\}$ satisfies conditions (M) and (N), then the norms $\|\phi\|_p' = \sup \{\int M_p(x)|D^\alpha\phi(x)|dx : |\alpha| \leqq p\}$ $(p = 1, 2, \cdots)$ generate the same locally convex topology as the sequence of norms $\{\| \ \|_p : p \geqq 1\}$. (Throughout we write $\int f$ to denote the integral of f over all R^n.)

The $K\{M_p\}$ spaces which we will consider will satisfy an additional condition. The $\{M_p\}$ are said to satisfy the factorization condition (F) if:

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(F) each M_p is symmetric, i.e., $M_p(x) = M_p(-x)$ and for each p there is a $p' > p$ and $C_{p'} > 0$ such that $M_p(x+h) \leqq C_{p'} M_{p'}(x) M_{p'}(h)$, for all $x, h \in R^n$.

EXAMPLES. 1. If $M_p(x) = (1+|x|^2)^p$, i.e., if $K\{M_p\} = \mathcal{S}$, then $\{M_p\}$ satisfies condition (F) since the inequality $(1+|x+h|^2) \leqq 2(1+|x|^2)(1+|h|^2)$ holds [4, Chapter 4, §11, Lemma 1].

2. If $M_p(x) = \exp(p\gamma(x))$, where $\gamma(x) = (1+|x|^2)^{1/2}$, the condition (F) is satisfied since $\gamma(x+h) \leqq \gamma(x) + \gamma(h)$ [**6**]. In this case $K\{M_p\}'$ is the space H of distributions of exponential order of [3].

3. The $W_{M,a}$ spaces of Gelfand and Shilov are also $K\{M_p\}$ spaces satisfying condition (F). Here $M_p(x) = \exp(M(a(1-1/p)x))$, where $M(X) = \int_0^x \mu$ with μ an increasing function such that $\mu(0) = 0$, $\mu(\infty) = \infty$. For each M_p the inequality $M_p(x+h) \leqq M_p(x) M_p(h)$ holds since M is a convex function (see [2, Chapter I, 1.1]).

We now consider translation on $K\{M_p\}$ spaces. If $\phi \in K\{M_p\}$ and $h \in R^n$, the translate of ϕ by h is denoted by $\tau_h\phi$ or $\tau_h\phi(x) = \phi(x+h)$. If $\{M_p\}$ satisfies conditions (M), (N) and (F), we have the following result.

LEMMA 1. *Let $\{M_p\}$ satisfy* (M), (N), *and* (F). *Then*

(i) *for each $h \in R^n$ the function $\phi \to \tau_h\phi$ is continuous from $K\{M_p\}$ to $K\{M_p\}$,*

(ii) *if B is a bounded subset of $K\{M_p\}$ and $\epsilon > 0$, the set $\{\tau_h\phi : |h| \leqq \epsilon, \phi \in B\}$ is also bounded in $K\{M_p\}$.*

PROOF. For $\phi \in K\{M_p\}$ and $h \in R^n$,

$$\begin{aligned} \int M_p(x)|D^\alpha \tau_h \phi(x)|dx &= \int M_p(t-h)|D^\alpha \phi(t)|dt \\ &\leqq C_{p'} M_{p'}(h) \int M_{p'}(t)|D^\alpha\phi(t)|dt. \end{aligned} \tag{1}$$

From (1), we obtain

$$\|\tau_h\phi\|_p' \leqq C_{p'} M_{p'}(h) \|\phi\|_{p'}' \tag{2}$$

so that (i) follows. Also since M_p is continuous on R^n, (ii) follows from (2).

REMARK. It follows from this lemma that translation is a continuous operation on $W_{M,a}$ spaces since conditions (M), (N) and (F) are satisfied for these spaces. In particular the result of [**1**, Chapter IV, 4.3, p. 191] follows from the lemma.

The results of Lemma 1 are the conditions set forth in [**1**, Chapter III, 3.1] for $K\{M_p\}$ to have a continuous translation and since differentiation is also continuous on $K\{M_p\}$, the condition of the lemma in [**1**, Chapter III, 3.3] are satisfied and we obtain

COROLLARY 2. *If $\{M_p\}$ satisfies* (M), (N) *and* (F), *then translation on $K\{M_p\}$ is differentiable* (*in the sense of* [**1**, *Chapter* III, 3.3]).

PROOF. Just note that $K\{M_p\}$ is "perfect" since $\{M_p\}$ satisfies condition (N) and therefore condition (P) [**1**, Chapter II, 2.3].

We use the definition of "convolute" given in [**1**, Chapter III, 3.2]. A generalized function $T \in K\{M_p\}'$ is said to be a convolute if for each $\phi \in K\{M_p\}$ the function $T * \phi : h \to \langle T, \tau_h \phi \rangle$ is in $K\{M_p\}$ and the map $\phi \to T * \phi$ is continuous from $K\{M_p\}$ into $K\{M_p\}$. If T is a convolute and $S \in K\{M_p\}'$, the convolution of T and S is given by $\langle S * T, \phi \rangle = \langle S, T * \phi \rangle$. From the definition of convolute, $S * T$ is in $K\{M_p\}'$ for each S in $K\{M_p\}'$.

We now give a characterization of the convolutes on $K\{M_p\}$ spaces which satisfy conditions (M), (N) and (F). In particular our result applies to the spaces in Examples 1–3. We thus obtain the characterization of $\mathcal{O}_c'$ given by L. Schwartz [**5**, Chapter VII, §5, Theorem IX] and the characterization of $\mathcal{P}_M'$ given by Yoshinaga [**6**, Proposition 11], and the characterization of $\mathcal{O}_c'(\mathcal{K}_1')$ in [**7**]. The result also gives a characterization of convolutes on $W_{M,a}$ spaces.

THEOREM 3. *Let $\{M_p\}$ satisfy conditions* (M), (N) *and* (F). *The following are equivalent for $T \in K\{M_p\}'$*:

(a) *T is a convolute,*

(b) *for each $\phi \in \mathcal{D}$, $T * \phi \in K\{M_p\}$,*

(c) *for each positive integer p, $\{M_p(h)\tau_{-h}T : h \in R^n\}$ is strongly bounded in $\mathcal{D}'$,*

(d) *for each positive integer k, $T = \sum_{|\alpha| \leqq n_k} D^\alpha f_\alpha$, where each $M_k f_\alpha$ is a continuous, bounded function on R^n.*

PROOF. (a) implies (b): this follows by the definition of a convolute since $\mathcal{D} \subseteq K\{M_p\}$.

(b) implies (c): First note that $\mathcal{D} \subseteq K\{M_p\}$, with $\mathcal{D}$ dense in $K\{M_p\}$ [**1**, Chapter II, 2.5] and the injection of $\mathcal{D}$ into $K\{M_p\}$ continuous with respect to the usual topology of $\mathcal{D}$. Thus $T \in K\{M_p\}'$ can be identified with a distribution and the statement in part (c) is meaningful. Now if $T * \phi \in K\{M_p\}$ for $\phi \in \mathcal{D}$,

$$\sup \{M_p(h)|T * \phi(h)| : h \in R^n\} = \sup \{|\langle M_p(h)\tau_{-h}T, \phi\rangle| : h \in R^n\} < \infty$$

so that $\{M_p(h)\tau_{-h}T : h \in R^n\}$ is weakly bounded in $\mathcal{D}'$ and is therefore strongly bounded in $\mathcal{D}'$ since $\mathcal{D}$ is a barrelled space.

(c) implies (d): Since $\{M_p(h)\tau_{-h}T : h \in R^n\}$ is bounded in $\mathcal{D}'$, there is a compact neighborhood K of $\mathcal{O}$ in R^n and a positive integer m such that if $\psi \in \mathcal{D}_K{}^m$, the family of continuous maps

$\{(M_p(h)\tau_{-h}T) * \psi : h \in R^n\}$ is bounded on K [5, Chapter VI, §7, Theorem XXII]. The elementary solution E of Δ^N is m-times continuously differentiable for large N so if we take $\gamma \in \mathfrak{D}_K$ such that γ is equal to 1 on a neighborhood of the origin, then $\gamma E \in \mathfrak{D}_K{}^m$ and $\delta = \Delta^N(\gamma E) - \phi$ where $\phi \in \mathfrak{D}$. Then

$$T = T * \delta = \Delta^N(T * \gamma E) - T * \phi. \tag{3}$$

Since $T \in \mathfrak{D}'$ and $\phi \in \mathfrak{D}, T * \phi \in \mathcal{E}$ and the hypothesis in (c) gives $\sup \{M_p(h)|T * \phi(h)| : h \in R^n\} < \infty$ so that the function $M_p(T * \phi)$ is bounded. Since $\gamma E \in \mathfrak{D}_K{}^m$ the family of continuous functions $\{M_p(h)\tau_{-h}T * \gamma E : h \in R^n\}$ is bounded on K so in particular the function $T * \gamma E$ is continuous and $\sup \{M_p(h)|T * \gamma E(h)| : h \in R^n\} < \infty$ since $0 \in K$. Therefore, $M_p(T * \gamma E)$ is a bounded continuous function and formula (3) gives the representation in part (d).

(d) implies (a): By Corollary 2 $K\{M_p\}$ has a differentiable translation so for each $\phi \in K\{M_p\}$ the function $\psi : h \rightarrow \langle T, \tau_h \phi \rangle$ is in $C^\infty(R^n)$ [1, Chapter III, 3.3] and $D^\alpha \psi = \langle T, \tau_h D^\alpha \phi \rangle$. Let p be a positive integer and $|\alpha| \leqq p$. Choose $q = p'$ as in condition (F) and then choose $r = q'$ as in condition (N). We then have

$$\begin{aligned} \int M_p(h)|D^\alpha \psi(h)|dh &= \int M_p(h)|\langle T, \tau_h D^\alpha \phi \rangle|dh \\ &\leqq \int M_p(h) \sum_{|\beta| \leqq n_r} \int |f_\beta(x)| D^{\alpha+\beta}\phi(x+h)|dxdh \\ &= \sum_{|\beta| \leqq n_r} \int |f_\beta(x)| \int M_p(u-x)|D^{\alpha+\beta}\phi(u)|dudx \\ &\leqq C_q \sum_{|\beta| \leqq n_r} \int |f_\beta(x)| M_q(x)dx \int M_q(u)|D^{\alpha+\beta}\phi(u)|\,du \end{aligned} \tag{4}$$

where we apply (d) to the integer r. Note that $\int |f_\beta| M_q < \infty$ since $|f_\beta| M_q \leqq |f_\beta| M_r m_{qr}$, $|f_\beta| M_r$ is bounded, and m_{qr} is summable on R^n. Also $\int M_q |D^{\alpha+\beta}\phi| \leqq \|\phi\|'_{q+n_r}$ so from (4) there is a constant L not depending on ϕ such that $\|\psi\|'_p \leqq L\|\phi\|'_{q+n_r}$ and therefore $\psi \in K\{M_p\}$ and the map $\phi \rightarrow \psi = T * \phi$ is continuous from $K\{M_p\}$ into $K\{M_p\}$, i.e., T is a convolute on $K\{M_p\}$.

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