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QUANTUM-MECHANICAL SCATTERING THEORY FOR SHORT-RANGE AND COULOMB INTERACTIONS

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ABSTRACT. A rigorous account is given of time-dependent nonrelativistic quantum-mechanical scattering problems involving one or many particles. The discussion is carried out in the framework of Hilbert space. Results are given both for short-range and for Coulomb interactions.

I. The description of particles in quantum mechanics. A nonrelativistic spinless quantum-mechanical particle of mass m is described by assigning to each real number t, $-\infty < t < \infty$, an element ψ_t of $L^2(R^3)$ such that $\|\psi_t\| = 1$. ψ_t is called the *wave-function* or sometimes the *state* of the particle *at time t*. Introduce the Fourier transform $\tilde{\psi_t}$ of ψ_t by

$$(1) \qquad \tilde{\psi_t}(\vec{k}) = \text{l.i.m.}\frac{1}{(2\pi)^{3/2}} \int_{R^3} e^{-i\vec{k}\cdot\vec{x}}\psi_t(\vec{x})d\vec{x} \ ,$$

so that

$$(2) \qquad \psi_t(\vec{x}) = \text{l.i.m.}\frac{1}{(2\pi)^{3/2}} \int_{R^3} e^{i\vec{k}\cdot\vec{x}}\tilde{\psi_t}(\vec{k})d\vec{k} \ .$$

Then the following partial interpretation of the wave-function can be given:

$|\psi_t(\vec{x})|^2$ is the position probability density (ppd) for the particle at time t,

$|\tilde{\psi}_t(\vec{k})|^2$ is the momentum probability density (mpd) for the particle at time t.

That is, if S is any (measurable) subset of R^3, then the probability that the particle is in S at time t is

$$(3) \qquad P_p(S,t) = \int_S |\psi_t(\vec{x})|^2 d\vec{x} \ ,$$

and the corresponding statement holds for momentum. Note that

$$(4) \qquad P_p(R^3,t) = \int_{R^3} |\psi_t(\vec{x})|^2 d\vec{x} = \|\psi_t\|^2 = 1 \ ,$$

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which expresses the fact that the particle is in R^3 with probability 1. A similar statement holds for the momentum since $\|\tilde{\psi_t}\| = \|\psi_t\| = 1$.

Of course the above does not provide much information about the particle. It is analogous to the statement in classical mechanics that a particle is described by assigning to each time t a position $\vec{x}(t)$ and a momentum $\vec{k}(t)$. The interesting information in the theory is that concerned with the prediction of the future motion of the particle if its present condition is known. To do this an equation of motion is needed. In quantum mechanics the equation of motion is the Schrödinger equation.

To say that ψ_t satisfies a Schrödinger equation means: there exists a selfadjoint linear transformation H (the Hamiltonian) of $L^2(R^3)$ into itself such that

$$\psi_t = e^{-iHt}\,\psi_0 \quad \text{for all } t \in R \ . \tag{5}$$

Note. (1) Any normalized $\psi_0 \in L^2$ is acceptable as the state of the particle at time $t = 0$. Once this is given, the state ψ_t is determined for all t.

(2) $\|\psi_0\| = 1 \Rightarrow \|\psi_t\| = 1$, since the operator e^{-iHt} is unitary.

(3) Characteristically, H is unbounded, and as a technical point it should be noted that, by Stone's Theorem, ψ_t is strongly differentiable if and only if ψ_0 belongs to the domain $D(H)$ of H. In this case

$$\frac{id\psi_t}{dt} = H\psi_t \ , \tag{6}$$

and it is this latter equation which is usually called the Schrödinger equation.

Detailed computation of the motion of the particle using Schrödinger's equation (5) is possible only when the operator H is known explicitly. Equation (5) is analogous to Newton's second law $\vec{F} = m\vec{a}$ governing the motion of a classical particle. The motion of the particle cannot be computed until the force $\vec{F}$ is known. The law is useful because we know what to put for $\vec{F}$ in interesting cases. For instance, if the classical particle is far from all other objects in the universe then adequate results are obtained by taking $\vec{F} = 0$, in which case the particle travels in a straight line with constant velocity:

Free classical particle:

$$\vec{F} = 0, \qquad \vec{x}(t) = \vec{x}(0) + \vec{v}(0)t \ . \tag{7}$$

Various classical theories are distinguished by what is taken for $\vec{F}$. In the same way various quantum-mechanical theories can be distinguished by what is taken for H, and the Schrödinger equation is useful because one knows what to put for H in interesting cases. As an example consider the case of a quantum-mechanical particle which is alone in the universe.

Quantum mechanical free particle: H is now denoted by H_0, and given by

$$H_0 \text{“}=\text{”} -\frac{\Delta}{2m}, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \ . \tag{8}$$

The equals sign has been put in quotation marks because H_0 is supposed to be a selfadjoint linear operator. Thus H_0 is not the differential operator $-\Delta/2m$ defined for suitably differentiable functions, but a selfadjoint extension of it. In order to make the definition complete, it is natural to pass to the Fourier transform, which converts differential operators into multiplicative ones. The operator H_0 is then defined as follows:

ψ belongs to the domain $D(H_0)$ of H_0 if and only if

$$\int \left| \frac{k^2}{2m} \psi^{\sim}(\vec{k}) \right|^2 d\vec{k} < \infty \ . \tag{9}$$

(Note: when the range of integration is not indicated explicitly, it is all of R^3.) If this condition is satisfied, then $H_0\psi$ is specified by giving its Fourier transform:

$$(H_0\psi)^{\sim}(\vec{k}) = \frac{k^2}{2m} \psi^{\sim}(\vec{k}) \ . \tag{10}$$

Because of the condition defining $D(H_0)$, the right-hand side of this last equation belongs to L^2, and $H_0\psi$ is well defined. With this definition, H_0 is indeed a selfadjoint linear operator. Of course, for "sufficiently smooth" functions ψ, H_0 reduces to the differential operator we began with:

$$H_0\psi = -\frac{\Delta\psi}{2m} \quad \text{for } \psi \text{ sufficiently smooth.} \tag{11}$$

In particular, it is convenient to introduce Schwartz's space S of C^∞ functions of fast decrease: a complex valued function ψ on R^3 belongs to S if ψ is a C^∞ function of its Cartesian coordinates and for any non-negative integers m_1, m_2, m_3, n_1, n_2, n_3 there exists a constant C_{m_i,n_i} such that

$$\left| x_1^{m_1} x_2^{m_2} x_3^{m_3} \frac{\partial^{n_1+n_2+n_3} \psi(x_1, x_2, x_3)}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} \right| \leqq C_{m_i,n_i} \ . \tag{12}$$

Briefly, the space S consists of very smooth functions which, along with all their derivatives, fall off rapidly in absolute value as the $|x_i|$'s increase. The space S has the property that it is mapped onto itself in a one-to-one fashion by the Fourier transform. (The equations defining the Fourier transform hold with the "l.i.m."'s erased when $\psi_t \in S$.) From the definition of the domain $D(H_0)$ it is then easy to see that $S \subseteq D(H_0)$, and further that

$$H_0\psi = -\frac{\Delta}{2m}\psi \quad \text{for } \psi \in S. \tag{13}$$

S is a linear space over the complex numbers, and it has the important property of being dense in L^2, i.e., if $f \in L^2$ and $\epsilon > 0$ are given, there is a $g \in S$ such that $\|f-g\| < \epsilon$. This fact has the pleasant consequence that in many problems in scattering theory it suffices to restrict one's attention to S.

Useful information about solutions of the free Schrödinger equation can be obtained by examining the case that $\psi_0 \in S$. Then

$$\begin{aligned} \psi_t(\vec{x}) &\equiv (e^{-iH_0t}\,\psi_0)(\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int e^{i\vec{k}\cdot\vec{x}}\, e^{-ik^2t/2m}\,\tilde{\psi_0}(\vec{k})d\vec{k} \\ &\underset{(t\neq 0)}{=} \left(\frac{m}{2\pi it}\right)^{3/2}\int e^{im(\vec{x}-\vec{x}\,')^2/2t}\,\psi_0(\vec{x}\,')d\vec{x}\,' \,. \end{aligned} \tag{14}$$

The first equality is essentially the definition of the left-hand side, and asserts that the Fourier transform of $e^{-iH_0t}\,\psi_0$ is obtained from that of ψ_0 by multiplication with $e^{-ik^2t/2m}$:

$$(e^{(-iH_0t)}\tilde{}\;\psi_0)(\vec{k}) = e^{-ik^2t/2m}\,\tilde{\psi_0}(\vec{k}) \,. \tag{15}$$

This of course corresponds to the definition

$$(H_0\psi_0)\tilde{}\,(\vec{k}) = \frac{k^2}{2m}\,\tilde{\psi_0}(\vec{k}) \,. \tag{16}$$

The second is obtained from the first by using the fact that the Fourier transform of a convolution is proportional to the product of the Fourier transforms. On the right-hand side is a convolution whose Fourier transform is the product in the integral on the left. A small amount of additional discussion is necessary because some of the functions involved are not in L^2, but this is easy.

From the formula for the Fourier transform of $e^{-iH_0t}\,\psi_0$ the mpd for a free particle at time t can be computed. It is found to be the same as the mpd at time $t = 0$:

$$|(e^{-iH_0t}\,\psi_0)\tilde{}\,(\vec{k})|^2 = |e^{-ik^2t/2m}\,\tilde{\psi_0}(\vec{k})|^2 = |\tilde{\psi_0}(\vec{k})|^2. \tag{17}$$

Thus the mpd of the particle is constant in time, in analogy with the fact that the momentum of a classical free particle is constant in time. ((15) actually holds for any $\psi \in L^2$, allowing the same conclusion. (14), however, does not—the integrals may not converge for an arbitrary $\psi_0 \in L^2$.) Suppose, however, that one decides to look at the position probability density for the particle. Is this ppd explainable in an intuitive manner by analogy with the behavior of a free classical particle? The answer is yes, at least for a very large times ($|t| \to \infty$), and since in scattering theory these are the principal interest, the necessary analysis is given.

LEMMA 1: *Suppose $t \neq 0$. Define two linear operators C_t and Q_t on L^2 by*

$$(Q_t\psi)(\vec{x}) = e^{imx^2/2t}\,\psi(\vec{x}), \tag{18}$$

$$(C_t\psi)(\vec{x}) = \left(\frac{m}{it}\right)^{3/2} e^{imx^2/2t}\,\psi^{\sim}\left(\frac{m\vec{x}}{t}\right). \tag{19}$$

Then C_t and Q_t are unitary operators, and

$$e^{-iH_0t}\,\psi = C_tQ_t\psi \tag{20}$$

for all $\psi \in L^2$.

PROOF. Q_t is clearly unitary and C_t is easily seen to be unitary using the unitarity of the Fourier transform. As for (20), one need only establish that it is correct for a set of ψ's which is dense in L^2, and it then immediately extends to any $\psi \in L^2$ by the continuity of the operators involved. For $\psi \in S$, however, $e^{-iH_0t}\,\psi$ is given by the last integral in (14) (with ψ_0 replaced by ψ). One has then only to expand the exponential:

$$e^{im(\vec{x}-\vec{x}')^2/2t} = e^{imx^2/2t}\,e^{-im\vec{x}\cdot\vec{x}'/t}\,e^{imx'^2/2t} \tag{21}$$

and recall the definition of C_t, Q_t, and (1) for the Fourier transform to see that (20) is correct. This completes the proof.

LEMMA 2. *Let $\psi_0 \in L^2$. Then*

$$\lim_{t\to\pm\infty} \|e^{-iH_0t}\,\psi_0 - C_t\psi_0\| = 0 .$$

PROOF. Using Lemma 1 and the unitarity of C_t gives

$$\begin{aligned}\|e^{-iH_0t}\,\psi_0 - C_t\psi_0\|^2 &= \|C_tQ_t\psi_0 - C_t\psi_0\|^2 \\ &= \|Q_t\psi_0 - \psi_0\|^2 = \int |e^{imx^2/2t} - 1|^2\,|\psi_0(\vec{x})|^2 d\vec{x} .\end{aligned} \tag{22}$$

The integrand on the right-hand side of (22) is bounded by the integrable function $4|\psi_0(\vec{x})|^2$ and converges to zero as $t \to \pm\infty$. Therefore the integral converges to zero as $t \to \pm \infty$, by Lebesgue's Dominated Convergence Theorem. This completes the proof of Lemma 2.

An easy estimate now shows that

$$\begin{aligned}\int_{R^3} \left| |(e^{-iH_0t}\,\psi_0)(\vec{x})|^2 - |(C_t\psi_0)(\vec{x})|^2 \right| d\vec{x} \\ \leqq 2\|e^{-iH_0t}\,\psi_0 - C_t\psi_0\| \xrightarrow[t\to\pm\infty]{} 0 \,, \end{aligned} \tag{23}$$

if $\|\psi_0\| = 1$. This implies, among other things, that when integrating over any measurable subset S of R^3, the ppd $|(e^{-iH_0t}\,\psi_0)(\vec{x})|^2$ can be replaced asymptotically by $|(C_t\psi_0)(\vec{x})|^2$:

$$\lim_{t\to\pm\infty} \int_S |(e^{-iH_0t}\,\psi_0)(\vec{x})|^2 d\vec{x} = \lim_{t\to\pm\infty} \int_S |(C_t\psi_0)(\vec{x})|^2 d\vec{x}\,, \tag{24}$$

in the sense that if the limit on one side of (24) exists, so does the other, and they are equal. Now

$$|(C_t\psi_0)(\vec{x})|^2 = \left|\frac{m}{t}\right|^3 \left|\psi_0^{\sim}\left(\frac{m\vec{x}}{t}\right)\right|^2, \tag{25}$$

so that asymptotically the ppd of the particle can be replaced by $|m/t|^3|\psi_0^{\sim}(m\vec{x}/t)|^2$. For large positive times, this means that up to the normalization factor $|m/t|^3$ the chance that the particle is at $\vec{x}$ at time t is the same as the chance that it had momentum $m\vec{x}/t$, i.e., the correct momentum to get from the origin to $\vec{x}$ in time t, which is the same result as if the particle had started from the origin and travelled in a straight line like a classical free particle, but one whose momentum is uncertain. (A similar interpretation holds for large negative times.) In fact, $|m/t|^3\ |\psi_0^{\sim}(m\vec{x}/t)|^2$ is just the ppd at time t of the classical free particle which starts from the origin at time $t = 0$ with mpd given by $|\psi_0^{\sim}(\vec{k})|^2$. This classical particle will be referred to as the classical particle *corresponding* to the quantum-mechanical particle with wavefunction $e^{-iH_0t}\,\psi_0$, since they have "the same" asymptotic ppd.

As an example of the use of (24), consider the asymptotic probability $P^{\pm}_{\text{free}}(\psi_0, C)$ that the particle will be found as $t \to \pm\infty$ in a cone C with apex at the origin. (24) and (25) imply

$$\begin{aligned} P^{\pm}_{\text{free}}(\psi_0, C) &= \lim_{t\to\pm\infty} \int_C |(e^{-iH_0t}\,\psi_0)(\vec{x})|^2 d\vec{x} \\ &= \lim_{t\to\pm\infty} \left|\frac{m}{t}\right|^3 \int_C \left|\psi_0^{\sim}\left(\frac{m\vec{x}}{t}\right)\right|^2 d\vec{x} = \int_{\pm C} |\psi_0^{\sim}(\vec{k})|^2 d\vec{k}\,. \end{aligned} \tag{26}$$

The last step follows by making the change of variables $\vec{k} = m\vec{x}/t$ in the integral over C. This change of variables maps C into itself or its reflection through the origin according as $t > 0$ or $t < 0$. This is the reason for the "$\pm C$" in the last integral. Equation (26), of course, makes the very reasonable statement that for large positive times the probability that the particle will be found in C is the same as the probability that its momentum lies in C, (recall that $|\tilde{\psi_0}(\vec{k})|^2$ is the mpd) and a similar statement for large negative times.

II. **Scattering theory for short range potentials.** Scattering theory deals with particles that are not free, but interact in some way with an obstacle. Basically, a scattering experiment is one in which the particle is hurled at an obstacle (also called a scattering center) by an experimenter, interacts with the obstacle, and flies off, usually in a different direction from its initial one.

Actually, it would be very hard to include in a simple theory an account of the experimenter and his apparatus. Instead, one tries to describe an idealized version of the above situation in which no experimenter is present but the particle just happens to come along, interacts with the obstacle, and finally goes off in another direction. It should be mentioned that one does not necessarily think of the particle as actually colliding with the obstacle—most typically it is deflected while it is still some distance off. In any case, quantum mechanics states that the entire history of the particle is governed by a wave-function of the form

$$\psi_t = e^{-iHt}\,\psi_0$$

where H is not the free Hamiltonian, because the particle is not alone in the universe. The case in which

$$H = H_0 + V \tag{27}$$

will be studied here, where V is the operation of multiplication by the real function $V(\vec{x})$. As a technical matter it is assumed that $V(\vec{x})$ can be written as

$$V(\vec{x}) = V_1(\vec{x}) + V_2(\vec{x}) \ , \tag{28}$$

where $V_1(\vec{x})$ is a real square-integrable function and $V_2(\vec{x})$ is a real bounded function. Then (Kato [**20**])

$$D(V) \supseteq D(H_0) = D(H) \tag{29}$$

and H is selfadjoint, *having the same domain as* H_0.

V is the part of the Hamiltonian representing the interaction of the particle with the obstacle, and is called the *potential.* The assumptions made on V are weak enough so that most potentials of physical interest (including the Coulomb potential) are covered. The split of H into H_0 and V corresponds to the classical division of energy into kinetic and potential.

Now a characteristic feature of what one thinks of as a scattering process is that if the history of the particle is traced further and further back in time it moves further and further from the scattering center. Therefore it seems plausible that it ought to behave more and more like a free particle, a thought which is encapsulated in the statement:

If $e^{-iHt}\psi_0$ represents a scattering process, then there should exist an element f_- in L^2 such that

$$\lim_{t\to-\infty} \|e^{iHt}\psi_0 - e^{-iH_0t}f_-\| = 0 \ . \tag{30}$$

It is quite easy to see that if this statement is true then asymptotically the ppd and the mpd of $e^{-iHt}\psi_0$ can be replaced by those of $e^{-iH_0t}f_-$, so that the ppd and the mpd of $e^{-iHt}\psi_0$ behave asymptotically like those of a free particle, and the conception of the initial motion of the particle as approximately "free" is justified. Of course, (30) is a stronger condition than is necessary to guarantee that initially the ppd and mpd of $e^{iHt}\psi_0$ behave like those of a free particle. In fact, one could ask why, instead of (30), one does not require only that initially the ppd and mpd behave in this way. One answer is that (30) is a slightly cleaner condition. Another is that one can usually get away with requiring (30). However, it will be shown later that when dealing with Coulomb potentials (30) cannot be satisfied and only the weaker condition mentioned above can be met. For the present, though, (30) is taken as the mathematical expression of the intuitive idea that originally the particle moved freely. For similar reasons one wants to require the following:

If $e^{-iHt}\psi_0$ represents a scattering process, then there exists a function $f_+ \in L^2$ such that

$$\lim_{t\to+\infty} \|e^{-iHt}\psi_0 - e^{-iH_0t}f_+\| = 0 \ . \tag{31}$$

If ψ_0 satisfies these conditions, $e^{-iHt}\psi_0$ will be said to describe a scattering experiment.

Now actually, the above discussion is a bit backwards, because it focuses initial attention on the state ψ_0—but ψ_0 is not the thing in the idealized scattering process that corresponds to the data the experimenter actually gets his hands on. Instead the experimenter prepares the particle a long way from the scattering center in what he perceives to be a good approximation of free motion. He then lets whatever is going to happen happen, and after a long while he detects the particle in a state which is again a good approximation of free motion. Hence the things with which the experimenter has direct contact are represented in the idealized picture by $e^{-iH_0t}f_\pm$:

$$\text{prepared state} \longleftrightarrow e^{-iH_0t} f_- ,$$
$$\text{detected state} \longleftrightarrow e^{-iH_0t} f_+ \quad .$$

$e^{-iH_0t} f_-$, for instance, has the form of a solution of the free Schröd-inger equation. *Which* solution is present initially is determined by f_-. Thus, f_- determines the initial free motion of the particle, and this is just what the experimenter determines when he prepares the particle. Just as f_- represents the state the experimenter prepares, so f_+ represents the state which he finally measures. The problem of scattering theory, of course, is to predict what one will measure if one knows what was prepared, i.e.,

$$\text{given } f_-, \text{ predict } f_+ \quad .$$

To a mathematician, of course, the problem is to show that each f_- uniquely determines an f_+ and, if possible, to give an explicit method of computing f_+ given f_-. In felicitous cases (this means for an appropriate class of potentials) it turns out that the problem is solvable and that the answer has this form:

$$f_+ = Sf_- \tag{32}$$

where S is a unitary operator usually called the "S-matrix" by physicists. We now proceed to the discussion of a felicitous case. The problem is best discussed in two parts:

(33) (1) Given $f_- \in L^2$, find $\psi_0 \in L^2$ such that
$$\lim_{t\to-\infty} \left\| e^{-iH_0t} f_- - e^{-iHt}\psi_0 \right\| = 0 \ .$$

The viewpoint taken here is that the experimenter should be able to prepare the particle in an essentially arbitrary state of free motion, i.e., achieve the initial approximate wave-function $e^{-iH_0t} f_-$ with *any* $f_- \in L^2$, and by doing this initiate a scattering process of the type outlined above, which ought to be described by a wave function of the form $e^{-iHt} \psi_0$ which converges strongly to $e^{-iH_0t} f_-$ as $t \to -\infty$.

(Note: technically, we need only require that (1) be solvable when $\|f_0\| = 1$, since only then is $e^{-iH_0t} f_0$ a suitable wave-function for a particle. However, if (1) is solvable with $\|f_0\| = 1$ then it is plainly solvable for any $f_0 \in L^2$. The norm of the ψ_0 obtained is equal to $\|f_0\|$, as follows from the isometry of $\Omega^\pm$ (see below). For convenience, we shall now and in the future frequently suppress references to such questions of normalization, which play no role in our proofs.)

More specifically, at large negative times a wave-function $e^{-iH_0t} f_-$ describes a particle localized far away from the scattering center, since the integral of the ppd for $e^{-iH_0t} f_-$ over any sphere about the scattering center approaches zero as $t \to -\infty$. Choosing a suitably large negative time, this wave-function describes a particle in a region of space where the effects of the scattering center are "negligible". Now the experimenter is a clever fellow, and if there were no scattering center around we grant him the ability to produce any state of free motion $e^{-iH_0t}\, g$ for the particle. When the scattering center is present, however, he can only produce states of the form $e^{iHt}\, g$.

The physical question is this: does there exist a wave-function $e^{-iHt}\psi_0$ which asymptotically agrees with $e^{-iH_0t}f_-$ as $t \to -\infty$? The answer should be "yes"; a physical argument is made for this answer by choosing a time such that $e^{-iH_0t}f_-$ describes a particle localized in a region of space where the effects of the obstacle are "negligible". Suppose that the experimenter performs in that region the same operations that he would have performed to create $e^{-iH_0t}\, f_-$ if there had been no obstacle. This produces a wave-function $e^{-iHt}\,\psi_0$ which is very close to $e^{-iH_0t}f_-$ (because the effects of the interaction are "negligible" where the operations are done). This does not produce exactly the desired ψ_0 satisfying (33), but by repeating the argument for larger and larger negative t, one works in regions of space where the interaction is more and more negligible, and the existence of the correct ψ_0 is deduced as a limiting case. Needless to say, the entire argument above is only suggestive. The reason for hoping that every "initially free" state $e^{iHt}\,\psi_0$ will again become "free" at large positive times, i.e., will describe a scattering process, is discussed later.

(34) (2) Given ψ_0 from (1), find $f_+ \in L^2$ such that

$$\lim_{t \to +\infty} \left\| e^{-iHt}\,\psi_0 - e^{-iH_0t}f_+ \right\| = 0 \;. \tag{34}$$

The point in requiring (2) is that since the ψ_0 from (1) is supposed to be such that $e^{-iHt}\,\psi_0$ describes a scattering process, this wave-function should again behave like that of a free particle at large positive times.

Whether or not these problems are solvable depends, as indicated above, on the potential V. Some examples of results (but not the only results) on these problems are:

If the function $V(\vec{x})$ belongs to L^2, then problem (1) is solvable (Cook [3]). If $V(\vec{x})$ belongs to $L^1 \cap L^2$, then problem (2) is solvable (Kuroda [22], [23]).

The more difficult problem (2) is not discussed in detail here. Instead problem (1) will be examined, and only in the relatively simple case that $V(\vec{x}) \in L^2$. First note that ψ_0 is unique if it exists and is then given by

(35) $$\psi_0 = \underset{t\to-\infty}{\text{s-lim}}\ e^{iHt}e^{-iH_0t}f_- \qquad \text{(s-lim means strong limit).}$$

This is because by the unitarity of e^{-iHt}

(36) $$\begin{aligned}\|e^{-iHt}\psi_0 - e^{-iH_0t}f_-\| &= \|e^{iHt}(e^{-iHt}\psi_0 - e^{-iH_0t}f_-)\| \\ &= \|\psi_0 - e^{iHt}e^{-iH_0t}f_-\| \ ,\end{aligned}$$

so that if (33) holds, then (35) must also hold, and in this case ψ_0 is clearly unique. Naturally one can equally well conclude (33) from (35). It is now easy to see that problem (1) is equivalent to showing that the limit

(37) $$\underset{t\to-\infty}{\text{s-lim}}\ e^{iHt}e^{-iH_0t}f_- \equiv \Omega^- f_-$$

exists for all $f_- \in L^2$. If this is true then problem (1) is solved by taking ψ_0 to be $\Omega^- f_-$. The limit in (37) will now be shown to exist when $V(\vec{x}) \in L^2$. In fact, slightly more will be shown.

THEOREM (COOK). *Let $H = H_0 + V$, with $V(\vec{x}) \in L^2$. Let $f \in L^2$. Then the strong limits*

(38) $$\lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t}f \equiv \Omega^{\pm} f$$

exist.

PROOF. Because the operator $e^{iHt}e^{-iH_0t}$ is unitary, it suffices to prove the convergence on a set dense in L^2, say S. Taking $f \in S \subseteq D(H_0) = D(H)$, a simple application of Stone's Theorem shows that the function $h(t)$ given by

(39) $$h(t) = e^{iHt}e^{-iH_0t}f$$

is strongly differentiable with derivative

(40) $$h'(t) = ie^{iHt}(H-H_0)e^{-iH_0t}f = ie^{iHt}Ve^{-iH_0t}f \ .$$

A short argument shows that $h'(t)$ is even strongly continuous in t, which ensures that the fundamental theorem of the calculus holds:

(41) $$h(t_2) - h(t_1) = \int_{t_1}^{t_2} h'(s)ds$$

where the integral on the right-hand side is a Bochner integral. (See, for instance, Yosida [**28**, pp. 132 ff].) Thus one obtains using the standard estimate for this integral,

$$\|h(t_2) - h(t_1)\| \leqq \int_{t_1}^{t_2} \|h'(s)\|ds = \int_{t_1}^{t_2} \|Ve^{-iH_0 s}f\|ds \ . \tag{42}$$

This last step is decisive, because one can now work with the expression on the right-hand side of (42), which no longer contains the very complicated operator e^{iHs}, but only the simple operator $e^{-iH_0 s}$. By a previous calculation, and using a standard inequality

$$\begin{aligned} |e^{-iH_0 s} f(\vec{x})| &= \left| \left(\frac{m}{2\pi is}\right)^{3/2} \int e^{im(\vec{x}-\vec{x}')^2/2s} f(\vec{x}')d\vec{x}' \right| \\ &\leqq \left| \left(\frac{m}{2\pi is}\right)^{3/2} \right| \int | e^{im(\vec{x}-\vec{x}')^2/2s} f(\vec{x}')|d\vec{x}' \\ &= \frac{C}{|s|^{3/2}} \ , \quad (s \neq 0) \ , \end{aligned} \tag{43}$$

where C is $|m/2\pi|^{3/2}$ times the L^1 norm of f, which exists since $f \in S$. Thus

$$\|Ve^{iH_0 s}f\| \leqq \frac{C}{|s|^{3/2}} \|V\| \tag{44}$$

and the integrals

$$\int_1^\infty \|Ve^{-iH_0 s}f\|ds \quad \text{and} \quad \int_{-\infty}^1 \|V e^{-iH_0 s} f\|ds \tag{45}$$

converge. This implies that

$$\lim \int_{t_1}^{t_2} \|Ve^{-iH_0 s} f\|ds = 0 \tag{46}$$

if both t_1 and t_2 go to $+\infty$ or both go to $-\infty$. The same is thus true of the norm $\|h(t_2) - h(t_1)\|$, so that the "sequence" $h(t)$ is Cauchy as $t \to \pm\infty$, i.e., (38) holds.

As the strong limits of sequences of unitary operators, the operators $\Omega^\pm$ are isometric:

$$\Omega^{\pm *}\Omega^\pm = I \ . \tag{47}$$

The operator $\Omega^\pm\Omega^{\pm *}$ is the projection $P_{R^\pm}$ on the range $R^\pm$ of the operator $\Omega^\pm$.

(*Note*: $R^\pm$ is a closed subspace because $\Omega^\pm$ is isometric.)

The operators $\Omega^\pm$ satisfy the *intertwining relations*

$$e^{iHt}\,\Omega^\pm = \Omega^\pm e^{iH_0 t} \, . \tag{48}$$

The proof of (48) is simple:

(49)
$$e^{iHt}\Omega^{\pm} = e^{iHt} \lim_{s\to\pm\infty} e^{iHs}e^{-iH_0s} = \lim_{s\to\pm\infty} e^{iH(s+t)}e^{-iH_0s}$$
$$= \lim_{s\to\pm\infty} \{e^{iH(s+t)}e^{-iH_0(s+t)}\}e^{iH_0t} = \Omega^{\pm}e^{iH_0t} ,$$

where the continuity of e^{iHt} has been used to move this operator past the $\lim_{s\to\pm\infty}$ sign. Taking adjoints and changing t to $-t$ in (48) gives

(50) $$\Omega^{\pm *}e^{iHt} = e^{iH_0t}\,\Omega^{\pm *}$$

and multiplying on the left by $\Omega^{\pm}$ and using (48) gives

(51) $$\Omega^{\pm}\Omega^{\pm *}\,e^{iHt} = e^{iHt}\Omega^{\pm}\Omega^{\pm *}$$

or

(52) $$P_{R^{\pm}}\,e^{iHt} = e^{iHt}\,P_{R^{\pm}}.$$

Stone's Theorem now yields

(53) $$P_{R^{\pm}}\,H \subseteq HP_{R^{\pm}}$$

so that the subspace $R^{\pm}$ reduces H. The *part of H in $R^{\pm}$* is defined as $HP_{R^{\pm}}$. It can now be shown that this latter operator, considered as an operator on the Hilbert space $R^{\pm}$,is unitarily equivalent to H_0 acting on L^2. For Stone's Theorem applied to (48) readily gives

(54) $$H\Omega^{\pm} = \Omega^{\pm}H_0 \ .$$

Multiplying on the left by $\Omega^{\pm *}$ gives

(55) $$\Omega^{\pm *}H\Omega^{\pm} = H_0 \ .$$

Because

(56) $$P_{R^{\pm}}\Omega^{\pm} = \Omega^{\pm} \ ,$$

one can rewrite this last equation as

(57) $$\Omega^{\pm *}(HP_{R^{\pm}})\Omega^{\pm} = H_0 \ ,$$

which is the desired unitary equivalence, since $\Omega^{\pm}$ is a unitary operator if considered as a map from the Hilbert space L^2 to the Hilbert space $R^{\pm}$. It follows that the spectrum of H in $R^{\pm}$ is the same as that of H_0 in L^2, hence is absolutely continuous.

Returning to the physical problem, we now assume that $V(\vec{x}) \in L^1 \cap L^2$ and take for granted the result of Kuroda that in this case problem (2) is solvable. It should be remarked that a necessary and sufficient condition for problem (2) to be solvable is that the ranges of $\Omega^{\pm}$ be equal, i.e.,

$$R^+ = R^- \ . \tag{58}$$

To prove this statement note that if (58) holds then the function $\psi_0 = \Omega^- f_-$ also belongs to R^+ so that

$$\psi_0 = P_{R^+}\ \psi_0 = \Omega^+\Omega^{+*}\Omega^- f_- \ . \tag{59}$$

But this last equation, along with (38), implies that problem (2) has the unique solution

$$f_+ = \Omega^{+*}\Omega^- f_- \ . \tag{60}$$

It is quite easy to see that if problem (2) is solvable then $R^- \subseteq R^+$. Moreover, this implies (58) as will be seen shortly.

It has now been seen that if $V \in L^1 \cap L^2$ then both problems (1) and (2) can be solved, and f_+ is determined by f_- according to equation (60). The operator $\Omega^{+*}\Omega^-$ is usually denoted by S. ("The S-matrix.") Naturally this operator is well defined whether or not R^+ and R^- are the same. It is a simple exercise to show that

$$R^+ = R^- \iff \text{S is unitary.} \tag{61}$$

Thus another way to formulate problem (2) is to say that a unitary S-matrix is needed. The proof that $R^+ = R^-$ when $V \in L^1 \cap L^2$ will not be given here. However, one amusing consequence of what a physicist would call the time-reversal invariance of the theory will be noted; namely, whether or not $V \in L^1 \cap L^2$, the set R^+ consists precisely of all complex conjugates of elements of R^-. This is a consequence of the fact that both H and H_0 are real operators in the sense that they commute with complex conjugation, which has as a consequence that for any $f \in L^2$

$$\overline{e^{iHt} f} = e^{-iHt}\bar{f}\ ; \qquad \overline{e^{iH_0t} f} = e^{-iH_0t}\bar{f} \ . \tag{62}$$

Using these facts gives

$$\begin{aligned}
\psi \in R^+ &\iff \psi = \lim_{t\to+\infty} e^{iHt}e^{-iH_0t} f, \quad \text{with } f \in L^2 \\
&\iff \bar{\psi} = \lim_{t\to+\infty} \overline{e^{iHt}e^{-iH_0t} f} \\
&\qquad = \lim_{t\to+\infty} e^{-iHt}e^{+iH_0t}\bar{f} = \lim_{t\to-\infty} e^{iHt}e^{-iH_0t}\bar{f} \iff \bar{\psi} \in R^- \ .
\end{aligned} \tag{63}$$

So showing that $R^+ = R^-$ is the same as showing that one of these subspaces is closed under complex conjugation. This may sound easier, but the author has no reason to believe that it is.

Under the assumption $V \in L^1 \cap L^2$ one can easily give the answer to a somewhat mathematicized version of a typical experimental question; namely, suppose that the initial condition of a particle in a scattering experiment is described by the free wave-function $e^{-iH_0t} f_-$. What is the probability $P(f_-, C)$ that at large positive times t the particle is found in a cone C with apex at the origin? To answer this question, note that the hypothesis is equivalent to saying that the wave-function of the particle at all times is $e^{-iHt}\,\Omega^- f_-$. The desired probability is the asymptotic value of the integral over the cone of the ppd determined by $e^{-iHt}\,\Omega^- f_-$:

$$(64) \qquad P(f_-, C) = \lim_{t\to+\infty} \int_C |(e^{-iHt}\Omega^- f_-)(\vec{x})|^2 d\vec{x} \ .$$

Since $e^{-iHt}\Omega^- f_-$ converges strongly to $e^{-iH_0t}\, Sf_-$ as $t \to +\infty$, one can replace the ppd of one by the other in the integral and use (26). This gives

$$(65) \qquad P(f_-, C) = \lim_{t\to+\infty} \int_C |(e^{-iH_0t}\, Sf_-(\vec{x})|^2 d\vec{x} = \int_C |(Sf_-)^{\sim}(\vec{k})|^2 d\vec{k} \ .$$

This result is very reasonable—the probability that the particle finally is found in the cone C is the same as the probability that the momentum determined from the final free wave-function $e^{-iH_0t}\, Sf_-$ lies in the cone. The result has been derived by Green and Lanford [11] by a different method.

Note also that the equation

$$(66) \qquad \lim_{t\to-\infty} \|\Omega^- f_- - e^{iHt}e^{-iH_0t}f_-\| = 0$$

implies

$$(67) \qquad \lim_{t\to+\infty} \|\Omega^- f_- - e^{-iHt}e^{iH_0t} f_-\| = 0 \ ,$$

so that also

$$(68) \qquad \lim_{t\to+\infty} \|e^{-iHt}\,\Omega^- f_- - e^{-2iHt}e^{iH_0t} f_-\| = 0 \ .$$

Because of (68), one can asymptotically replace the ppd of the wave-function $e^{-iHt}\,\Omega^- f_-$ by $|(e^{-2iHt}e^{iH_0t} f_-)(\vec{x})|^2$ in (64), to obtain

$$(69) \qquad P(f_-, C) = \lim_{t\to+\infty} \int_C |(e^{-2iHt}e^{iH_0t} f_-)|^2\, d\vec{x} \ .$$

(*Note*: (65) and (69) can be proved whenever the potential is such that $\Omega^\pm$ exist and $R^+ = R^-$.)

The formula (69) is of some interest because the expression given for the probability no longer involves the operator Ω^-. Of course, heavy use of the existence of Ω^- has been made in deriving (69), but since

Ω^- has been eliminated in the final result it is conceivable that the formula (69) can be used to define the probability $P(f_-, C)$ in a situation where Ω^- does not exist. The meaning of this statement is not really clear because $P(f_-, C)$ was a probability based on the hypothesis that the particle had a wave-function approaching $e^{-iH_0t}f_-$ as $t \to -\infty$. But if Ω^- does not exist, this will not happen. To see what interpretation might be given to this statement, consider the trivial case of a constant potential

$$V(\vec{x}) = V_0 = \text{const.} \tag{70}$$

Then the Hamiltonian

$$H = H_0 + V \tag{71}$$

is not one for which the Møller wave-matrices $\Omega^{\pm}$ exist, because

$$e^{iHt}e^{-iH_0t} = e^{iV_0t} \quad , \tag{72}$$

and e^{iV_0t} does not converge as $t \to \pm\infty$. On the other hand, the behavior of the wave-function $e^{-iHt}\psi_0$ is no different from that of the wave-function $e^{-iH_0t}\psi_0$—their ppd's and mpd's are identical for all times—so a particle described by a wave-function $e^{-iHt}\psi_0$ behaves just like a free particle.

Now since Ω^- does not exist, given $f_- \in L^2$ there will *not* be a $\psi_0 \in L^2$ such that (33) holds. However, there certainly *will* be a ψ_0 such that the ppd and mpd of $e^{-iHt}\psi_0$ agree with those of $e^{-iH_0t}f_-$ as $t \to -\infty$. In fact one can take

$$\psi_0 = f_- \ . \tag{73}$$

This suggests that, in the case of a constant potential, the statement that the particle has initial state f_- be interpreted to mean that the state for all times is $e^{-iHt}f_-$. The question "What is the probability $P(f_-, C)$ that if the particle has the initial state f_- it will asymptotically be found in the cone C for $t \to +\infty$?" can now be answered. The answer is

$$\begin{aligned} P(f_-, C) &= \lim_{t\to+\infty} \int_C |(e^{-iHt}f_-)(\vec{x})|^2 d\vec{k} \\ &= \lim_{t\to+\infty} \int_C |(e^{-2iHt}e^{iH_0t}f_-)(\vec{x})|^2 d\vec{x} = \int_C |f_-^{\sim}(\vec{k})|^2 d\vec{k} \end{aligned} \tag{74}$$

so that the answer is correctly given by (69).

Note that $e^{-iHt}f_-$ is not the only function whose ppd and mpd asymptotically agree with those of $e^{-iH_0t}f_-$. However, if $e^{-iHt}f_-'$ is another such function, it is easy to see that f_- and f_-' must have the same mpd.

Hence if $P(f_-, C)$ is computed, using f_-' instead of f_- in (74), the same result is obtained. Of course, the numerical value of $P(f_-, C)$ given by (74) is exactly what would have been obtained for a free particle ($V_0 = 0$), and any physicist will object that this example was pointless on the grounds that V_0 ought to have been set equal to zero to begin with. However, formula (69), correctly interpreted, again gives correct results for the nontrivial case of the Coulomb potential, so that it does provide a useful generalization of the usual formalism.

Returning now to the discussion of the formalism of scattering theory, it may happen that the Hamiltonian H has eigenvalues E_n such that

(75) $$H\psi_n = E_n\psi_n \ , \quad \text{where } E_n \in R \text{ and } \psi_n \in L^2 \quad .$$

In this case the corresponding functions ψ_n are called (normalizable) stationary states or sometimes bound states for the Hamiltonian H. If (75) holds, then

(76) $$e^{-iHt}\,\psi_n = e^{-iE_nt}\psi_n \quad .$$

It is clear by inspection of (76) that if the state of the particle at $t = 0$ is given by a bound state then the ppd and the mpd for the particle never change, since propagating the wave-function in time consists in multiplying this wave-function by e^{-iE_nt}.

The wave-function describing an electron in a hydrogen atom is of this form, for instance. It is not true that the electron does not move, but it is true that its ppd and mpd are constant. In our case one can think of a bound state as describing a particle which, in virtue of its interaction with the scattering center, is constrained to motion in a neighborhood of the scattering center. It should be clear that the behavior of these functions is radically different from that of the wave-functions describing scattering experiments. The latter eventually have ppd's like those of free particles, with the result that asymptotically the probability for finding a particle described by such a wave-function inside any sphere approaches zero. For a bound state, however, this probability is constant. Denote by B the closed subspace of L^2 spanned by all the bound states of H. Then it can be shown that whenever $\Omega^\pm$ exist,

(77) $$B \perp R^\pm \quad .$$

This is actually clear from the fact that B is by definition the subspace H_p spanned by all the eigenfunctions of H, while it is a consequence of (57) and the following discussion that $R^\pm$ is contained in the absolutely continuous subspace H_{ac} of H. However, another proof is given because it is instructive.

To prove (77) one need only show that each bound state ψ_n is orthogonal to $R^{\pm}$. To avoid complicated notation, the proof is given for R^{+}: let $g = \Omega^{+}f \in R^{+}$. Then

$$(78)\qquad \begin{aligned}(\psi_n, g) &= \lim_{t\to+\infty} (\psi_n, e^{iHt}e^{-iH_0t}f) = \lim_{t\to+\infty} (e^{-iHt}\psi_n, e^{-iH_0t}f) \\ &= \lim_{t\to+\infty} e^{iE_nt}(\psi_n, e^{-iH_0t}f)\,.\end{aligned}$$

However, the last limit is zero as follows directly from the fact that the operator e^{-iH_0t} converges weakly to zero as $t\to\pm\infty$. For completeness, a proof of this fact is given. Let h_1, $h_2 \in$ S. Then using the estimate (43) on $|(e^{-iH_0t}h_2)(\vec{x})|$ gives

$$(79)\qquad \begin{aligned}|(h_1, e^{-iH_0t}h_2)| &\leqq \int |h_1(\vec{x})|\,|(e^{-iH_0t}h_2)(\vec{x})|d\vec{x} \\ &\leqq \frac{C}{|t|^{3/2}}\int |h_1(\vec{x})|d\vec{x} = \frac{D}{|t|^{3/2}} \xrightarrow[t\to\pm\infty]{} 0\,.\end{aligned}$$

One now shows in a straightforward manner (using the fact that S is dense in L^2) that $(h_1, e^{-iH_0t}h_2)$ tends to zero as $t\to\pm\infty$ for any h_1, $h_2 \in L^2$.

Again, consider a "nice" theory in which the Hamiltonian H is such that the operators $\Omega^{\pm}$ exist and $R^{+} = R^{-}$. Denote $R^{\pm}$ by R. The last result above shows that R is orthogonal to B; hence

$$(80)\qquad L^2 = R\oplus B\oplus X$$

where X is defined by (80). In the subspace X, H may have some of its absolutely continuous spectrum and must have all of its singular continuous spectrum (if any).

Now the experience of physicists with classical particles suggests the hope that the bound states and the scattering states (i.e., wavefunctions from B and R) and their linear combinations ought to exhaust L^2, i.e., one may expect that

$$(81)\qquad L^2 = R\oplus B\,.$$

This equation expresses the idea that essentially two distinct things can happen to a particle: either it is bound once and for all in a neighborhood of the scattering center, or else it is initially free and will again become free with the passage of time. There should not be any situations in which the particle is initially free and finally is "captured" by the scattering center, thereafter remaining localized near the scattering center. This is suggested by many situations in classical mechanics—for instance the case of the motion of a "small" celestial object with respect to our sun. (It is unnecessary to insist that the object be "small"—however, if its mass is large enough to disturb the motion of the sun appreciably, then one cannot think of the sun as "fixed", and has instead to describe the relative motion of object and sun.) There are two cases: either the object is trapped near the sun, like the planets or Halley's comet, or else it is not, and if it is not, it never will be. (This ignores the effects of the planets on the "object", which would merely confuse the issue. It is also true that there is a rare kind of orbit having the form of a parabola, in which the "object" is not trapped by the sun, but still its path does not approach a straight line at large times, so that this orbit has some features of both the "eventually free" and the "forever interacting" cases. The $1/x$ potential may be trusted to produce trouble and confusion.) This and many other examples in classical mechanics suggest that a similar situation should prevail in quantum mechanics, and this is the reason for hoping that (81) holds. (In quantum mechanics, if one grants that the two situations described are possible, then he must also allow linear superpositions of the states describing these situations, as is seen in (81). This is a quantum-mechanical fact of life. The hope is, then, that the situation is no more complicated than is apparent from (81).)

If equation (81) holds, then the theory is said to be *asymptotically complete,* and (81) is called the requirement of *asymptotic completeness.* It should be clear from the discussion of the spectrum of H that (81) is equivalent to the pair of conditions

(82) $H_{\mathrm{ac}} = R$ and H has no singular continuous spectrum.

The condition $H_{\mathrm{ac}} = R$ is what is called in Professor Kato's terminology "completeness of the wave operators $\Omega^{\pm}$". Conditions under which the requirement of asymptotic completeness is satisfied have been found by Ikebe [**16**].

This introductory section will now be ended with a brief recital of the expected connection between the "stationary state" approach to scattering theory and the present approach.

In the stationary-state approach to scattering theory, one seeks non-square-integrable solutions of the equation

$$H_\psi = \frac{k^2}{2m}\psi \tag{83}$$

by replacing this equation with the integral equation

$$\begin{aligned} \psi^{(\pm)}_{\vec{k}}(\vec{x}) &= e^{i\vec{k}\cdot\vec{x}} - \frac{m}{2\pi}\int \frac{e^{\mp ik|\vec{x}-\vec{x}\,'|}}{|\vec{x}-\vec{x}\,'|}V(\vec{x}\,')\,\psi^{(\pm)}_{\vec{k}}(\vec{x}\,')d\vec{x}\,' , \\ \psi^{(+)}_{\vec{k}} &= \overline{(\psi^{(-)}_{-\vec{k}})}. \end{aligned} \tag{84}$$

If all goes well (e.g., Ikebe [**16**] finds sufficient conditions) then $\psi^{(\pm)}_{\vec{k}}$ "span $R^{\pm}$" in the sense that

(85) $$\psi \in R^{\pm} \Longleftrightarrow \exists h^{\pm} \in L^2 \text{ such that}$$
$$\psi(\vec{x}) = \text{l.i.m.} \frac{1}{(2\pi)^{3/2}} \int \psi_{\vec{k}}^{(\pm)}(\vec{x}) h^{\pm}(\vec{k}) d\vec{k}$$

Furthermore, for any $f \in L^2$

(86) $$(\Omega^{\pm} f)(\vec{x}) = \text{l.i.m.} \frac{1}{(2\pi)^{3/2}} \int \psi_{\vec{k}}^{(\pm)}(\vec{x}) f^{\sim}(\vec{k}) d\vec{k}.$$

Comparing this with

(87) $$f(\vec{x}) = \text{l.i.m.} \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{k}\cdot\vec{x}} f^{\sim}(\vec{k}) d\vec{k},$$

it is seen that $\Omega^{\pm} f$ is obtained from f by replacing $e^{i\vec{k}\cdot\vec{x}}$ with $\psi_{\vec{k}}^{(\pm)}(\vec{x})$ in the Fourier integral for f.

The functions $\psi_{\vec{k}}^{(\pm)}(\vec{x})$ have the asymptotic form for large $|\vec{x}|$

(88) $$\psi_{\vec{k}}^{(\pm)}(\vec{x}) \sim e^{i\vec{k}\cdot\vec{x}} + \frac{e^{\pm ikx}}{x} f^{(\pm)}(\vec{k}, \theta) + \cdots$$

where $x = |\vec{x}|$, $k = |\vec{k}|$, θ is the angle between $\vec{k}$ and $\vec{x}$, and $\cdots$ denotes terms which fall off faster than $1/x$ for large x. $f^{(-)}$ is usually called by physicists the *scattering amplitude*. A physical interpretation can be made of $\psi_{\vec{k}}^{(\pm)}$ and $f^{(-)}$, but it will not be given here.

III. **Coulomb potential scattering.** The Coulomb potential $V_c(\vec{x})$ is defined by the equation

(89) $$V_c(\vec{x}) = e_1 e_2 / x$$

where e_1 and e_2 are real numbers which represent the electric charges of two physical objects, e.g., in our previous picture the particle and the scattering center. Keeping this picture in mind, the Hamiltonian of the particle is now

(90) $$H_c = H_0 + V_c \quad .$$

The theory of the Hamiltonian (90) has long been of interest to physicists. All bound states for H_c can be found by solving the eigenvalue problem, and the resulting eigenfunctions ψ_n give the possible ppd's and mpd's for a particle bound by a Coulomb potential (e.g., an electron in a hydrogen atom). There are no eigenvalues if $e_1 e_2 > 0$. When $e_1 e_2 < 0$, the corresponding eigenvalues (of the form c/n^2, $c < 0$, $n = 1, 2, 3, \cdots$) are interpreted as the energies associated with the motions of the particle described by the wave-functions ψ_n.

Scattering by a Coulomb potential was studied by Rutherford, using classical mechanics. The quantum-mechanical stationary state scattering problem for the Coulomb potential was solved by Gordon [**10**]—that is to say, Gordon found two "complete" sets $\psi_{c\vec{k}}^{(\pm)}$ of non-square-integrable solutions of the equation

$$H_c\psi_{c\vec{k}}^{(\pm)} = \frac{k^2}{2m}\psi_{c\vec{k}}^{(\pm)} \quad . \tag{91}$$

The solution sets are complete in the sense discussed in the paper by Shenk and Thoe in this issue. That is, if ψ_n denotes the nth bound state for H_c, then for any $f \in L^2$

$$f = \text{l.i.m.} \sum_{n=1}^{\infty} c_n\psi_n + \text{l.i.m.} \frac{1}{(2\pi)^{3/2}} \int \psi_{c\vec{k}}^{(\pm)}(\vec{k})\hat{f}^{\pm}(\vec{k})d\vec{k} \tag{92}$$

where

$$\hat{f}^{\pm}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int \psi_{c\vec{k}}^{(\pm)}(\vec{x})f(\vec{x})d\vec{x} \ . \tag{93}$$

The completeness of the functions $\psi_{c\vec{k}}^{(\pm)}$ was proved by Titchmarsh (see for instance Titchmarsh [**27**, Vol. I]).

If wave operators $\Omega_c^{\pm}$ and an S-matrix are defined by

$$(\Omega_c'^{\pm}f)(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \psi_{c\vec{k}}^{(\pm)}(\vec{x})f^{\sim}(\vec{k})d\vec{k} \tag{94}$$

and

$$S = \Omega_c'^{+*}\Omega_c'^{-} \tag{95}$$

then satisfactory physical results are obtained using the operator S, i.e., the theory predicts correctly.

(Nothing has been said as yet about the time-dependent theory. However, there is a method of obtaining the relevant predictions of experimental data without making reference to the time-dependent theory, and it is this method whose application yielded the correct results. Although it is not described here, it is worthwhile remarking that this method had to be deformed somewhat for the Coulomb case because of the peculiar behavior of the $\psi_{c\vec{k}}^{(\pm)}$'s, which will be described shortly.)

This is as expected and entirely satisfactory. There are, however, some unusual features to the situation. Gordon did not obtain the $\psi_{c\vec{k}}^{(\pm)}$ by means of the integral equation (84). Instead, he proceeded by a straightforward attack on the differential equation (91), which he managed to solve exactly. The integral equation (84) would, in fact, be quite misleading for the functions $\psi_{c\vec{k}}^{(\pm)}$, since these functions do not have the asymptotic form $e^{i\vec{k}\cdot\vec{x}} + O(1/|\vec{x}|)$ when $|\vec{x}|$ is large. In fact, one can convince oneself that no solution of (91) can have this form. Gordon attempted to find solutions of (91) which behaved "as much as possible" like ordinary stationary state wave-functions at large $|\vec{x}|$. As will be seen, he came close. Gordon's solution to the equation is as follows. Let

(96) $$n(k) = me_1e_2/k \quad \text{(hereafter simply denoted } n\text{).}$$

Then

(97) $$\psi_{c\vec{k}}^{(-)}(\vec{x}) = e^{-\pi n/2}\,\Gamma(1+in)e^{i\vec{k}\cdot\vec{x}}\,{}_1F_1(-in, 1, i(kx - \vec{k}\cdot\vec{x}))$$

and

(98) $$(\psi_{c\vec{k}}^{(+)}(\vec{x})) = \overline{(\psi_{c,-\vec{k}}^{(-)}(\vec{x}))}$$

where ${}_1F_1(a, b, z)$ is Kummer's confluent hypergeometric function:

(99 $$ {}_1F_1(a, b, z) = 1 + \sum_{s=1}^{\infty} \frac{a(a+1)\cdots(a+s-1)}{b(b+1)\cdots(b+s-1)}\frac{z^s}{s!} \quad .$$

The definition (99) looks complicated, and it is not necessary for our present purposes to study it in detail. Note that the coefficients of the power series defining ${}_1F_1$ fall off as $s \to \infty$ about like $1/s!$, so that ${}_1F_1$ is an entire function. It is also possible to show that the particular ${}_1F_1$ appearing in (97) is bounded as a function of $\vec{x} \in R^3$. The asymptotic form of ${}_1F_1$ for large values of its argument is given by Slater [26]. As one sees from equation (97), it is less natural to ask for the asymptotic form of $\psi_{c\vec{k}}^{(-)}(\vec{x})$ for large x than for large $|kx - \vec{k}\cdot\vec{x}|$. Of course, if $\vec{x}$ makes an angle greater than or equal to some fixed angle θ_0 with $\vec{k}$, then

(100) $$x \geqq |kx - \vec{k}\cdot\vec{x}|/k \geqq x(1-\cos\theta_0)$$

so that large $|\vec{x}|$ will then mean much the same thing as large $|kx - \vec{k}\cdot\vec{x}|$. In any case, when $|kx - \vec{k}\cdot\vec{x}|$ is large,

(101) $$\psi_{c\vec{k}}^{(-)}(\vec{x}) = e^{i(\vec{k}\cdot\vec{x} + n\log(kx - \vec{k}\cdot\vec{x}))}\left[1 + \frac{n^2}{i(kx - \vec{k}\cdot\vec{x})}\right] + \frac{e^{i(kx - n\log 2kx)}}{x} f_c(\theta) + \cdots$$

where θ is the angle between $\vec{k}$ and $\vec{x}$. It should be evident at a glance that this asymptotic form is unnatural when x is large but $|kx - \vec{k}\cdot\vec{x}|$ is small. In (101) an attempt has been made to make the expansion for $\psi_{c\vec{k}}^{(-)}(\vec{x})$ look as much like the expansion (88) as possible. The expression which most closely resembles $e^{i\vec{k}\cdot\vec{x}}$ has been written first, multiplied by the coefficient $[1 + n^2/i(kx-\vec{k}\cdot\vec{x})]$ which blows up when the direction of $\vec{x}$ is the same as that of $\vec{k}$ ($\theta = 0$). The next term is the one containing the expression which most closely resembles e^{ikx}/x. As a matter of interest, $f_c(\theta)$ also blows up when $\theta = 0$. Aside from this sort of misbehavior of the expansion, the interesting deviation from (88) comes, of course, in the additional logarithmic "distortion" of the phase of $e^{i\vec{k}\cdot\vec{x}}$ and e^{ikx}. This distortion is connected with the long range of the Coulomb potential, and the logarithm gets into the picture essentially because it is the indefinite integral of the potential with respect to x. However, the origins of the distorting factor will not be investigated here. The main point to be absorbed is the fact that although the stationary state theory is well understood, it is "peculiar" in that the stationary state wave-functions do not have the usual properties.

Now consider the time-dependent theory. As was pointed out by Professor Kato, the existence proof given earlier for the Møller wave-matrices $\Omega^{\pm}$ when V is square-integrable can be generalized to cover other cases. In particular, if the potential $V(\vec{x})$ in the Hamiltonian $H = -\Delta/2m + V$ is locally square-integrable and falls off like $1/|\vec{x}|^{1+\epsilon}$ for large $|\vec{x}|$, then $\Omega^{\pm}$ exist (Hack [**12**] and Jauch and Zinnes [**19**]). However, the proof given above does not extend to the Coulomb potential. The situation as $t \to +\infty$ will be discussed. The case $t \to -\infty$ is similar. Let $f \in S$ and consider the expression

$$\|V_c\, e^{-iH_0t}\, g\| = H(t)\ . \tag{102}$$

If the previous proof is to work, $H(t)$ must be integrable over $(1, \infty)$, say. Now using the equation (14) for $(e^{-iH_0t}\, g)(\vec{x})$, with the quadratic in $e^{im(\vec{x}-\vec{x}')^2/2t}$ expanded, gives

$$\begin{aligned} H(t)^2 = \int & \frac{(e_1e_2)^2}{x^2} \\ & \cdot \left| \left(\frac{m}{2\pi i t}\right)^{3/2} e^{imx^2/2t} \int e^{-im\vec{x}\cdot\vec{x}'/t} e^{imx'^2/2t}\, g(\vec{x}')\,d\vec{x}' \right|^2 d\vec{x}. \end{aligned} \tag{103}$$

Change variables to $\vec{y} = m\vec{x}/t$, giving

$$\begin{aligned} H(t)^2 &= \left(\frac{me_1e_2}{|t|}\right)^2 \int \frac{1}{y^2} \left| \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{y}\cdot\vec{x}'} e^{imx'^2/2t}\, g(\vec{x}')\,d\vec{x}' \right|^2 d\vec{y} \\ &\equiv \left(\frac{me_1e_2}{|t|}\right)^2 h(t)^2\ . \end{aligned} \tag{104}$$

Letting

$$F(\vec{y}, t) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{y}\cdot\vec{x}'} e^{imx'^2/2t} g(\vec{x}')d\vec{x}' \tag{105}$$

it is easy to see (convert y^2 to $-\Delta_{x'}$ on $e^{-i\vec{y}\cdot\vec{x}'}$ and integrate by parts) that $(1 + y^2)F(\vec{y}, t)$ is bounded in t. Further, $F(\vec{y}, t)$ converges pointwise to $g^{\sim}(\vec{y})$ as $t \to \infty$. Thus a simple application of Lebesgue's Dominated Convergence Theorem gives

$$h(t)^2 \underset{t\to\infty}{\longrightarrow} \int \frac{|g^{\sim}(\vec{y})|^2}{y^2} d\vec{y} \equiv h_\infty^2 . \tag{106}$$

Note that h_∞^2 cannot vanish unless $g = 0$. Also

$$H(t) = h(t) \cdot |me_1e_2|/|t| . \tag{107}$$

Because of (106), there exists a t_0 such that

$$H(t) \geqq h_\infty |me_1e_2|/2|t|, \qquad t \geqq t_0 , \tag{108}$$

whence

$$\int_{t_0}^{\infty} H(t)dt = \infty . \tag{109}$$

The integral diverges logarithmically. It will be seen that expressions involving logarithms occur again and again in dealing with Coulomb potentials. Note that the outcome of the above argument could have been deduced heuristically from our knowledge of the behavior of the ppd of a free particle, as follows:

$$\begin{aligned} \frac{1}{x^2} |(e^{-iH_0 t} g)(\vec{x})|^2 &\underset{\text{large } |t|}{\sim} \frac{1}{x^2} \left| \frac{m}{t} \right|^3 \left| g^{\sim}\left(\frac{m\vec{x}}{t} \right) \right|^2 \\ &= \left| \frac{m}{t} \right|^2 \cdot \left| \frac{m}{t} \right|^3 \left| \frac{g^{\sim}(m\vec{x}/t)}{|m\vec{x}/t|} \right|^2 \\ &= \left| \frac{m}{t} \right|^2 \cdot \left| \frac{m}{t} \right|^3 \left| f^{\sim}\left(\frac{m\vec{x}}{t} \right) \right|^2 \end{aligned} \tag{110}$$

where

$$f^{\sim}(\vec{k}) = g^{\sim}(\vec{k})/k . \tag{111}$$

Integrating the right-hand side of (110) with respect to $\vec{x}$ and changing variables as before, gives the previous result. Equations (110) and (111) suggest the heuristic conclusion that multiplying a solution of the free Schrödinger equation by $1/x$ is asymptotically like multiplying the Fourier transform of the solution by $(m/|t|)\cdot(1/k)$, i.e., "multiplying" the solution itself by the operator $(m/|t|)(1/(-\Delta)^{1/2})$. An (outrageous?) extension of this observation is that multiplying the solution by

$$e^{i(H_0+e_1e_2/|\vec{x}|)t} = e^{i\int^t (H_0+e_1e_2/|\vec{x}|)ds}$$

is like multiplying it by

$$e^{i\int^t (H_0 + me_1e_2/|s|(-\Delta)^{1/2})ds} = e^{iH_0t}e^{i(me_1e_2/(-\Delta)^{1/2})\log|t|}.$$

The vague reasoning here is that multiplying by $H_c = H_0 + e_1e_2/|\vec{x}|$ is like multiplying by $H_0 + me_1e_2/(-\Delta)^{1/2}|t|$ so multiplying by the solution e^{iH_ct} of the equation

$$-idU(t)/dt = H_cU(t) \tag{112}$$

is like multiplying by the solution

$$e^{iH_0t}e^{i(me_1e_2/(-\Delta)^{1/2})\log|t|}$$

of the equation

$$-i\frac{dW(t)}{dt} = \left(H_0 + \frac{me_1e_2}{(-\Delta)^{1/2}|t|}\right)W(t)\ . \tag{113}$$

(It was assumed above that $t > 0$; if $t < 0$ the solution of (113) has a minus sign multiplying the log $|t|$ in the exponential. Account is taken of this in the final results. The present discussion is only illustrative.)

If this is by chance true, then it is easy to see that the Møller wave-matrices cannot exist, because

$$e^{iH_ct}e^{-iH_0t} \sim e^{iH_0t}e^{i(me_1e_2/(-\Delta)^{1/2})\log|t|}e^{-iH_0t} = e^{i(me_1e_2/(-\Delta)^{1/2})\log|t|} \tag{114}$$

and it is not difficult to show that the operator on the right-hand side of (114) converges weakly to zero as $t \to \infty$, in essentially the same manner as we earlier showed that e^{-iH_0t} converges weakly to zero as $t \to \pm\infty$. This of course would mean that strong convergence could not take place in (114). But it is also now very plausible that the sequence

$$e^{iH_ct}e^{-iH_0t}e^{-i(me_1e_2/(-\Delta)^{1/2})\log|t|} \tag{115}$$

will converge strongly as $t \to \infty$. This heuristic motivation suggests the

THEOREM. *Let*

$$H_c = -\Delta/2m + e_1e_2/x \tag{116}$$

be the Hamiltonian for a charged particle in the neighborhood of a *fixed charge. Let*

$$H_{0c}(t) = H_0 t + \epsilon(t)\, \frac{m e_1 e_2}{(-\Delta)^{1/2}} \log\left(\frac{-2|t|\Delta}{m}\right) \tag{117}$$
$$\equiv H_0 t + A(t) \qquad (t \neq 0)$$

where $A(t)$, *called the "anomalous" term, is defined by the last equation and*

$$\begin{aligned} \epsilon(t) &= 1, \quad t > 0\ , \\ &= -1, \quad t < 0\ . \end{aligned} \tag{118}$$

Then the strong limits

$$\Omega_c^{\pm} = \lim_{t \to \pm\infty} e^{iH_c t}\, e^{-iH_{0c}(t)} \tag{119}$$

exist on all of L^2.

REMARKS. The operator $e^{-iH_{0c}(t)}$ is defined by

$$(e^{-iH_{0c}(t)} h)(\vec{x}) = \text{l.i.m.}\, \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{k}\cdot\vec{x}} e^{-ik^2 t/2m} e^{-i\epsilon(t)(me_1e_2/k)\log(2k^2|t|/m)}\, h^{\sim}(\vec{k})\, d\vec{k} \tag{120}$$

and is unitary by inspection. The factor $\epsilon(t)$ is 1 for the case $t \to +\infty$. For $t \to -\infty$ an additional minus sign is necessary, because the indefinite integral $\int^t |s|^{-1}\, ds$ is $-\log|t|$ for $t < 0$. Writing

$$\log\left(-\frac{(2|t|\Delta)}{m}\right) = \log|t| + \log\left(-\frac{2\Delta}{m}\right) \tag{121}$$

it is seen that the logarithmic factor in (117) is the one found before plus a constant operator. The constant operator, of course, cannot affect the convergence properties, and is included for convenience, as will be explained later. Thus the theorem is consistent with the earlier heuristic considerations. The proof of this theorem is accomplished by the method of Cook, but due to the peculiar character of the propagation operator $e^{-iH_{0c}(t)}$, a certain amount of refinement of the earlier calculations is required. First, it is necessary to analyse the behavior of a function of the type $e^{-iH_{0c}(t)}h$, where h is some convenient sort of function. The operator $e^{-iH_{0c}(t)}$ is evaluated in terms of the Fourier integral as in equation (120).

In order to be able to erase the l.i.m's and manipulate freely with the integration, it is convenient to take $h \in S$. Because time-derivatives will introduce factors of $1/k$ in the integral in (120), it is convenient to have $\tilde{h}$ vanish in a neighborhood of the origin. Hence the definition: h is a *C-function* if $h \in S$ and $\tilde{h}$ vanishes in some neighborhood of the origin. (The neighborhood can depend on h.) The C-functions are dense in L^2. Consider $e^{-iH_{0c}(t)}h$, where h is a C-function. A first goal is to show that for large times ($t \to \pm\infty$) the ppd and mpd of this function can be replaced by the ppd and mpd of the solution $e^{-iH_0 t}h$. In fact, even more detailed information about $e^{-iH_{0c}(t)}h$ will be obtained. Note that according to definition (117) $H_{0c}(t)$ is the sum of two parts $H_0 t$ and $A(t)$ which permute, since they are both functions of the operator Δ. Thus

(122) $$e^{-iH_{0c}(t)} = e^{-iH_0 t}\, e^{-iA(t)}$$

and using equation (20) gives

(123) $$e^{-iH_{0c}(t)} = C_t Q_t e^{-iA(t)}, \qquad (t \neq 0)\ .$$

LEMMA 1. *Let h be a C-function. Then*

(124) $$\lim_{t \to \pm\infty} \|e^{-iH_{0c}(t)}\, h - C_t\, e^{-iA(t)}\, h\| = 0\ ,$$

and in fact the norm in (124) *decreases "almost as fast as* $1/t$"; *namely, it is bounded by an expression of the form*

(125) $$C(\log|t|)^n/|t| \quad \text{as } t \to \pm\infty\ .$$

PROOF. In view of (123),

(126) $$\begin{aligned} \|e^{-iH_{0c}(t)}h - C_t e^{-iA(t)}h\|^2 &= \|Q_t e^{-iA(t)}h - e^{-iA(t)}h\|^2 \\ &= \int |e^{imx^2/2t} - 1|^2\, |(e^{-iA(t)}h)(\vec{x})|^2 d\vec{x}\,. \end{aligned}$$

The last expression can be estimated, by making use of two facts, namely

(127) $$|e^{imx^2/2t} - 1| \leqq mx^2/2|t|$$

and the fact that for any nonnegative integer p there exists a constant C such that

(128) $$|(e^{-iA(t)}\, h)(\vec{x})| \leqq C(\log|t|)^{2p}/(1+x^2)^p\ .$$

(It is assumed throughout that $|t| \geqq e$ so that $\log|t| \geqq 1$. This is just a convenience.) The inequality (127) is standard. (128) is obtained in a standard way referred to before, namely write

$$(1+x^2)^p(e^{-iA(t)}h)(\vec{x})$$

$$= \frac{1}{(2\pi)^{3/2}}\int(1+x^2)^p e^{i\vec{k}\cdot\vec{x}}$$

$$\cdot \exp\left(-i\epsilon(t)\frac{me_1e_2}{k}\log\left(\frac{2k^2|t|}{m}\right)\right) h^{\sim}(\vec{k})d\vec{k} \tag{129}$$

$$= \frac{1}{(2\pi)^{3/2}}\int\{(1-\Delta_{\vec{k}})^p e^{i\vec{k}\cdot\vec{x}}\}$$

$$\cdot \exp\left(-i\epsilon(t)\frac{me_1e_2}{k}\log\left(\frac{2k^2|t|}{m}\right)\right) h^{\sim}(\vec{k})d\vec{k}.$$

Now integrate by parts to shift the differential operator $(1-\Delta_{\vec{k}})^p$ away from $e^{i\vec{k}\cdot\vec{x}}$. All surface terms vanish because $h \in S$. Differential operators like $(1-\Delta_{\vec{k}})^p$ acting on

$$\exp\left(-i\epsilon(t)\frac{me_1e_2}{k}\log\left(\frac{2k^2|t|}{m}\right)\right)$$

bring down various powers of $1/k$, $\log k^2$, and $\log|t|$. (The highest power of $\log|t|$ will be the $2p$th.) Multiplied by $h^{\sim}(\vec{k})$, which vanishes in a neighborhood of $\vec{k}=0$, neither $1/k$ nor $\log k^2$ produces a singularity. Acting on $h^{\sim}(\vec{k})$, differentiation produces another C-function. All told, the result of the integration by parts is an equation of the form

$$(1+x^2)^p(e^{-iA(t)}h)(\vec{x})$$

$$= \frac{1}{(2\pi)^{3/2}}\int e^{i\vec{k}\cdot\vec{x}}\exp\left(-i\epsilon(t)\frac{me_1e_2}{k}\log\left(\frac{2k^2|t|}{m}\right)\right) \tag{130}$$

$$\cdot\sum_{\ell=0}^{2p}(\log|t|)^{\ell}\, h_{\ell}^{\sim}(\vec{k})d\vec{k}$$

with $h_{\ell}^{\sim}(\vec{k}) \in S$, $h_{\ell}^{\sim}(\vec{k})$ vanishing in a neighborhood of $\vec{k}=0$. (130) implies the existence of constants $C_1 \cdots C_{2p}$ such that

$$|(1+x^2)^p(e^{-iA(t)}h)(\vec{x})| \leqq \sum_{\ell=0}^{2p} C_{\ell}(\log|t|)^{\ell}. \tag{131}$$

Since $\log|t| \geqq 1$, there is a constant C such that

$$|(1+x^2)^p(e^{-iA(t)}h)(\vec{x})| \leqq C(\log|t|)^{2p}. \tag{132}$$

(132) is the desired inequality. Taking $p=2$ and using (127) and

(128) gives

$$\begin{aligned}(133)\qquad &\int |e^{imx^2/2t} - 1|^2\, |(e^{-iA(t)}h)(\vec{x})|^2 d\vec{x} \\ &\leqq (\log|t|)^8 \left(\frac{m}{2|t|}\right)^2 \int x^4 \frac{d\vec{x}}{(1+x^2)^8} = \frac{C^2(\log|t|)^8}{|t|^2}\end{aligned}$$

and this proves the lemma, since the first expression in (133) is the square of the norm we wanted to estimate: i.e.,

$$(134)\qquad \|e^{iH_{0c}(t)} h - C_t e^{iA(t)} h\| \leqq \frac{C(\log|t|)^n}{|t|}$$

where $n = 4$. (No use will be made of the value of n.)

From this lemma it follows immediately (using the unitarity of all operators involved and the fact that the C-functions are dense in L^2) that (124) (but not (134)) holds for all $H \in L^2$. From the definition (19) of C_t and the fact that

$$(135)\qquad (e^{-iA(t)}h)^{\sim}(\vec{k}) = \exp\left(-i\epsilon(t)\frac{me_1e_2}{k}\log\left(\frac{2k^2|t|}{m}\right)\right)h^{\sim}(\vec{k}),$$

it follows that

$$\begin{aligned}(136)\qquad &(C_t e^{iA(t)}h)(\vec{x}) \\ &= \left(\frac{m}{it}\right)^{3/2} e^{imx^2/2t} \exp\left(-i\epsilon(t)\frac{e_1e_2|t|}{|\vec{x}|}\log\left(\frac{2mx^2}{|t|}\right)\right)h^{\sim}\left(\frac{m\vec{x}}{t}\right).\end{aligned}$$

The point to notice in (136) is that except for the unusual phase factor, the right-hand side is exactly the same as $(C_th)(\vec{x})$, and that the absolute square of the right-hand side is

$$(137)\qquad |(C_t e^{-iA(t)}h)(\vec{x})|^2 = \left|\frac{m}{t}\right|^3 \left|h^{\sim}\left(\frac{m\vec{x}}{t}\right)\right|^2.$$

From (124) and (137) it follows that the ppd of $|(e^{-iH_{0c}(t)}h)(\vec{x})|^2$ can be replaced by $|m/t|^3 |h^{\sim}(m\vec{x}/t)|^2$ as $t \to \pm\infty$, i.e., the same result holds as in the case of a solution of the free Schrödinger equation.

In the future $e^{-iH_{0c}(t)}h - C_t e^{-iA(t)}h$ is often denoted by $R_h(t)$, R standing for "remainder".

We remark here, without proof, that similar techniques can be used to derive the estimate

$$\begin{aligned}(138)\qquad &\left\|\frac{me_1e_2}{x} R_h(t)\right\| \\ &\equiv \left\|\frac{me_1m_2}{x}(e^{-iH_{0c}(t)}h - C_t e^{-iA(t)}h)\right\| \leqq \frac{C(\log|t|)^n}{t^2}\end{aligned}$$

when h is a C-function. Here again, C is a constant and n a non-negative integer. Heuristically, the factor $V_c = e_1e_2/x$ behaves like $me_1e_2/(|t|(-\Delta)^{1/2})$ for large $|t|$. The factor $1/(-\Delta)^{1/2}$ does not cause any trouble, and an extra factor of $1/|t|$ multiplying the estimate of (134) is obtained. The theorem stated earlier that $e^{iH_0t}e^{-iH_{0c}(t)}$ converges strongly as $t \to \pm\infty$ on all of L^2 can now be proved. The technique is the same as in the earlier proof of the existence of $\Omega^{\pm}$: Let f be a C-function, and define

$$h(t) = e^{iH_ct}e^{-iH_{0c}(t)}f \,. \tag{139}$$

The function $h(t)$ is strongly differentiable, with strong derivative

$$\begin{aligned} h'(t) &= e^{iH_ct}\left(H_c - \left(H_0 + \frac{me_1e_2}{|t|(-\Delta)^{1/2}}\right)\right)e^{-iH_{0c}(t)}f \\ &= e^{iH_ct}\left(\frac{e_1e_2}{|\vec{x}|} - \frac{me_1m_2}{|t|(-\Delta)^{1/2}}\right)e^{-iH_{0c}(t)}f \,. \end{aligned} \tag{140}$$

Note that f is in the domain of $1/(-\Delta)^{1/2}$ because f is a C-function. The derivative $h'(t)$ is strongly continuous in t, if the point $t=0$ is avoided. It will be shown that $h(t)$ converges as $t \to +\infty$ by showing that

$$\int_e^{\infty} \|h'(t)\|dt < \infty \,. \tag{141}$$

(The lower limit e is chosen so that $\log|t| \geqq 1$ as before.) The case $t \to -\infty$ is handled in the same way, so only (141) will be proved. Now

$$\|h'(t)\| = \left\|\left(\frac{e_1e_2}{x} - \frac{me_1e_2}{|t|(-\Delta)^{1/2}}\right)e^{-iH_{0c}(t)}f\right\| . \tag{142}$$

To see why the right-hand side of (142) should be integrable with respect to t, recall that $e^{-iH_{0c}(t)}f$ behaves much like a solution of the free Schrödinger equation at large $|t|$, thus the norm of $(e_1e_2/x)e^{-iH_{0c}(t)}f$ should behave like $C/|t|$ at large t, in analogy with the behavior of the norm of $(1/x)e^{-iH_0t}$ (analyzed earlier). Likewise the norm of

$$\frac{me_1e_2}{|t|(-\Delta)^{1/2}}e^{-iH_{0c}(t)}f$$

is easily seen to behave like $C/|t|$ at large $|t|$. The point is that the two terms in (142) tend to cancel each other (recall the heuristic statement that asymptotically multiplying a solution of the free Schrödinger equation by $1/x$ is like multiplying it by $m/|t|(-\Delta)^{1/2}$) and the result is that the norm falls off faster than $1/|t|$, producing integrability.

Henceforth we take $t \geqq e$ and drop the absolute value signs on t. Consider first the function $(me_1e_2/t(-\Delta)^{1/2})e^{-iH_{0c}(t)}f$. The operator $1/(-\Delta)^{1/2}$ commutes with $e^{-iH_{0c}(t)}$, so that we have

$$\frac{me_1e_2}{t(-\Delta)^{1/2}}e^{-iH_{0c}(t)}f = \frac{me_1e_2}{t}e^{-iH_{0c}(t)}g \tag{143}$$

with

$$g = \frac{1}{(-\Delta)^{1/2}}f\,. \tag{144}$$

Recall that (144) merely means

$$g^{\sim}(\vec{k}) = f^{\sim}(\vec{k})/k. \tag{145}$$

Now

$$\frac{e_1e_2}{x}\,e^{-iH_{0c}(t)}f = \frac{e_1e_2}{x}\{C_te^{-iA(t)}f + R_f(t)\}\ , \tag{146}$$

and

$$\frac{me_1e_2}{t}\,e^{-iH_{0c}(t)}g = \frac{me_1e_2}{t}\,\{C_te^{-iA(t)}g + R_g(t)\}. \tag{147}$$

Because of (145), it follows from the definition of C_t that

$$\frac{e_1e_2}{x}\,C_te^{-iA(t)}f = \frac{me_1e_2}{t}\,C_te^{-iA(t)}g\ . \tag{148}$$

Thus

$$\begin{aligned} &\left\|\left(\frac{e_1e_2}{x} - \frac{me_1e_2}{t(-\Delta)^{1/2}}\right)e^{-iH_{0c}(t)}f\right\| \\ &\qquad = \left\|\frac{e_1e_2}{x}\,R_f(t) + \frac{me_1e_2}{t}\,R_g(t)\right\| \\ &\qquad \leqq \left\|\frac{e_1e_2}{x}\,R_f(t)\right\| + \left|\frac{me_1e_2}{|t|}\right|\,\|R_g(t)\|. \end{aligned} \tag{149}$$

But using (138) on the first norm on the right-hand side of (149) and (134) on the second gives

$$\|h'(t)\| = \left\|\left(\frac{e_1e_2}{x} - \frac{me_1e_2}{t}\right)e^{-iH_{0c}(t)}f\right\| \leqq \frac{C(\log t)^n}{|t|^2} \tag{150}$$

and (150) shows that $\|h'(t)\|$ is indeed integrable from e to ∞, proving the theorem.

The familiar results of scattering theory can now be established as before; namely,

$\Omega_c^{\pm}$ are isometric operators which satisfy the intertwining relations

$$e^{iH_c t}\,\Omega_c^{\pm} = \Omega_c^{\pm} e^{iH_0 t}\ . \tag{151}$$

The subspaces $R_c^{\pm} = \Omega_c^{\pm} L^2$ reduce H_c, *and* H_0 is unitarily equivalent to the part of H_c in $R_c^{\pm}$. Isometry of the operators $\Omega_c^{\pm}$ is clear. As before, all the later statements will follow if (151) is proved. It may seem strange that (151) should hold in view of the fact that the convergence of $e^{iHt}e^{-iH_0 t}$ has not been proved but only that of $e^{iHt}e^{-iH_0 t}e^{-iA(t)}$. The reason that the relations (151) still hold is best found in the proof itself. The proof of (151) is as follows:

$$\begin{aligned} e^{iH_c t}\,\Omega_c^{\pm} &= e^{iH_c t} \lim_{s\to\pm\infty} e^{iH_c s}\, e^{-iH_{0c}(s)} \\ &= \lim_{s\to\pm\infty} e^{iH_c (t+s)}\, e^{-iH_{0c}(s)}\ . \end{aligned} \tag{152}$$

Recall that

$$H_{0c}(s) = H_0 s + \epsilon(s)\frac{me_1e_2}{(-\Delta)^{1/2}} \log\left(\frac{-2|s|\Delta}{m}\right)\ . \tag{153}$$

Thus (assuming, as is eventually true, that neither s nor $s+t$ is zero, and that s and $s+t$ have the same sign)

$$H_{0c}(s) = H_{0c}(s+t) - H_0 t + L(s,t) \tag{154}$$

with

$$\begin{aligned} L(s,t) = \epsilon(s)&\ \frac{me_1e_2}{(-\Delta)^{1/2}} \log\left(\frac{-2|s|\Delta}{m}\right) \\ &-\epsilon(s+t)\ \frac{me_1e_2}{(-\Delta)^{1/2}} \log\left(\frac{-2|s+t|\Delta}{m}\right) \\ &= \epsilon(s)\ \frac{me_1e_2}{(-\Delta)^{1/2}} \left\{\log\left(\frac{-2|s|\Delta}{m}\right) - \log\left(\frac{-2|s+t|\Delta}{m}\right)\right\} \\ &= \epsilon(s)\ \frac{me_1e_2}{(-\Delta)^{1/2}} \log\left(\frac{s}{s+t}\right)\ . \end{aligned} \tag{155}$$

Clearly in some sense (best expressed, probably, by equation (157) below) $L(s,t)\to 0$, $s\to\infty$, an intuitive result which means that for large s and fixed t the difference between $H_{0c}(s+t)$ and $H_{0c}(s)$ is adequately represented by $H_0 t$, and the difference in the anomalous parts of $H_{0c}(s+t)$ and $H_{0c}(s)$ is asymptotically unimportant. This is what makes possible the proof of (151), which can now be given as follows: using (152) and (154) gives

$$
(156) \qquad \begin{aligned} e^{iH_ct}\,\Omega_c^{\pm} &= \lim_{s\to\pm\infty} e^{iH_c(t+s)}\, e^{-iH_{0c}(t+s)}\, e^{-iH_0t} e^{-iL(s,t)} \\ &= \Omega_c^{\pm}\, e^{-iH_0t} \ , \end{aligned}
$$

since as is easily shown

$$
(157) \qquad \operatorname*{s-lim}_{s\to\pm\infty} e^{-iL(s,t)} = I \ .
$$

Using the existence theorem for $\Omega_c^{\pm}$ it can now be seen that in fact the sequence $e^{iHt}\,e^{-iH_0t}$ does not converge as $t\to\pm\infty$. Namely, by using techniques similar to those employed in the proof, the following can be made rigorous:

$$
(158) \qquad \begin{aligned} e^{iH_ct}\;e^{-iH_0t} &= e^{iH_ct}\;e^{-iH_0t}\;e^{-iA(t)}\;e^{iA(t)} \\ &= e^{iH_ct}\;e^{-iH_{0c}(t)}\;e^{iA(t)} \underset{t\to\pm\infty}{\cong} \Omega_c^{\pm}\;e^{iA(t)} \ . \end{aligned}
$$

More precisely, it can be shown that for any $f \in L^2$ we have

$$
(159) \qquad \lim_{t\to\pm\infty} \|e^{iH_ct}\;e^{-iH_0t}f - \Omega_c^{\pm}e^{iA(t)}f\| = 0 \, .
$$

Let $g \in L^2$. Then by (159)

$$
(160) \qquad \lim_{t\to\pm\infty}\;(g,\,e^{iH_ct}\,e^{-iH_0t}f) = \lim_{t\to\pm\infty}(\Omega_c^{\pm *}g,\,e^{iA(t)}f) = 0
$$

because, as stated before, the operator $e^{iA(t)}$, like the operator e^{-iH_0t}, converges weakly to zero as $t\to\pm\infty$. From (160) it is seen that $e^{iH_ct}\;e^{-iH_0t}$ converges weakly to zero as $t\to\pm\infty$, and hence, being unitary, cannot converge strongly to anything.

If B_c denotes the set of bound states of the Hamiltonian H_c, then as before

$$
(161) \qquad B_c \perp R_c^{\pm}.
$$

It is the weak convergence to zero of $e^{-iH_{0c}(t)}$ as $t\to\pm\infty$ which is used in the proof of (161) just as in the proof of (77) $e^{-iH_0t} \rightharpoonup 0$ was used. Otherwise the proof of (161) is no different. However, even more is true; namely,

$$
(162) \qquad (\Omega_c^{\pm}f)(\vec{x}) = (\Omega_c'^{\pm}f)(\vec{x}) = \text{l.i.m.}\frac{1}{(2\pi)^{3/2}}\int \psi_{c\vec{k}}^{(\pm)}(\vec{x})f^{\sim}(\vec{k})d\vec{k} \ .
$$

It is not hard to see how an attempt to prove this equation should go. It is a consequence of Titchmarsh's work [27] on the Coulomb stationary state wave-functions $\psi_{c\vec{k}}^{(\pm)}(\vec{x})$ that $\Omega_c'^{\pm}$ is an isometric transformation of $f \in L^2$. In terms of the functions $\psi_{c\vec{k}}^{(\pm)}$, the operator $\Omega_c'^{\pm *}$ is given by

$$(\Omega_c'^{\pm *}f)^{\sim}(\vec{k}) = \mathrm{l.i.m}\frac{1}{(2\pi)^{3/2}}\int \overline{\psi_{c\vec{k}}^{(\pm)}(\vec{x})}f(\vec{x})d\vec{x} \ . \tag{163}$$

Now $\Omega_c^{\pm}$ is also an isometry. Thus (162) may be proved on a dense set and extended to L^2. Naturally, it is a good idea to take f to be a C-function, so the dense set will be taken to be the set of all C-functions. Because of the strong convergence of $e^{iH_c t}e^{-iH_{0c}(t)}f$ to $\Omega_c^{\pm}f$,

$$\begin{aligned}(\Omega_c'^{\pm *}\Omega_c^{\pm}f)^{\sim}(\vec{k}) &= \operatorname*{s-lim}_{t\to\pm\infty}(\Omega_c'^{*}e^{iH_c t}e^{-iH_{0c}(t)}f)^{\sim}(\vec{k}) \\ &= \operatorname*{s-lim}_{t\to\pm\infty}\mathrm{l.i.m.}\frac{1}{(2\pi)^{3/2}}\int\overline{\psi_{c\vec{k}}^{(\pm)}(\vec{x})}(e^{iH_c t}e^{-iH_{0c}(t)}f)(\vec{x})d\vec{x} \\ &= \lim_{t\to\pm\infty}\mathrm{l.i.m}\frac{1}{(2\pi)^{3/2}}\int\overline{e^{-ik^2t/2m}\psi_{c\vec{k}}^{(\pm)}(\vec{x})}(e^{-iH_{0c}(t)}f)(\vec{x})d\vec{x} \ .\end{aligned} \tag{164}$$

Here use has been made of the fact intuitively expressed by saying that $e^{-iH_c t}$ acting on $\psi_{c\vec{k}}^{(\pm)}$ is the same as $e^{-ik^2t/2m}$ acting on $\psi_{c\vec{k}}^{(\pm)}$ and nonintuitively expressed by writing

$$\Omega_c'^{\pm *}e^{-iH_c t} = e^{-iH_0 t}\Omega_c'^{\pm *} \ . \tag{165}$$

In any case, if f is a C-function, the l.i.m.'s in (163) can be erased. The problem is then relatively straightforward, namely to find the limit

$$\lim_{t\to\pm\infty}\frac{1}{(2\pi)^{3/2}}e^{ik^2t/2m}\int\overline{\psi_{c\vec{k}}^{(\pm)}(\vec{x})}\,(e^{-iH_{0c}(t)}f)(\vec{x})d\vec{x} \ . \tag{166}$$

Only relatively familiar things like $e^{-iH_{0c}(t)}f$ and some special functions appear in (164), so the problem is to do an integral and find its asymptotic value. In fact it can be shown that the pointwise limit of the expression in (164) is $f^{\sim}(\vec{k})$. Hence, since strong convergence takes place in (164) by construction, the expression in (164) converges strongly to $f^{\sim}$. Thus finally

$$\Omega_c'^{\pm *}\Omega_c^{\pm}f = f \tag{167}$$

for any C-function f and hence for any $f \in L^2$. Hence

$$\Omega_c'^{\pm}\Omega_c'^{\pm *}\Omega_c^{\pm}f = \Omega'^{\pm}f \ . \tag{168}$$

Now $\Omega_c'^{\pm}\Omega_c'^{\pm *}$ is the projection P_{R_c} on the common range R_c of $\Omega_c'^{+}$ and $\Omega_c'^{-}$. However, Titchmarsh's results, discussed before, imply that if P_{B_c} is the projection on the subspace B_c spanned by the bound states then

$$P_{B_c} + P_{R_c} = I, \qquad B_c \oplus R_c = L^2 \ . \tag{169}$$

Furthermore, (161) implies that

(170) $$P_{B_c}\Omega_c^{\pm} = 0 \ .$$

Thus using all these facts gives

(171) $$\Omega_c^{\pm} = (P_{B_c} + P_{R_c})\Omega_c^{\pm} = P_{R_c}\Omega_c^{\pm} = \Omega_c'^{\pm} \ ,$$

where the last equality is just (168). The identification (171) permits the identification of $R_c^{\pm}$ with R_c, whence (169) implies that the theory is asymptotically complete. (Of course, this sentence is intended to mean that the theory is asymptotically complete if the wave operators $\Omega_c^{\pm}$ are defined as done here, instead of as the strong limits of $e^{iH_ct}e^{-iH_0t}$, which do not exist.) It can now be seen what would have happened if the term $\epsilon(t)\log(-2\,\Delta/m)$ in the operator $H_{0c}(t)$ had been omitted (see equation (121)). An operator $\Omega_c^{x\pm}$ satisfying

(172) $$\Omega_c^{x\pm} = \Omega_c'^{\pm}\, e^{i\epsilon(t)\log(-2\,\Delta/m)}$$

would have been obtained. Equation (171) is clearly more appealing. Of course, the correction factor $\epsilon(t)\log(-2\,\Delta/m)$ was introduced precisely in order that (171) should hold, and in order to know what factor to put it is necessary to know all about the stationary state theory. One could therefore still argue that inserting this factor is "cheating". However, as will be clear presently, the presence or absence of this factor is not crucial to the interpretation of the time-dependent theory, so that its inclusion is somewhat a matter of taste.

A physical interpretation of the results can now be given. Namely, the asymptotic behavior of states of the form $e^{-iHt}\psi$, $\psi \in L^2$, can now be classified. Because of (169) it suffices to consider the two cases $\psi \in B_c$ and $\psi \in R_c$. If $\psi \in B_c$ then f is a bound state or a linear combination of bound states. If ψ is a bound state ψ_{cn}, then

(173) $$e^{-iH_ct}\psi_{cn} = e^{-iE_nt}\psi_{cn}$$

and the ppd given by $e^{-iHt}\psi$ never changes. The behavior of a more general $\psi \in B_c$ is easily deduced from (173). If $\psi \in R_c$, then by the theorem given above there exist elements $f_{\pm} \in L^2$ such that

(174) $$\lim_{t\to\pm\infty} \|e^{-iH_ct}\psi - e^{-iH_{0c}(t)}f_{\pm}\| = 0 \ .$$

In fact,

(175) $$f_{\pm} = \Omega_c^{\pm *}\psi \ .$$

The above analysis shows that there are no states ψ in L^2 for which $e^{-iH_ct}\psi$ eventually behaves like a solution of the free Schrödinger equation. Aside from the states $\psi \in B_c$, there are only the states with the "anomalous" asymptotic behavior given by (174). Because of this fact, we reformulate Coulomb scattering theory as follows: A state

$\psi \in L^2$ is said to describe a scattering experiment if there exist $f_\pm \in L^2$ such that (174) holds. For simplicity f_- is called the initial state of the experiment and f_+ the final state. One may visualize the experimenter as preparing the state initially in a good approximation of free motion, described by $e^{-iH_{0c}(t)} f_-$. Note that, as stated before, the ppd and mpd of this wave-function are identical to those of $e^{-iH_0 t} f_-$, so that in this sense the behavior of the particle is a good approximation of free motion. The experimenter then waits for scattering to occur, and finds a final state $e^{-iH_{0c}(t)} f_+$. The problem of scattering theory is, of course, to predict f_+ if one is given f_-. The answer is already known. It is

$$f_+ = S_c f_- \tag{176}$$

with

$$S_c = (\Omega_c^+)^* \Omega_c^- \ . \tag{177}$$

It can be seen at this point that the deletion of the term $\epsilon(t)\log(-2\,\Delta/m)$ from $H_{0c}(t)$ would not have changed the substance of this interpretation. Instead, it would merely have changed the "names" of all the initial and final states. Let $H_{0c}^x(t)$ be $H_{0c}(t)$ with the term $\epsilon(t)\log(-2\,\Delta/m)$ deleted. Then if all the theorems had been proved with $H_{0c}^x(t)$ instead of $H_{0c}(t)$, the interpretation would now read that at large negative times the experimenter prepares a state described by the wave-function $e^{-iH_{0c}^x(t)} f_-^x$. f_-^x would be called the initial state. To convert to the theory given above, one need only write

$$\begin{aligned} e^{-iH_{0c}^x(t)} f_-^x &= e^{-iH_{0c}^x(t)}\, e^{-i\epsilon(t)\log(-2\,\Delta/m)}\, e^{i\epsilon(t)\log(-2\,\Delta/m)} f_-^x \\ &= e^{-iH_{0c}(t)}\, e^{-i\log(-2\,\Delta/m)} f_-^x \ , \end{aligned} \tag{178}$$

where $\epsilon(t) = -1$ has been used since $t < 0$ initially. Then write

$$f_- = e^{-i\log(-2\,\Delta/m)} f_-^x \tag{179}$$

and call f_- the initial state. Hence the same physical situation would just be going by another name, so to speak, and the formulas would look slightly different for this reason, but the results would be the same. With this, an acceptable Coulomb scattering theory has been achieved. Several remarks are in order:

First, it is possible to compute the probability $P(f_-, C)$ that a Coulomb particle with initial state $e^{-iH_{0c}(t)} f_-$ eventually emerges in a cone C. Mimicking the previous discussion for short-range potentials and recalling that the ppd for $e^{-iH_{0c}(t)} f_-$ behaves asymptotically like that for a solution of the free Schrödinger equation, it is easy to deduce that

(180) $$P(f_-, C) = \int_C |(S_c f_-)^\sim(\vec{k})|^2 d\vec{k} \,,$$

as expected. It is also true but less easy to prove, that $P(f_-, C)$ is also correctly given by (69), i.e.,

(181) $$P(f_-, C) = \lim_{t \to +\infty} \int_C |(e^{-2iH_c t} e^{iH_0 t} f_-)(\vec{x})|^2 d\vec{x} \,.$$

Thus (69) actually provides a formula that works for both Coulomb and short-range potentials.

To close the section on Coulomb potential scattering, a few additional remarks are given. First, if V is a potential which is either square-integrable or else is locally square-integrable and bounded for large x by $C/x^{1+\beta}$, $\beta > 0$, then the Møller wave-matrices are known to exist for the Hamiltonian $H = H_0 + V$ [12], [19]. This has been shown for square-integrable V, and the proof extends to the other type mentioned. If V satisfies one of the two conditions stated, for the purposes of the present discussion V is called a "short-range" potential. The sum $V_c + V'$ of the Coulomb potential V_c and another potential V' is called "Coulomb-like" if V' is short range. Then the following theorem holds.

THEOREM. *Let*

(182) $$V = V_c + V'$$

be a Coulomb-like potential, and let

(183) $$H = H_0 + V \,.$$

Let $H_{0c}(t)$ be defined as before. Then the limit

(184) $$\operatorname*{s-lim}_{t \to \pm\infty} e^{iHt} e^{-iH_{0c}(t)} = \Omega^{\pm}$$

exists on all of L^2.

The proof of this theorem parallels closely the proof of the earlier theorem for the pure Coulomb potential. Essentially it shows that the addition of a short-range potential to the Coulomb potential produces no new complexities at large times. The old complexities are, however, very much there to haunt us.

Second, the kind of technique outlined here offers an example of a potential V for which the integral $\int_1^\infty \|Ve^{-iH_0 t} f\| dt$ diverges (at least for $f \in S$) while the usual Møller wave-matrices nevertheless exist. Put

(185) $$V(\vec{x}) = \frac{e_1 e_2 \sin x}{x} \,.$$

(The e_1e_2 is unnecessary, of course, and is inserted for comparison with the Coulomb case.) Then, as in the proof of (109), it can be shown that indeed $\int_1^\infty \|Ve^{-iH_0t}f\|dt$ diverges for $f \in S$. However, one can obtain a wave-operator as before by introducing an anomalous factor. Namely, define

$$H_0'(t) = H_0 t + \frac{me_1e_2}{(-\Delta)^{1/2}} \int_1^t \frac{\sin\{(-\Delta)^{1/2}|t'|/m\}}{|t'|} dt'. \tag{186}$$

(This formula holds for $t > 0$. The lower limit $t' = 1$ was chosen arbitrarily. Any number greater than zero could have been taken. For $t < 0$ a lower limit less than zero, say $t' = -1$, should be chosen.)

Mimicking the proof given above for the Coulomb potential, it is found that if $H = H_0 + V$, with V given by (185), then the limits

$$\underset{t\to\pm\infty}{\text{s-lim}}\ e^{iHt}\, e^{-iH_0'(t)} = \Omega'^{\pm} \tag{187}$$

exist on all of L^2. However, the integral on the right-hand side of (186) converges as $t \to +\infty$. (A similar statement holds as $t \to -\infty$.)

Write

$$I(\Delta) = \frac{me_1e_2}{(-\Delta)^{1/2}} \int_1^\infty \frac{\sin\{(-\Delta)^{1/2}|t'|/m\}}{|t'|} dt' . \tag{188}$$

Then it is not hard to show that

$$\underset{t\to+\infty}{\text{s-lim}} \left\{ \exp\left(-ime_1e_2 \int_1^t \frac{\sin\{(-\Delta)^{1/2}|t'|/m\}dt'}{|t'|} \right) - e^{iI(\Delta)} \right\} = 0. \tag{189}$$

But this implies that

$$\underset{t\to+\infty}{\text{s-lim}}\ \{e^{iHt}\, e^{-iH_0'(t)} - e^{iHt}\, e^{-iH_0t}\, e^{-iI(\Delta)}\} = 0 . \tag{190}$$

Thus by (187)

$$\underset{t\to+\infty}{\text{s-lim}}\ e^{iHt}\, e^{-iH_0t}\, e^{-iI(\Delta)} = \Omega'^{+} \tag{191}$$

and thus

$$\underset{t\to+\infty}{\text{s-lim}}\ e^{iHt}\, e^{-iH_0t} = \Omega'^{+}\, e^{iI(\Delta)} \equiv \Omega^{+} . \tag{192}$$

Of course, similar statements hold for $t \to -\infty$. Thus the usual strong limits $\lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t}$ do exist.

Third, one can write down a canonical guess[1] for the anomalous factor to be used to try to obtain the wave operators for a long-range potential V. Recall the heuristic argument for obtaining the anomalous

[1] NOTE ADDED IN PROOF. The canonical guess requires modification if the range of the potentials becomes too great. See [30].

factor in the Coulomb case: at large t we have

$$V(\vec{x})\{e^{-iH_0t}f\}(\vec{x}) \underset{t\to\pm\infty}{\cong} \{e^{-iH_0t}g\}(\vec{x}) \tag{193}$$

where

$$\tilde{g}(\vec{k}) = V\left(\frac{\vec{k}t}{m}\right)f^{\sim}(\vec{k}) \tag{194}$$

i.e.,

$$g = V\left(\frac{-i\vec{\nabla}t}{m}\right)f \quad \left(\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)\right). \tag{195}$$

(*Note*: in the Coulomb case $V(\vec{x}) = 1/x = 1/(x^2)^{1/2}$, whence

$$V\left(\frac{-i\vec{\nabla}t}{m}\right) = \frac{1}{(-\nabla^2 t^2/m^2)^{1/2}} = \frac{m}{(-\Delta)^{1/2}|t|} .)$$

The operator $-i\vec{\nabla}/m$ is called the "velocity operator" in quantum mechanics. This name is motivated by the fact that if ψ is the state of a particle then

$$\left(\psi, \frac{-i\vec{\nabla}}{m}\psi\right) = \int \frac{\vec{k}}{m}|\psi^{\sim}(\vec{k})|^2 d\vec{k} \tag{196}$$

and the right-hand side of (196) can be recognized as the expectation value of the momentum of the particle divided by m, i.e., the expectation value of the velocity. If this name is used for $-i\vec{\nabla}/m$, then (193) can be written

$$\begin{aligned} V(\vec{x})\, e^{-iH_0t} f &\underset{t\to\pm\infty}{\cong} e^{-iH_0t}\, V\left(\frac{-i\vec{\nabla}t}{m}\right)f = V\left(\frac{-i\vec{\nabla}t}{m}\right)e^{-iH_0t} f \\ &= V\,(\text{velocity operator times } t)\; e^{-iH_0t} f . \end{aligned} \tag{197}$$

Equation (197) states that asymptotically the position can be replaced by the velocity operator times time when acting on solutions of the free Schrödinger equation. Define

$$H_0'(t) = H_0 t + \int^t V\left(\frac{-i\vec{\nabla}t}{m}\right)dt' . \tag{198}$$

Then if $H = H_0 + V$ it may be expected that

$$\begin{aligned} \frac{d}{dt}(e^{iHt}\, e^{-iH_0'(t)}) &= e^{iHt}\left(H - \left(H_0 + V\left(\frac{-i\vec{\nabla}t}{m}\right)\right)\right)e^{-iH_0'(t)} \\ &= e^{iHt}\left(V(\vec{x}) - V\left(\frac{-i\vec{\nabla}t}{m}\right)\right)e^{-iH_0'(t)} . \end{aligned} \tag{199}$$

It is hoped that the two terms on the right-hand side of (199) will cancel against each other, and that the norm of this right-hand side will be integrable from 1 to ∞ and $-\infty$ to -1, proving convergence of $e^{iHt}e^{-iH_0'(t)}$ by the method of Cook. Naturally, all this requires a special analysis, and the above is merely a heuristic conjecture, not a proof. In fact, the above conjecture can be confirmed for potentials of the form $1/x^\beta$, $3/4 < \beta < 1$. Whether it can be pushed any further than this is, as far as the author knows, an open question. (Note added in proof: an answer is given in reference [30].) It is interesting to note that if one introduces an anomalous term of the form of the integral in (198) for a potential V for which Cook's original proof works (e.g. $V \in L^2$) then in general "faster" convergence is obtained for $e^{iHt}e^{-iH_0'(t)}$ than for $e^{iHt}e^{-iH_0 t}$ (applied to a function in S, say). That is, both $e^{iHt}e^{-iH_0'(t)}f$ and $e^{iHt}e^{-iH_0 t}f$ will tend to limits as $t \to \pm\infty$, but for $f \in$ S the rate of convergence is faster for $e^{iHt}e^{-iH_0'(t)}f$.

The basic idea of much of the theory above can be encapsulated in the sentence: "In scattering theory, position equals velocity times time"—i.e., it is often permissible, in determining the asymptotic behavior for $t \to \pm\infty$, to replace $\vec{x}$ by $-i\vec{\nabla}t/m$—and this produces by the techniques shown the anomalous factors for the Coulomb problem, etc.

IV. ***n*-body scattering problems.** "n-body problems" are problems which concern n particles. We begin with a word on the quantum-mechanical description of n particles. The description closely parallels that for one particle, and the discussion is shortened for this reason.

n nonrelativistic spinless quantum-mechanical particles of masses $m_1, \cdots, m_n$ are described by assigning to each $t \in R^1$ an element ψ_t of $L^2(R^{3n})$ such that $\|\psi_t\| = 1$. ψ_t is called the *state* or *wave function* of the particles at time t. (Note that now ψ_t has $3n$ variables. The first three, written $\vec{x}_1$, refer to the first particle, the second three, written $\vec{x}_2$, refer to the second particle, etc.) Introduce the Fourier transform $\psi_t^\sim$ by

$$(200)\quad \begin{aligned} &\psi_t^\sim(\vec{k}_1, \cdots, \vec{k}_n) \\ &= \text{l.i.m} \frac{1}{(2\pi)^{3n/2}} \int e^{-i\sum_{j=1}^n \vec{k}_j\cdot\vec{x}_j}\, \psi_t(\vec{x}_1, \cdots, \vec{x}_n)d\vec{x}_1 \cdots d\vec{x}_n , \end{aligned}$$

$$(201)\quad \begin{aligned} &\psi_t(\vec{x}_1, \cdots, \vec{x}_n) \\ &= \text{l.i.m} \frac{1}{(2\pi)^{3n/2}} \int e^{i\sum_{j=1}^n \vec{k}_j\cdot\vec{x}_j}\psi_t^\sim(\vec{k}_1, \cdots, \vec{k}_n)d\vec{k}_1 \cdots d\vec{k}_n . \end{aligned}$$

In partial interpretation of the wave-function, the following statements can be made: $|\psi_t(\vec{x}_1, \cdots, \vec{x}_n)|^2$ is the joint ppd at time t that particle

#1 should be at $\vec{x}_1, \cdots$, particle #n at $\vec{x}_n$.

$|\psi_t^{\sim}(\vec{k}_1, \cdots, \vec{k}_n)|^2$ is the joint mpd at time t that particle #1 should have momentum $\vec{k}_1, \cdots$, particle #n should have momentum $\vec{k}_n$.

The time development of ψ_t is described by requiring that ψ_t satisfy a Schrödinger equation:

$$\psi_t = e^{-iHt}\,\psi_0 \tag{202}$$

where H is a selfadjoint linear transformation on L^2 called the Hamiltonian for the particles. If the particles are alone in the universe *and do not exert any influence on one another,* then they are called "free". The Hamiltonian describing n free particles is a generalization of the Hamiltonian describing one free particle. It is denoted by H_0 and given by

$$H_0 \text{ "="} \sum_{j=1}^{n} \frac{-\Delta_j}{2m_j} \tag{203}$$

where Δ_j is the Laplacian in the variable $\vec{x}_j$, and m_j is the mass of the jth particle. Actually, (203) is not to be taken quite literally, as indicated by the quotation marks. The definition of H_0 is as follows: Let $D(H_0)$ denote the domain of H_0. Then

$$D(H_0) = \left\{ \psi \in L^2 \;\middle|\; \int \left| \sum_{j=1}^{n} \frac{k_j^2}{2m_j} \right|^2 |\psi^{\sim}(\vec{k}_1, \cdots, \vec{k}_n)|^2 \, d\vec{k}_1 \cdots d\vec{k}_n < \infty \right\} \tag{204}$$

and for $\psi \in D(H_0)$

$$(H_0\psi)^{\sim}(\vec{k}_1, \cdots, \vec{k}_n) = \sum_{j=1}^{n} \frac{k_j^2}{2m_j}\, \psi^{\sim}(\vec{k}_1, \cdots, \vec{k}_n)\,. \tag{205}$$

(There is no index n on the operator H_0 to recall how many particles are referred to. This should be clear from the context. We could write H_{0n}, but this symbol will be used to mean something else, as in (207) below.)

S is defined for functions of $3n$ variables in complete analogy with the definition for three variables. We find $S \subseteq D(H_0), f \in S \Longleftrightarrow f^{\sim} \in S$, etc., as before. If $\psi \in S$ then

$$\begin{aligned}
&(e^{-iH_0t}\,\psi)(\vec{x}_1, \cdots, \vec{x}_n) \\
(206) \quad &= \frac{1}{(2\pi)^{3n/2}} \int e^{i\Sigma_{j=1}^{n} \vec{k}_j\cdot\vec{x}_j}\, e^{-i\Sigma_{j=1}^{n} k_j^2 t/2m_j}\, \psi^{\sim}(\vec{k}_1, \cdots, \vec{k}_n)\, d\vec{k}_1 \cdots d\vec{k}_n \\
&\underset{(t \neq 0)}{=} \frac{(m_1 \cdots m_n)^{3/2}}{(2\pi i t)^{3n/2}} \int e^{i\Sigma_{j=1}^{n} m_j(\vec{x}_j - \vec{x}'_j)^2/2t}\, \psi(\vec{x}_1, \cdots, \vec{x}_n)\, d\vec{x}_1 \cdots d\vec{x}_n
\end{aligned}$$

so that the formulas for H_0 and e^{-iH_0t} are in complete analogy with those for the case of a single free particle. The notation

$$H_{0j} \text{“}=\text{”} -\Delta_j/2m_j \tag{207}$$

is used frequently below where (207) means that H_{0j} is the (selfadjoint) operation of multiplication by $k_j^2/2m_j$ after Fourier transformation.

(*Note:* The operator H_{0j} also has an interpretation on the space $L^2(R^3)$ as $-\Delta/2m_j$. H_{0j} is used frequently in this sense below without this fact being mentioned explicitly. The first instance of this is mentioned after (211), but thereafter it is assumed to be clear from the context whether H_{0j} is being considered as a selfadjoint linear transformation on $L^2(R^{3n})$ or $L^2(R^3)$. Similar ambiguities occur with other operators, as is noted below occasionally.)

The operators $H_{0j}, j = 1, \cdots, n$, all permute. The equation

$$H_0 = \sum_{j=1}^{n} H_{0j} \tag{208}$$

is literally true as an equation between operators on $L^2(R^{3n})$, and

$$e^{-iH_0t} = \prod_{j=1}^{n} e^{-iH_{0j}t} , \tag{209}$$

the order of the terms on the right-hand side of (209) being irrelevant.

If $\psi \in L^2$ has the special form

$$\psi(\vec{x}_1, \cdots, \vec{x}_n) = \psi_1(\vec{x}_1) \cdots \psi_n(\vec{x}_n) \tag{210}$$

with $\psi_k \in L^2(R^3)$, then $e^{-iH_0t}\,\psi$ takes the simple form

$$e^{-iH_0t}\,\psi = \{e^{-iH_{01}t}\,\psi_1\} \cdots \{e^{-iH_{0n}t}\,\psi_n\} . \tag{211}$$

(On the right-hand side of (211) the H_{0j} are considered as operators on $L^2(R^3)$ instead of $L^2(R^{3n})$.)

The operations F_j on S are defined by

$$\begin{aligned}(F_j\psi)(\vec{x}_1, \cdots, \vec{x}_{j-1}, \vec{k}_j, \vec{x}_{j+1}, \cdots, \vec{x}_n)\\ = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{k}_j\cdot\vec{x}_j}\,\psi(\vec{x}_1, \cdots, \vec{x}_n) d\vec{x}_j ,\end{aligned} \tag{212}$$

and can be extended to all of L^2 by continuity. (This method of defining F_i avoids some delicate questions concerning sets of measure 0.) F_j is the Fourier transformation in the jth variable; all the F_j's commute, and

$$F_1 \cdots F_n\psi = \psi^{\sim} . \tag{213}$$

Operators Q_{jt} and C_{jt} can now be defined in analogy to the one-particle case by

(214) $$(Q_{jt}\psi)(\vec{x}_1, \cdots, \vec{x}_n) = e^{imx_j^2/2t}\ \psi(\vec{x}_1, \cdots, \vec{x}_n)$$

and (for $t \neq 0$)

(215) $$(C_{jt}\psi)(\vec{x}_1, \cdots, \vec{x}_n) = \left(\frac{m_j}{it}\right)^{3/2} e^{imx_j^2/2t}(F_j\psi)\left(\vec{x}_1, \cdots, \frac{m\vec{x}_j}{t}, \cdots, \vec{x}_n\right).$$

Then from (206) we have for $t \neq 0$

(216) $$e^{-iH_0t} = C_{1t}Q_{1t} \cdots C_{nt}Q_{nt}$$

and it is easy to show that for any $\psi \in L^2$

(217) $$\lim_{t\to\pm\infty} \|e^{-iH_0t}\,\psi - C_{1t}C_{2t} \cdots C_{nt}\psi\| = 0$$

so that asymptotically the ppd determined by $e^{-iH_0t}\,\psi$ can be replaced by the absolute square of $C_{1t}C_{2t} \cdots C_{nt}\psi$:

(218) $$|(e^{-iH_0t}\psi)(\vec{x})|^2 \quad \text{can be replaced by} \quad \frac{(m_1 \cdots m_n)^3}{|t|^{3n}} \left| \psi^\sim \left(\frac{m_1\vec{x}_1}{t}, \cdots, \frac{m_n\vec{x}_n}{t}\right)\right|^2 \text{ when } t \to \pm\infty .$$

Formula (218) can be used to compute various quantities of interest. For instance, consider the probability $P^{\pm}_{\text{free}}$ $(\psi;\ C_1, \cdots, C_n)$ that at large positive or negative times particle 1 will be found in a cone C_1 in R^3, $\cdots$, particle n in a cone C_n in R^3, if the wave-function of the n particles is $e^{-iH_0t}\,\psi$. This probability is given by

(219) $$\begin{aligned} P^{\pm}_{\text{free}}(\psi; C_1, \cdots, C_n) &= \lim_{t\to\pm\infty} \int_{C_1\times\cdots\times C_n} |(e^{-iH_0t}\psi)(\vec{x}_1, \cdots, \vec{x}_n)|^2 d\vec{x}_1 \cdots d\vec{x}_n \\ &= \lim_{t\to\pm\infty} \int_{C_1\times\cdots\times C_n} \frac{(m_1m_2 \cdots m_n)^3}{|t|^{3n}} \left| \psi^\sim\left(\frac{m_1\vec{x}_1}{t}, \cdots, \frac{m_n\vec{x}_n}{t}\right)\right|^2 d\vec{x}_1 \cdots d\vec{x}_n \\ &= \int_{\pm C_1\times\cdots\times\pm C_n} |\psi^\sim(\vec{k}_1, \cdots, \vec{k}_n)|^2\, d\vec{k}_1 \cdots d\vec{k}_n \end{aligned}$$

where the last step follows from the change of variables $\vec{k}_j = m_j\vec{x}_j/t_j$, $j = 1, \cdots, n$. As before, $-C$ stands for the reflection of the cone C

through the origin. From (219) it is seen that as $t \rightarrow +\infty$ the required probability $P_{\text{free}}(\psi; C_1, \cdots, C_n)$ is just the probability that the various particles have their momenta in the appropriate cones. A similar statement holds as $t \rightarrow -\infty$.

Of course, the case that is of most interest is that in which the particles are not free. The particles can fail to be free in various ways. Each of the particles may, of course, interact with a fixed center of force, just as in the case of a single particle. However, in the case of n particles there is also the possibility that the particles may interact with each other. To describe a situation in which both these possibilities occur, a Hamiltonian of the form

$$H = H_0 + V \tag{220}$$

is considered where V is a multiplicative operator:

$$(V\psi)(\vec{x}_1, \cdots, \vec{x}_n) = V(\vec{x}_1, \cdots, \vec{x}_n)\psi(\vec{x}_1, \cdots, \vec{x}_n) \tag{221}$$

and $V(\vec{x}_1, \cdots, \vec{x}_n)$ is a real function of the form

$$V(\vec{x}_1, \cdots, \vec{x}_n) = \sum_{j=1}^{n} V_{0j}(\vec{x}_j) + \sum_{1 \leqq i < j \leqq n} V_{ij}(\vec{x}_i - \vec{x}_j) \tag{222}$$

where all the V_{ij}'s, $0 \leqq i < j \leqq n$, are also real functions. V_{0j} is thought of as representing the interaction of particle j with a fixed center of force, while for $1 \leqq i < j \leqq n$, V_{ij} is thought of as representing a mutual interaction of particles i and j. Note that V_{ij} is written as a function of $\vec{x}_i - \vec{x}_j$, the difference in the position of the particles i and j. This is in accord with a general physical principle saying that the interaction between two particles should not depend on their absolute positions in space but only on how far one is from the other and (possibly) in what direction. Kato [**20**] has shown that if each of the potentials V_{ij}, $0 \leqq i < j \leqq n$, can be written as the sum of a bounded function and a square-integrable (over R^3) function, then the operator H defined by (220) is selfadjoint with the same domain as H_0, in complete analogy with the one body theory. It is always assumed below that any potentials which occur are "Kato potentials" (i.e., they can be written as the sum of a bounded and a square-integrable function).

Of course, equation (222) is general enough to cover the case in which there are no fixed centers of force (set $V_{0j} = 0$, $j = 1, \cdots, n$) or no interaction between particles (set $V_{ij} = 0$, $1 \leqq i < j \leqq n$).

To begin with a simple but instructive case is considered; namely, that of two particles interacting with each other, with no fixed centers of force present. Then the Hamiltonian has the form

(223) $$H = \frac{-\Delta_1}{2m_1} + \frac{-\Delta_2}{2m_2} + V_{12}(\vec{x}_1 - \vec{x}_2) = H_{01} + H_{02} + V_{12}.$$

The fact that V_{12} depends only on the difference $\vec{x}_1 - \vec{x}_2$ of coordinates 1 and 2 is now exploited as follows: introduce the "relative" coordinate $\vec{x}$ and "center of mass" coordinate $\vec{X}$ just as in classical mechanics:

(224) $$\vec{x} = \vec{x}_1 - \vec{x}_2 \ , \qquad \vec{X} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2} \ .$$

(The reader should note that the Jacobian of the transformation (224) is 1.) A formal computation along with an examination of domains now shows that (Laplacians always mean the natural selfadjoint extensions of the differential operators)

(225) $$-\frac{\Delta_1}{2m_1} - \frac{\Delta_2}{2m_2} = -\frac{\Delta_{\vec{X}}}{2M} - \frac{\Delta_{\vec{x}}}{2\mu}$$

where

(226) $$M = m_1 + m_2 \ , \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \ .$$

The operators $\Delta_{\vec{X}}$ and $\Delta_{\vec{x}}$ permute. Moreover,

(227) $$H = -\Delta_{\vec{X}}/2M + H'$$

where

(228) $$H' = \frac{-\Delta_{\vec{x}}}{2\mu} + V(\vec{x}) \, .$$

Because $\Delta_{\vec{x}}$ permutes with $\Delta_{\vec{X}}$ and V is a function of $\vec{x}$ alone, $\Delta_{\vec{X}}$ permutes with H'. Hence, if $K_0 = -\Delta_{\vec{X}}/2M$, then

(229) $$H = K_0 + H'$$

and

(230) $$e^{-iHt} = e^{-iK_0 t} e^{-iH't} \ .$$

Apply (230) to a function $\psi \in L^2(R^6)$ of the form

(231) $$\psi(\vec{x}_1, \vec{x}_2) = f(\vec{X}) g(\vec{x}), \qquad f, g \in L^2(R^3) \ .$$

Then

(232) $$e^{-iHt}\psi = \{e^{-iK_0 t} f\}\{e^{-iH't} g\} \ .$$

(*Note.* in (232) H', as well as K_0, is being considered as an operator on $L^2(R^3)$. It is useful to have a name for operators like e^{-iHt}, $e^{-iK_0 t}$, $\cdots$. They are called "propagators" below. If the time-dependence of a function is obtained by acting on it with a propagator, the operation of acting on the function with the propagotor is called "propagating the function in time.") From (232) it is seen that the time-dependence of ψ can be computed by assigning a time-dependence to f and g separately: The function f of the center-of-mass variable is to be propagated by the operator $e^{-iK_0 t}$, i.e., as a solution of the free Schrödinger equation with mass M. The function g of the relative coordinate, on the other hand, is to be propagated by the operator $e^{-iH't}$. The time-developments of the two functions do not interfere with each other—if ψ is originally a product, as in (231), then it remains a product when the operator e^{-iHt} is applied.

The above facts are often expressed by physicists by saying that "the center of mass travels freely," while the behavior of the relative coordinate is influenced by the potential. Of course, since linear combinations of functions of the form (231) are dense in $L^2(R^6)$, the case analyzed above is "typical" and the findings can be generalized. The situation can be described economically by the tensor product notation

$$L^2(R^6) = L^2_X(R^3) \otimes L^2_x(R^3) \ , \tag{233}$$

(where the $\otimes$ in (233) means the closed tensor product) and correspondingly

$$e^{-iHt} = e^{-iK_0 t} \otimes e^{-iH't} \ . \tag{234}$$

To analyze e^{-iHt} it is only necessary to analyse $e^{-iK_0 t}$ and $e^{-iH't}$ separately. The operator $e^{-iK_0 t}$ describes free motion, which has already been analyzed. Moreover, the operator $e^{-iH't}$ also describes something which has already been analyzed, because the operator

$$H' = -\Delta_{\vec{x}}/2\mu + V(\vec{x}) \tag{235}$$

is formally identical with the Hamiltonian of a particle of mass μ interacting with a fixed center of force. It is this identification which causes physicists to think in terms of a "fictitious particle of mass μ" when discussing a two-body problem of the type described above. To exploit the knowledge of potential scattering gained in §2, write

$$H_{0x} = -\Delta_{\vec{x}}/2\mu \tag{236}$$

and assume that H' is a Hamiltonian giving rise to an asymptotically complete scattering theory, i.e., assume that

$$\lim_{t \to \pm\infty} e^{iH't} e^{-iH_{0x} t} = \Omega_x^{\pm} \tag{237}$$

exists, that the ranges $R_x^{\pm}$ of $\Omega_x^{\pm}$ are equal,

$$R_x^{\pm} \equiv R_x \tag{238}$$

and that R_x is the orthogonal complement in $L^2(R^3)$ of the subspace B_x spanned by the bound states of H':

$$L^2(R^3) = R_x \oplus B_x \ . \tag{239}$$

The behavior of two typical kinds of wave-functions in $L^2(R^6)$ is now investigated. First, take

$$\psi = f(\vec{X})g(\vec{x}) \tag{240}$$

where $f \in L^2(R^3)$ and $g \in R_x$. In this case if

$$g_{\pm} = \Omega_{\bar{x}}^{\mp *} g \ , \tag{241}$$

then

$$\lim_{t \to \pm \infty} \|e^{-iH't} g - e^{-iH_{0x}t} g_{\pm}\| = 0 \ . \tag{242}$$

(*Note*: In (242) it is the norm in $L^2(R^3)$ which is intended. Whenever a norm is written in the future, it is the norm over $L^2(R^{3K})$ for some K. The reader should be able to decide from the context which value of K is intended.)

Now

$$e^{-iHt} \psi = e^{-iK_0t} f e^{-iH't} g \tag{243}$$

and it follows from (241) that

$$\lim_{t \to \pm \infty} \|e^{\ iK_0t} f e^{-iH't} g - e^{-iK_0t} f e^{-iH_{0x}t} g_{\pm}\| = 0 \ . \tag{244}$$

Now (225) implies

$$\begin{aligned} K_0 + H_{0x} &\equiv -\Delta_{\bar{X}}/2M - \Delta_{\bar{x}}/2\mu = -\Delta_1/2m_1 - \Delta_2/2m_2 \\ &= H_{01} + H_{02} \equiv H_0 \ . \end{aligned} \tag{245}$$

Thus with a small amount of manipulation (243) can be rewritten so that it becomes

$$\lim_{t \to \pm \infty} \|e^{-iHt} \psi - e^{-iH_0t} f g_{\pm}\| = 0 \ . \tag{246}$$

This familiar-looking equation asserts that as $t \to \pm \infty$, $e^{-iHt} \psi$ approaches a solution of the free Schrödinger equation (for two particles). The same assertion extends to wave-functions which are linear combinations of ones of the special form (240) with $f \in L^2(R^3)$ and $g \in R_x$, and to strong limits of such combinations. In summary: If

$$\psi \in L_X^2(R^3) \otimes R_x \equiv \mathcal{S} \tag{247}$$

then there exist functions $F_{\pm} \in L^2(R^6)$ such that

$$\lim_{t \to \pm \infty} \|e^{-iHt} \psi - e^{-iH_0t} F_{\pm}\| = 0 \ . \tag{248}$$

If $\psi \in \mathcal{S}$, then asymptotically the two particles described by $e^{-iHt}\psi$ become free, each going its own way. It is easy to see using (229) and (245) that

$$e^{iHt}\, e^{-iH_0 t} = e^{i(K_0+H')t}\, e^{-i(K_0+H_{0x})t} \tag{249}$$

$$= e^{iH't}\, e^{-iH_{0x}t} \xrightarrow[t\to\pm\infty]{} \Omega_x^{\pm}\,, \tag{250}$$

the convergence of the operators in (250) (now all interpreted as acting in $L^2(R^6)$) being strong, because it was strong in $L^2(R^3)$. The operator $\Omega_x^{\pm}$ is an isometry mapping $L^2(R^6)$ onto $\mathcal{S}$.

As a second typical kind of wave-function in $L^2(R^6)$, consider

$$\psi = f(\vec{X})\psi_\delta(\vec{x}) \tag{251}$$

where $f \in L^2(R^3)$ and ψ_δ is one of the bound states for the operator H':

$$H'\psi_\delta = E_\delta \psi_\delta\,. \tag{252}$$

The index $\delta = 1, 2, \cdots, \delta_{\max} \leqq \infty$ is used to distinguish the various bound states of H'. In this case

$$\begin{aligned} e^{-iHt}\psi = e^{-i(K_0+H')t}\psi &= (e^{-iK_0 t}f)(\vec{X})(e^{-iH't}\psi_\delta)(\vec{x}) \\ &= (e^{-iK_0 t}f)(\vec{X})(e^{-iE_\delta t}\psi_\delta)(\vec{x})\,. \end{aligned} \tag{253}$$

From (253) it is seen that again the center-of-mass coordinate is propagated freely. The time-dependence of the function of the relative coordinate $\vec{x}$ is, however, trivial. If $|e^{-iHt}\psi|^2$ is regarded as providing the joint ppd for $\vec{X}$ and $\vec{x}$ instead of for $\vec{x}_1$ and $\vec{x}_2$ (this is clearly justified—either view is permissible as long as account is taken of the functional dependence of $e^{-iHt}\psi$ on the different variables) then (253) states that while the ppd for $\vec{X}$ behaves like that of a free particle of mass M, the ppd for $\vec{x}$ never changes. This represents a new asymptotic behavior of the particles, in which they drift away bound together, with their relative coordinate having the fixed ppd $|\psi_\delta(\vec{x})|^2$.

It is frequently convenient to rewrite the right-hand side of (253) as

$$(e^{-iK_0 t}f)(e^{-iE_\delta t}\psi_\delta) = e^{-iH_\delta t}(f\psi_\delta) \tag{254}$$

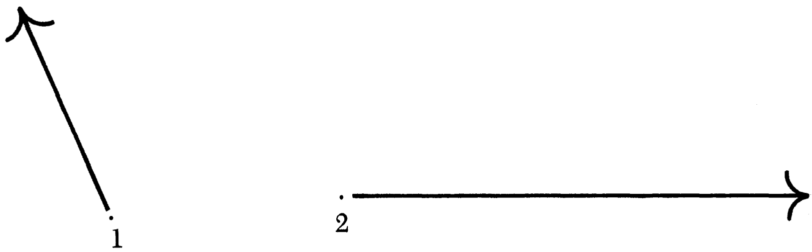
where, of course,

$$H_\delta = K_0 + E_\delta = -\Delta_{\vec{X}}/2M + E_\delta\,. \tag{255}$$

Thus the asymptotic behavior of the wave-function in the case at hand is governed by the operator $e^{-iH_\delta t}$. H_δ is called a "channel Hamiltonian" below.

From these two typical cases it is clear that pictorially at large times one of the following situations exists. (A linear superposition of the two is also possible.)

Case 1. $\psi \in \mathcal{S}$. Then the particles are far from everything, including each other:



This is the situation as it should appear near $t = +\infty$. Near $t = -\infty$, the particles should be far apart but getting closer as t increases.

Case 2.

$$\psi = f(\vec{X})\psi_{\delta'}(\vec{x}) , \qquad f \in L^2(R^3), \psi_{\delta'} \in B_x .$$

In this case, the particles stay together but are far from everything else:

The exact final behavior of the particles in Case 2 depends on which bound state $\psi_{\delta'}$ is chosen, so there are really a number of different cases covered by (2). One may think of an electron and a proton, which may travel separately (Case 1) or combine as a hydrogen atom and travel together (Case 2). (Of course this example has the defect that the particles interact through a Coulomb potential, rendering the discussion somewhat more complicated, but it is hoped that the physical picture is clear.)

A more complicated example—the problem of two bodies which interact not only with each other but also with fixed centers of force—is considered next. To describe this situation, write

(256) $$H = -\Delta_1/2m_1 - \Delta_2/2m_2 + V_1(\vec{x}_1-\vec{x}_2) + V_{01}(\vec{x}_1) + V_{02}(\vec{x}_2) .$$

The variables can still be changed to the pair $\vec{X}$, $\vec{x}$, but this does not bring about the same simplification it did before. Performing this change gives

$$H = -\frac{\Delta_{\vec{X}}}{2M} - \frac{\Delta_{\vec{x}}}{2\mu} + V_{12}(\vec{x}) \tag{257}$$
$$+ V_{01}\left(\vec{X} + \frac{m_2}{M}\vec{x}\right) + V_{02}\left(\vec{X} - \frac{m_1}{M}\vec{x}\right),$$

where

$$\vec{x}_1 = \vec{X} + \frac{m_2}{M}\vec{x}, \qquad \vec{x}_2 = \vec{X} - \frac{m_1}{M}\vec{x}. \tag{258}$$

Thus

$$H = K_0 + H' + V'' \tag{259}$$

where K_0 and H' are defined exactly as before (note, however, that H now stands for something new—namely the old H plus V''), and V'' represents the effect of the two scattering centers.

Now consider a two-body scattering process with the Hamiltonian H of (256). According to the general principles laid down in §I, any such process is described by a wave-function of the form $e^{-iHt}\psi_0$. However, as before, interest will center on the specification of scattering processes by the asymptotic behavior of the wave-function as $t \to \pm\infty$. Two broad classes of asymptotic scattering situations may be distinguished physically as follows: (*Note*: The purpose of the following discussion is to classify possible simple behaviors of the wave-function at large times. No attempt is being made to assert that if the wave-function has a certain behavior at large negative times it will continue to have such behavior at large positive times.)

Class A. Asymptotically, both particles are far from the fixed scattering centers (and hence are presumably unaffected by them). Note that it is not required that asymptotically the particles should be far from each other. In some situations this is so and in some it is not, as is shown below. A mathematical description of the asymptotic behavior of the wave-function in the Class A situation is developed next. Note that since both particles are supposed to be far from the fixed scattering centers asymptotically, the potentials V_{01} and V_{02} should eventually have little effect on them, and thus the wave-function describing them should eventually have time-dependence of the form $e^{-i(K_0+H')t}f$ with $f \in L^2(R^6)$. This is the form that the wave-function would have if $V_{01} = V_{02} = 0$—i.e., if there were no fixed potentials—hence this is presumably the form the wave-function should have asymptotically if the particles are far from the fixed potentials. However, even more information about the behavior of the particles can be given because it is assumed that $t \to \pm \infty$ and the behavior of functions of the form $e^{-i(K_0+H')t} f$ for such times has been analyzed. Indeed, if the scattering theory for H' is satisfactory (as is assumed), then there are various possible types of asymptotic behavior; namely, free motion, in which the wave-function asymptotically has the form $e^{-iH_0t} f$, with $f \in L^2(R^6)$ and

$$H_0 = -\Delta_1/2m_1 - \Delta_2/2m_2 , \tag{260}$$

and motions in which the particles travel together, in which the wave-function takes the form $e^{-iH_\delta t} g_\delta$, where g_δ belongs to the closed subspace D_δ of $L^2(R^6)$ defined by

$$D_\delta = \{g_\delta \in L^2(R^6) | g_\delta(\vec{x}_1, \vec{x}_2) = g(\vec{X})\psi_\delta(\vec{x})\} \tag{261}$$

where $\psi_\delta(\vec{x})$ is an eigenfunction of H' with eigenvalue E_δ. H_δ is of course defined by

$$H_\delta = -\Delta_{\vec{X}}/2M + E_\delta . \tag{262}$$

A linear combination of these situations is also possible. Thus in situations of Class A the wave-function should have the asymptotic form

$$e^{-iHt}\psi \sim e^{-iH_0t} f + \sum_{\delta=1}^{\delta \max} e^{-iH_\delta t} g_\delta , \tag{263}$$

with $g_\delta \in D_\delta$.

Class B. Asymptotically, one of the particles is far from the fixed scattering centers, while the other remains "trapped" in a neighborhood of the fixed potentials which interacts with it. In the discussion which follows it is assumed that it is particle 1 which is far from the scattering centers, while particle 2 is "trapped." (The discussion of the opposite case is of course similar.) Note that in the situation under discussion, since particle 2 is "near the scattering center V_{02}" and particle 1 is "far from the scattering centers," particle 1 is also far from particle 2. Therefore it is reasonable to assume that particle 1 is asymptotically not influenced by *any* of the potentials V_{01}, V_{02} or $V_{12}(\vec{x}_1-\vec{x}_2)$; i.e., particle 1 should behave asymptotically like a free particle. Particle 2, on the other hand, should not be influenced by the potential V_{12} of the distant particle 1—nor is it influenced by the fixed potential V_{01}, which pertains only to the particle 1. Thus particle 2 should behave as if it is alone except for the potential V_{02}—the time-dependence of particle 2 should then be governed by the Hamiltonian

(264) $$H_2' = -\Delta_2/2m_2 + V_{02}(\vec{x}_2) \ .$$

Further, by the statement that particle 2 is "trapped" by the potential V_{02} it is natural to mean that particle 2 is described by a wave-function of the form $(e^{-iH_2't}\psi_{2\beta})(\vec{x}_2)$, where $\psi_{2\beta}$ is an eigenfunction of H_2' with eigenvalue $E_{2\beta}$:

(265) $$H_2'\psi_{2\beta} = E_{2\beta}\psi_{2\beta} \ .$$

(These eigenfunctions are labelled by the index $\beta = 1, \cdots, \beta_{\max} \leqq \infty$.) Because of (265), we have of course

(266) $$e^{-iH_2't}\psi_{2\beta} = e^{-iE_{2\beta}t}\psi_{2\beta} \ .$$

The above considerations suggest the following asymptotic form for the wave-function in this case:

(267) $$e^{-iHt}\psi_0 \sim (e^{-iH_{01}t}f)(\vec{x}_1)(e^{-iE_{2\beta}t}\psi_{2\beta})(\vec{x}_2)$$

where H_{01} is the operator $-\Delta_1/2m_1$ as usual, and $e^{-iH_{01}t}f$ represents the free propagation of particle 1, while $e^{-iE_{2\beta}t}\psi_{2\beta}$ represents the "trapped" behavior of particle 2. Of course it is also possible to have a superposition of functions of the form on the right-hand side of (267), with different values of β. To simplify the notation, define

(268) $$D_{2\beta} = \{g \in L^2(R^6) | g(\vec{x}_1, \vec{x}_2) = f(\vec{x}_1)\psi_{2\beta}(\vec{x}_2)\}$$

where $\psi_{2\beta}$ is an eigenfunction of H_2' with eigenvalue $E_{2\beta}$ and $f \in L^2(R^3)$. Also set

(269) $$H_{2\beta} = H_{01} + E_{2\beta} \ .$$

Then the situation in which particle 1 is asymptotically far from the fixed potentials and particle 2 is asymptotically trapped by the fixed potential V_{02} is described by a wave-function with the following asymptotic form:

(270) $$e^{-iHt}\psi_0 \sim \sum_{\beta=1}^{\beta_{\max}} e^{-iH_{2\beta}t} g_{2\beta}$$

with $g_{2\beta} \in D_{2\beta}$.

To describe the situation in which the roles of the particles are reversed, define

(271) $$H_1' = -\Delta_1/2m_1 + V_{01}(\vec{x}_1) \ ,$$

(272) $$D_{1\gamma} = \{g \in L^2(R^6) | g(\vec{x}_1, \vec{x}_2) = \psi_{1\gamma}(\vec{x}_1)f(\vec{x}_2)\}$$

where $f \in L^2(R^3)$ and $\psi_{1\gamma}$ is an eigenfunction of H_1' with eigenvalue

$E_{1\gamma}$ $(\gamma = 1, \cdots, \gamma_{\max} \leqq \infty)$:

(273) $$H_1' \psi_{1\gamma} = E_{1\gamma} \psi_{1\gamma} \, .$$

Also set

(274) $$H_{1\gamma} = -\, \Delta_2/2m_2 + E_{1\gamma} = H_{02} + E_{1\gamma} \, .$$

Then the situation in which particle 1 is asymptotically "trapped" by the fixed potential V_{01} and particle 2 is asymptotically far from the fixed potentials is described by a wave-function with the asymptotic form

(275) $$e^{-iHt} \psi_0 \sim \sum_{\gamma=1}^{\gamma_{\max}} e^{-iH_{1\gamma} t} g_{1\gamma}$$

with $g_{1\gamma} \in D_{1\gamma}$.

Having listed the asymptotic situations which we believe to be characteristic of scattering states, it is now possible to say mathematically what is meant by a scattering experiment. The wave-function $e^{-iHt} \psi_0$ is said to describe a scattering experiment if as $t \to \pm\infty$ it converges strongly to a linear combination of functions of the forms (263), (270), and (275), i.e., if there exist functions $f^\pm \in L^2(R^6)$, $g_\delta \in D_\delta$ $(\delta = 1, \cdots, \delta_{\max})$, $g_{2\beta}^\pm \in D_{2\beta}$ $(\beta = 1, \cdots, \beta_{\max})$, $g_{1\gamma}^\pm \in D_{1\gamma}$ $(\gamma = 1, \cdots, \gamma_{\max})$, such that

(276) $$\lim_{t\to\pm\infty} \Bigg\| e^{-iHt} \psi_0 - e^{-iH_0 t} f^\pm - \sum_{\delta=1}^{\delta_{\max}} e^{-iH_\delta t} g_\delta^\pm - \sum_{\beta=1}^{\beta_{\max}} e^{-iH_{2\beta} t} g_{2\beta}^\pm - \sum_{\gamma=1}^{\gamma_{\max}} e^{-iH_{1\gamma} t} g_{1\gamma}^\pm \Bigg\| = 0 \, .$$

Of course equation (276) is complicated, and it is simpler to think of special cases of asymptotic behavior for the wave-function rather than the general case represented by (276). The special cases which are simplest are those in which only one term of the sum of asymptotic functions in (276) is nonzero, i.e., either $e^{-iHt} \psi$ converges to something of the form $e^{-iH_0 t} f$ or to something of the form $e^{-iH_\delta t} g_\delta$ or to one of the other terms in the sums in (275). A list can be made of possible simple asymptotic behaviors for $e^{-iHt} \psi$, starting with the "free" behavior $e^{-iH_0 t} f$, continuing through the entire collection of things of the form $e^{-iH_\delta t} g_\delta$, then the $e^{-iH_{2\beta} t} g_{2\beta}$, and finally the $e^{-iH_{1\gamma} t} g_{1\gamma}$. Each such possible simple asymptotic behavior is called a "channel" of the system. In order to specify a channel it is necessary to specify a propagator ($e^{-iH_0 t}$ or $e^{-iH_\delta t}$ or $\cdots$) and in the case of a propagator other than

e^{-iH_0t}, the bound state (ψ_δ or $\psi_{2\beta}$ or $\psi_{1\gamma}$) that goes with the propagator. (It may be that the Hamiltonian H' has more than one bound state with a particular eigenvalue $E_{\delta 0}$. In this case we will always take an *orthonormal* set of eigenfunctions $\psi_{\delta 1}$, $\psi_{\delta 2}$, $\cdots$, to define the corresponding channels, $\{e^{-iH_{\delta 0}t}, \psi_{\delta 1}\}$, $\{e^{-iH_{\delta 0}t}, \psi_{\delta 2}\}$, $\cdots$, all of whose propagators will, as indicated, be identical. One may ask whether or not it is reasonable to call the channels specified in this way "different". The answer is largely a matter of taste. Current taste is to emphasize the fact that the propagators are the same and say that the channels are the same, while not losing sight of the fact that the bound states are orthogonal. Here the channels will be called "different". Most future statements on channels hold good whichever convention is used. Exceptions are convention-dependent statements about the labelling of channels, etc. Similar remarks apply to the Hamiltonians H_1' and H_2'.) The operators H_0, H_δ, $H_{2\beta}$, $H_{1\gamma}$ are called "channel Hamiltonians."

It should be clear that some new notation is needed at this point. Note that since each of the operators H', H_1', H_2', has an at most countable number of bound states, the number of channels is at most countable. Introduce a multi-index α:

$$\alpha = (p, q) \tag{277}$$

where p is a channel Hamiltonian and q is a bound state that goes with the channel Hamiltonian.

Note: In the case of the channel Hamiltonian H_0, there is no bound state that "goes with it." Here q is taken to be 1 by definition.

Each α identifies a unique channel. Henceforth all operators and other objects arising in the theory are labelled with the appropriate index α. Thus H_α is the channel Hamiltonian for the channel α. D_α is the closed subspace of $L^2(R^6)$ appropriate to the channel (e.g., if p is H_δ and q is ψ_δ then D_α means D_δ of (261). If p is H_0 and q is 1, then by definition D_α means $L^2(R^6)$).

Note that $e^{-iH_\alpha t}$ maps D_α into itself. The advantages of using the α-notation are many. For instance, instead of writing the clumsy equation (276) one can now say that e^{-iHt} is a scattering state if for each α there exist functions $g_\alpha^\pm \in D_\alpha$ such that

$$\lim_{t\to\pm\infty} \left\| e^{-iHt}\psi_0 - \sum_\alpha e^{-iH_\alpha t} g_\alpha^\pm \right\| = 0 . \tag{278}$$

This is much more concise than the earlier notation, but it is a good idea not to forget what it means in detail.

The problem of scattering theory can now be stated. Of course the intuitive statement is that, given all the functions g_α^- in (278), all the functions g_α^+ should be identifiable—i.e., the behavior of the wave-function at large positive times should be predictable provided that the behavior at large negative times is given. As before, the problem is split into two parts:

Problem I. Given a collection $\{g_\alpha^-\}$, $g_\alpha \in D_\alpha$, such that

$$\sum_\alpha \|g_\alpha^-\|^2 < \infty \;, \tag{279}$$

show that there exists a $\psi_0 \in L^2$ such that

$$\lim_{t\to-\infty} \left\| e^{-iHt}\,\psi_0 - \sum_\alpha e^{-iH_\alpha t}\, g_\alpha^- \right\| = 0\,. \tag{280}$$

The condition (279) is intended to guarantee that $\sum_\alpha e^{-iH_\alpha t} g_\alpha^-$ has finite norm. However, it is not perfectly obvious that (279) guarantees this, because the subspaces D_α and hence the vectors g_α need not be orthogonal to each other. (The nonorthogonality of the D_α is most easily seen from the fact that for the "free" channel D_α is $L^2(R^6)$.) In order to show the finiteness of the norm of $\sum_\alpha e^{-iH_\alpha t} g_\alpha^-$ one can split the sum into four smaller sums, one (containing one term) for the free channel, one containing all channels where particle 1 is trapped and particle 2 is free, one for channels with particle 2 trapped and particle 1 free, and one for channels with the particles moving bound together, far from fixed force centers. In each of these smaller sums the D_α's involved are orthogonal (see the definition of these D_α's) and since $e^{-iH_\alpha t}$ maps D_α into itself, each smaller sum consists of orthogonal terms. The condition (279) is thus seen to guarantee a finite norm for $\sum_\alpha e^{-iH_\alpha t} g_\alpha^-$. In fact a small extension of the argument shows that (279) is a necessary and sufficient condition that the sum $\sum_\alpha e^{-iH_\alpha t} g_\alpha$ should converge unconditionally, i.e., independently of the order in which the sum is done. Eventually the sum in (279) should be required to have the value unity, because this gives $\|\psi_0\| = 1$. This should become clear later. (See eqs. (331) and (337) and the note on normalization shortly after eq. (33).)

It is not difficult to see that Problem I can be solved channel by channel. That is, Problem I can be posed equivalently as follows:

Problem I. Given any function $g_\alpha^- \in D_\alpha$, show that there exists a $\psi_0 \in L^2$ such that

$$\lim_{t\to-\infty} \|e^{-iHt}\,\psi_0 - e^{-iH_\alpha t} g_\alpha^-\| = 0\,. \tag{281}$$

The rest of the problem of scattering theory is, of course, this:

Problem II. Given ψ_0 from Problem I, show that there exist functions $\{g_\alpha^+\}$, $g_\alpha^+ \in D_\alpha$, such that $\sum_\alpha \|g_\alpha^+\|^2 < \infty$ and

$$\lim_{t\to+\infty}\left\|e^{-iHt}\psi_0-\sum_{\alpha}e^{-iH_\alpha t}g_\alpha^+\right\|=0\,. \tag{282}$$

Although it is possible to get rid of the infinite sum occurring in the formulation of Problem I, this is not possible in Problem II—the infinite sum is inevitable. The reason is that while one can (by experimental manipulation) select the initial state at will, and hence send in the particles in the channel α, after this is done there is no more choice—whatever happens will happen. Since the $e^{-iHt}\,\psi_0$ of Problem I is believed to represent a scattering state if the particles are sent in in channel α, as $t\to+\infty$ this state should have the form shown in (282)—but this is as far as one can go; "the worst" must be expected, i.e., a sum over all channels for a final state. Note that the most recent statement of Problem I is equivalent to the following:

***Problem* I. Prove that for any α, the strong limits**

$$\lim_{t\to-\infty}e^{iHt}\,e^{-iH_\alpha t}\equiv\Omega_\alpha^- \tag{283}$$

exist *on the subspace* D_α.

An example of sufficient conditions under which Problem I can be solved is provided by the following theorem due to Hack [13]:

THEOREM. *Suppose that* V_{01}, V_{02} *and* V_{12} *all belong to* $L^2(R^3)$. *Then for each channel index* α *the strong limits*

$$\lim_{t\to\pm\infty}e^{iHt}\,e^{-iH_\alpha t}\equiv\Omega_\alpha^\pm \tag{284}$$

exist on the subspace D_α.

PROOF. For the proof of this theorem one has to abandon the uniform notation and deal with various cases separately. Consider first the free channel ($\alpha=(H_0,1)$). Here the problem is to show that the strong limits

$$\lim_{t\to\pm\infty}e^{iHt}\,e^{-iH_0t}\equiv\Omega^\pm_{(H_0,1)} \tag{285}$$

exist on all of $L^2(R^6)$. As usual, the method of Cook is used. Let $f\in S$ and write

$$h(t)=e^{iHt}\,e^{-iH_0t}f\,. \tag{286}$$

Then $h(t)$ is strongly differentiable, and

$$\begin{aligned}h'(t)&=ie^{iHt}(H-H_0)\,e^{-iH_0t}f\\&=ie^{iHt}(V_{01}+V_{02}+V_{12})\,e^{-iH_0t}f\,.\end{aligned} \tag{287}$$

$h'(t)$ is strongly continuous, and the convergence proof can be completed in the usual way if it can be shown that $\|h'(t)\|$ is integrable over the intervals $(1, \infty)$ and $(-\infty, -1)$. Now

$$\begin{aligned} \|h'(t)\| &= \|(V_{01} + V_{02} + V_{12})e^{-iH_0t} f\| \\ &\leqq \|V_{01}\, e^{-iH_0t} f\| + \|V_{02}\, e^{-iH_0t} f\| + \|V_{12}\, e^{-iH_0t} f\| \ . \end{aligned} \tag{288}$$

Here, the terms can be estimated separately, the entire discussion going very much as in the case of potential scattering, with the difference that f now belongs to $S(R^6)$ instead of $S(R^3)$. This requires only minor modifications. In the term $\|V_{01}\, e^{-iH_0t} f\|$, one needs the estimate

$$|(e^{-iH_0t} f)(\vec{x}_1, \vec{x}_2)| \leqq \frac{g(\vec{x}_2)}{|t|^{3/2}}\,, \qquad (t \neq 0) \tag{289}$$

where $g \in L^2(R^3)$. Such an estimate is easy to obtain (using (206), for example), and once obtained it gives

$$\begin{aligned} \|V_{01}\, e^{-iH_0t} f\|^2 &= \int |V_{01}(\vec{x}_1)(e^{-iH_0t} f)(\vec{x}_1, \vec{x}_2)|^2\, d\vec{x}_1 d\vec{x}_2 \\ &\leqq \frac{1}{|t|^3} \int |V_{01}(\vec{x}_1)|^2\, |g(\vec{x}_2)|^2 d\vec{x}_1 d\vec{x}_2 = \frac{c}{|t|^3}\,, \end{aligned} \tag{290}$$

since $V_{01} \in L^2(R^3)$. Thus $\|V_{01}\, e^{-iH_0t} f\|$ falls off like $1/|t|^{3/2}$ and is integrable as desired. The term $\|V_{02}\, e^{-iH_0t} f\|$ is handled similarly. The term $\|V_{12} e^{-iH_0t} f\|$ is slightly different because V_{12} is a function of $\vec{x}_1 - \vec{x}_2$, but after changing variables to $\vec{x}$, $\vec{X}$ all goes as before, and this finishes the proof of (285) (since convergence has been proved on S, which is dense in L^2) and the discussion of the free channel.

Consider next the case of the channels in which both particles are far from the fixed centers of force, but close to each other. That is, consider the case of

$$\alpha = (H_\delta, \psi_\delta) \tag{291}$$

with

$$H_\delta = -\ \Delta_{\vec{X}}/2M + E_\delta = K_0 + E_\delta \tag{292}$$

and

$$D_\alpha = D_\delta = \{g_\delta \in L^2(R^6) | g_\delta(\vec{x}_1, \vec{x}_2) = g(\vec{X})\psi_\delta(\vec{x})\} \tag{293}$$

with $g \in L^2(R^3)$ and

$$H'\psi_\delta = E_\delta \psi_\delta \ . \tag{294}$$

It is necessary in this case to prove that the limits

$$\lim_{t \to \pm\infty} e^{iHt} e^{-iH_\delta t}\, g_\delta \equiv \Omega_\alpha^\pm g_\delta \tag{295}$$

exist for $g_\delta = g\psi_\delta$ with any $g \in L^2(R^3)$. To do this it suffices to give the proof with $g \in S(R^3)$. Therefore let $g \in S(R^3)$ and set

$$h(t) = e^{iHt}\, e^{-iH_\delta t}\, g_\delta\,. \tag{296}$$

Then $h(t)$ is strongly differentiable and use of (259) gives

$$\begin{aligned} h'(t) &= ie^{iHt}(H - H_\delta)\, e^{-iH_\delta t}\, g\psi_\delta \\ &= ie^{iHt}(K_0 + H' + V'' - (K_0 + E_\delta))\, e^{-iH_\delta t}\, g\psi_\delta\,. \end{aligned} \tag{297}$$

Now the operator H' pertains only to the coordinate $\vec{x}$ and thus permutes with $e^{-iH_\delta t}$ and acts on ψ_δ, "paying no attention" to g. Thus by (294)

$$(H' - E_\delta)\, e^{-iH_\delta t}\, g\psi_\delta = 0 \tag{298}$$

and hence

$$h'(t) = ie^{iHt}\, V''\, e^{-iH_\delta t}\, g_\delta = ie^{iHt}(V_{01} + V_{02})\, e^{-iH_\delta t}\, g\psi_\delta\,. \tag{299}$$

$h'(t)$ is strongly continuous. Also

$$\begin{aligned} \|h'(t)\| &= \|(V_{01} + V_{02})\, e^{-iH_\delta t}\, g\psi_\delta\| \\ &\leqq \|V_{01}\, e^{-iH_\delta t}\, g\psi_\delta\| + \|V_{02}\, e^{-iH_\delta t}\, g\psi_\delta\|\,. \end{aligned} \tag{300}$$

It will be shown how to estimate the first term on the right-hand side of (300), the second being little different. Using the expression (292) for H_δ and noting that the factor $e^{-iE_\delta t}$ cancels out on taking absolute values, gives

$$\begin{aligned} &\|V_{01}\, e^{-iH_\delta t}\, g\psi_\delta\|^2 \\ &\quad = \int \left| V_{01}\left(\vec{X} + \frac{m_2\vec{x}}{M}\right)\right|^2 |(e^{-iK_0 t}g)(\vec{X})|^2\, |\psi_\delta(\vec{x})|^2 d\vec{x}\, d\vec{X}\,. \end{aligned} \tag{301}$$

Now $e^{-iK_0 t}\, g$ is a solution of the free Schrödinger equation with $g \in S(R^3)$, so by the usual estimates

$$|(e^{-iK_0 t}\, g)(\vec{X})| \leqq c/|t|^{3/2}\,. \tag{302}$$

Thus

$$\begin{aligned} &\|V_{01}\, e^{-iH_\delta t} g\psi_\delta\|^2 \\ &\quad \leqq \frac{c^2}{|t|^3} \int \left| V_{01}\left(\vec{X} + \frac{m_2\vec{x}}{M}\right)\right|^2 |\psi_\delta(\vec{x})|^2 d\vec{x}\, d\vec{X}\,. \end{aligned} \tag{303}$$

Doing the $\vec{X}$-integral first in (303) and then the $\vec{x}$-integral, the right-hand side of (303) is seen to be just $(c^2/|t|^3)\ \|V_{01}\|^2\ \|\psi_\delta\|^2$, the norms being in $L^2(R^3)$. Hence

(304) $$\|V_{01}\, e^{-iH_\delta t} g\psi_\delta\| \leqq c'/|t|^{3/2} .$$

Combining (304) with the similar estimate for the second norm on the right-hand side of (300), it is seen that $\|h'(t)\|$ is integrable with respect to t over $(1, \infty)$ and $(-\infty, -1)$, proving convergence of $h(t)$ by Cook's method when $g \in S(R^3)$ and hence for any $g \in L^2(R^3)$. This completes the proof of the theorem when $\alpha = (H_\delta, \psi_\delta)$.

The remaining parts of the theorem deal with the channels in which there are asymptotically one free and one trapped particle, i.e., $\alpha = (H_{2\beta}, \psi_{2\beta})$ or $\alpha = (H_{1\gamma}, \psi_{1\gamma})$. The reader should be able to give these proofs by the methods used above and they are not given here. Thus we may now consider the theorem as proved in its entirety.

The theorem just proved provides operators $\Omega_\alpha^\pm$ defined on the domains D_α. In order to be able to speak of adjoints, it is convenient to extend the domain of $\Omega_\alpha^\pm$ to all of $L^2(R^6)$ by setting $\Omega_\alpha^\pm$ equal to zero on the orthogonal complement $D_\alpha^\perp$ of D_α.

(305) $$\Omega_\alpha^\pm f \equiv 0 , \qquad f \in D_\alpha^\perp .$$

Assume that this has been done, while remembering that the existence of the limit of $e^{iHt}\, e^{-iH_\alpha t}$ has only been proved on D_α. Denote by $R_\alpha^\pm$ the range of $\Omega_\alpha^\pm$:

(306) $$R_\alpha^\pm = \Omega_\alpha^\pm L^2 = \Omega_\alpha^\pm D_\alpha .$$

$R_\alpha^\pm$ are closed subspaces of L^2. In addition the subspaces R_α^+ are pairwise orthogonal, as are the subspaces R_α^-. As an example of how this can be shown, take the case in which α_1 is the free channel $(H_0, 1)$ and α_2 is a channel $(H_{2\beta}, \psi_\beta)$ in which particle 1 is asymptotically free and particle 2 is bound in the state ψ_β. A vector in $R_{\alpha_1}^+$ has the form $\Omega_{\alpha_1}^+ f$, $f \in L^2(R^6)$. A vector in $R_{\alpha_2}^+$ has the form $\Omega_{\alpha_2}^+ g_{\alpha_2}$, where

(307) $$g_{\alpha_2}(\vec{x}_1, \vec{x}_2) = f(\vec{x}_1)\psi_{2\beta}(\vec{x}_2)$$

with $f \in L^2(R^3)$. $H_{\alpha 2}$ has the form

(308) $$H_{\alpha_2} = -\Delta_1/2m_1 + E_{2\beta} = H_{01} + E_{2\beta} .$$

Now

(309) $$\begin{aligned}(\Omega_{\alpha_1}^+ f, \Omega_{\alpha_2}^+ g_{\alpha_2}) &= \lim_{t\to+\infty} (e^{iHt}\, e^{-iH_{\alpha_1}t} f, e^{iHt}\, e^{-iH_{\alpha_2}t}\, g_{\alpha_2}) \\ &= \lim_{t\to+\infty} (f, e^{iH_{\alpha_1}t}\, e^{-iH_{\alpha_2}t}\, g_{\alpha_2}) .\end{aligned}$$

Moreover, H_{α_1} is H_0:

(310) $$H_{\alpha_1} = -\Delta_1/2m_1 - \Delta_2/2m_2 = H_{01} + H_{02}$$

so that

$$(\Omega_{\alpha}^{+}f, \Omega_{\alpha_2}^{+}g_{\alpha_2}) = \lim_{t\to+\infty} e^{-iE_{2\beta}t}(f, e^{iH_{02}t}g_{\alpha_2}) = 0 \ , \tag{311}$$

where the last equality in (308) holds because $e^{iH_{02}t}$ converges weakly to zero as $t \to +\infty$. This shows orthogonality of $R_{\alpha_1}^{+}$ and $R_{\alpha_2}^{+}$. The proofs of cases other than the one considered are similar. (In the case of channels with the *same* channel Hamiltonian, resulting from degeneracy of an eigenvalue for a Hamiltonian such as H', orthogonality of the corresponding subspaces R_{α}^{+} (or R_{α}) results from the fact that an orthonormal set of eigenfunctions was used to define the corresponding channels. See the discussion of this point after the definition of "channels," in the paragraph preceding equation (277). Also in the case, say, of $H_{2\beta}$ and $H_{2\beta'}$, $E_{\beta} \neq E_{\beta'}$, the bound states are orthogonal.)

Remarks similar to the ones made at this point in the case of potential scattering can now be made. Writing E_{α} and $F_{\alpha}^{\pm}$ for the projections on the subspaces D_{α} and $R_{\alpha}^{\pm}$, the fact that $\Omega_{\alpha}^{\pm}$ is the strong limit of unitary operators on D_{α} can be used to conclude that $\Omega_{\alpha}^{\pm}$ is a partial isometry with initial set D_{α} and final set $R_{\alpha}^{\pm}$, so that

$$\Omega_{\alpha}^{\pm *}\Omega_{\alpha}^{\pm} = E_{\alpha} \ , \qquad \Omega_{\alpha}^{\pm}\Omega_{\alpha}^{\pm *} = F_{\alpha}^{\pm} \ . \tag{312}$$

Further

$$\Omega_{\alpha}^{\pm} E_{\alpha} = \Omega_{\alpha}^{\pm} \ , \qquad E_{\alpha}\Omega_{\alpha}^{\pm *} = \Omega_{\alpha}^{\pm *} \ , \tag{313}$$

and

$$F_{\alpha}^{\pm}\Omega_{\alpha}^{\pm} = \Omega_{\alpha}^{\pm} \ , \qquad \Omega_{\alpha}^{\pm *} F_{\alpha}^{\pm} = \Omega_{\alpha}^{\pm *} \ . \tag{314}$$

From (314) one can deduce the identity

$$\begin{aligned} \Omega_{\alpha}^{\pm *}\Omega_{\alpha'}^{\pm} &= \Omega_{\alpha}^{\pm *} F_{\alpha}^{\pm}F_{\alpha'}^{\pm}\Omega_{\alpha'}^{\pm} = \Omega_{\alpha}^{\pm *}\delta_{\alpha\alpha'}F_{\alpha}^{\pm}\Omega_{\alpha'}^{\pm} \\ &= \delta_{\alpha\alpha'}\Omega_{\alpha}^{\pm *}\Omega_{\alpha'}^{\pm} = \delta_{\alpha\alpha'} E_{\alpha} \ , \end{aligned} \tag{315}$$

where $F_{\alpha}^{\pm}F_{\alpha'}^{\pm}$ has been replaced by $\delta_{\alpha\alpha'}F_{\alpha}^{\pm}$ because the operators F_{α}^{+} are a family of pairwise orthogonal projections, as are the F_{α}^{-}. Of course, equation (315) must be read taking either all plus signs or all minus signs, but not a mixture of the two. The intertwining relations

$$e^{iHt}\Omega_{\alpha}^{\pm} = \Omega_{\alpha}^{\pm} e^{iH_{\alpha}t} \tag{316}$$

are proved just as in the case of potential scattering. (316) implies

$$\Omega_{\alpha}^{\pm *} e^{iHt} = e^{iH_{\alpha}t}\Omega_{\alpha}^{\pm *} \ . \tag{317}$$

Multiplying (317) on the left by $\Omega_{\alpha}^{\pm}$ and using (316) gives

$$F_{\alpha}^{\pm} e^{iHt} = e^{iHt} F_{\alpha}^{\pm} \ , \tag{318}$$

whence by Stone's Theorem

$$F_\alpha^\pm H \subseteq HF_\alpha^\pm \ , \tag{319}$$

i.e., $R_\alpha^\pm$ reduces H.

Equation (316) can also be differentiated. In fact (316) and Stone's Theorem imply that if f belongs to the domain $D(H_\alpha)$ of H_α then $\Omega_\alpha^\pm f$ belongs to the domain $D(H)$ of H and $H\Omega_\alpha^\pm f = \Omega_\alpha^\pm H_\alpha f$, so that

$$H\Omega_\alpha^\pm \supseteq \Omega_\alpha^\pm H_\alpha \ . \tag{320}$$

If $\Omega_\alpha^\pm f \in D(H)$ it follows that the left-hand side and therefore the right-hand side of (316) is strongly differentiable. This does not prove that $f \in D(H_\alpha)$, though, unless $\Omega_\alpha^\pm$ preserves the norm of $e^{-iH_\alpha t}f$. This happens only if $f \in D_\alpha$, by the definition of $\Omega_\alpha^\pm$ (and the fact, mentioned previously, that $e^{-iH_\alpha t}$ maps D_α into itself). Thus if $f \in D_\alpha$ and $\Omega_\alpha^\pm f \in D(H)$, then $f \in D(H_\alpha)$ and $H\Omega_\alpha^\pm f = \Omega_\alpha^\pm H_\alpha f$. Multiplying (320) on the right by the projection E_α on D_α gives

$$H\Omega_\alpha^\pm E_\alpha \supseteq \Omega_\alpha^\pm H_\alpha E_\alpha \ . \tag{321}$$

Suppose f belongs to the domain of the operator on the left-hand side of (321). This means that $\Omega_\alpha^\pm(E_\alpha f) \in D(H)$. By definition $E_\alpha f \in D_\alpha$. By the previous discussion this means $E_\alpha f \in D(H_\alpha)$ and $H\Omega_\alpha^\pm(E_\alpha f) = \Omega_\alpha^\pm H_\alpha(E_\alpha f)$. Hence equality holds in (321):

$$H\Omega_\alpha^\pm E_\alpha = \Omega_\alpha^\pm H_\alpha E_\alpha \tag{322}$$

and because of (313) the E_α on the left-hand side of (322) can be deleted:

$$H\Omega_\alpha^\pm = \Omega_\alpha^\pm H_\alpha E_\alpha \ . \tag{323}$$

Now use (314) to insert $F_\alpha^\pm$ in front of $\Omega_\alpha^\pm$ on the left-hand side of (323) and multiply (323) on the left by $\Omega_\alpha^{\pm *}$. This gives

$$\Omega_\alpha^{\pm *}(HF_\alpha^\pm)\Omega_\alpha^\pm = \Omega_\alpha^{\pm *}\Omega_\alpha^\pm H_\alpha E_\alpha = E_\alpha H_\alpha E_\alpha = H_\alpha E_\alpha \ . \tag{324}$$

The last equality in (324) holds because D_α reduces H_α (recall that $e^{-iH_\alpha t}$ maps D_α into itself).

Considered as a map from the Hilbert space D_α to the Hilbert space $R_\alpha^\pm$, $\Omega_\alpha^\pm$ is unitary. Thus equation (324) states this:

The part of H_α in D_α is unitarily equivalent to the part of H in $R_\alpha^\pm$.

Denote by $R^\pm$ the orthogonal direct sum of the subspaces $R_\alpha^\pm$:

$$R^\pm = \bigoplus_\alpha R_\alpha^\pm \ . \tag{325}$$

With the information now collected some additional remarks on Problems I and II can be made. The theorem just proved shows that

Problem I is solvable and that given $g_\alpha^- \in D_\alpha$ there is exactly one ψ_0 such that

$$\lim_{t\to-\infty} \|e^{-iHt}\psi_0 - e^{-iH_\alpha t}\, g_\alpha^-\| = 0 \ , \tag{326}$$

namely

$$\psi_0 = \Omega_\alpha^- g_\alpha^- \ . \tag{327}$$

Now consider Problem II. It is not difficult to see that $e^{-iHt}\,\psi_0$ will satisfy

$$\lim_{t\to+\infty} \left\| e^{-iHt}\,\psi_0 - \sum_\alpha e^{-iH_\alpha t}\, g_\alpha^+ \right\| = 0 \tag{328}$$

where $g_\alpha^+ \in D_\alpha$ and $\sum_\alpha \|g_\alpha^+\|^2 < \infty$, if and only if $\psi_0 \in R^+$. For if (328) holds then by our theorem (extended to the case of an infinite sum)

$$\psi_0 = \sum_\alpha \Omega_\alpha^+ g_\alpha^+ \in R^+ \ . \tag{329}$$

Applying $\Omega_{\alpha'}^{+*}$ on the left in (329) and using (315) gives

$$\Omega_{\alpha'}^{+*}\psi_0 = \sum_\alpha \Omega_{\alpha'}^{+*}\Omega_\alpha^+ g_\alpha^+ = \sum_\alpha \delta_{\alpha\alpha'} E_\alpha g_\alpha^+ = E_{\alpha'} g_{\alpha'}^+ = g_{\alpha'}^+ \ . \tag{330}$$

Thus if ψ_0 satisfies the condition (328) then $\psi_0 \in R^+$ and the g_α's are uniquely determined by (330). From (330) it also follows that

$$\begin{aligned} \sum_\alpha \|g_\alpha^+\|^2 &= \sum_\alpha \|\Omega_\alpha^{+*}\psi_0\|^2 \\ &= \sum_\alpha \|\Omega_\alpha^+\Omega_\alpha^{+*}\psi_0\|^2 = \sum_\alpha \|F_\alpha^+\psi_0\|^2 = \|\psi_0\|^2 \ . \end{aligned} \tag{331}$$

In deriving this the facts that Ω_α^+ is isometric on D_α and $\Omega_\alpha^{+*}\psi_0 \in D_\alpha$ were used, and also the fact that ψ_0 belongs to R^+ and is thus the sum of the orthogonal parts $F_\alpha^+\psi_0$.

If $\psi_0 \in R^+$ then defining $g_{\alpha'}^+$ by (330) it is easy to derive (328).

For Problem II to be solvable means by definition that any ψ_0 obtained from Problem I satisfies (328), i.e., belongs to R^+. But the ψ_0's obtained from Problem I have the form (327) and thus belong to R^-. If each ψ_0 of the form (327), for any α, belongs to R^+, then clearly $R^- \subseteq R^+$.

Just as in the case of potential scattering it can be shown that R^- consists precisely of the set $(R^+)^{cc}$ of complex conjugates of things in R^+, and thus that $R^- \subseteq R^+ \Rightarrow R^- = R^+$. Thus Problem II is solvable if and only if $R^- = R^+$.

Assume that $R^- = R^+$. Then the solution to the problem of scattering theory can be expressed in terms of the operators $\Omega_\alpha^\pm$.

For if the wave-function behaves like $e^{-iH_\beta t} g_\beta^-$, $g_\beta^- \in D_\beta$, (β is now being used as a multi-index like α) as $t \to -\infty$, this means that for all times the wave-function is $e^{-iHt}\psi_0$ with $\psi_0 = \Omega_\beta^- g_\beta^-$, and therefore at large positive times the wave-function behaves like $\sum_\alpha e^{-iH_\alpha t} g_\alpha^+$, with

$$g_\alpha^+ = \Omega_\alpha^{+*}\psi_0 = \Omega_\alpha^{+*}\Omega_\beta^- g_\beta^- = S_{\alpha\beta} g_\beta^- , \tag{332}$$

where the *partial S-matrices* $S_{\alpha\beta}$ are defined by

$$S_{\alpha\beta} = \Omega_\alpha^{+*}\Omega_\beta^- . \tag{333}$$

Thus, "sending in the function g_β^- in the channel β," one will "get out the function $S_{\alpha\beta}g_\beta^-$ in the channel α." If the initial state is assumed to have the form $\sum_\beta e^{-iH_\beta t} g_\beta^-$ with $g_\beta^- \in D_\beta$ and $\sum_\beta \|g_\beta^-\|^2 < \infty$, then the final state has the form $\sum_\beta e^{-iH_\alpha t} g_\alpha^+$ with

$$g_\alpha^+ = \sum_\beta S_{\alpha\beta} g_\beta^- . \tag{334}$$

Note that if g_α^+ satisfies (334) then

$$\|g_\alpha^+\|^2 = \Big\|\sum_\beta \Omega_\alpha^{+*}\Omega_\beta^- g_\beta^-\Big\|^2 = \Big\|\Omega_\alpha^{+*}\sum_\beta \Omega_\beta^- g_\beta^-\Big\|^2 = \Big\|F_\alpha^+ \sum_\beta \Omega_\beta^- g_\beta^-\Big\|^2 . \tag{335}$$

The sum $\sum_\beta \Omega_\beta^- g_\beta^-$ is an orthogonal sum, which converges because

$$\sum_\beta \|\Omega_\beta^- g_\beta^-\|^2 = \sum_\beta \|g_\beta^-\|^2 < \infty . \tag{336}$$

Clearly $\sum_\beta \Omega_\beta^- g_\beta^-$ belongs to R^-. Writing F^+ for the projection on R^+ and recalling that by assumption $R^+ = R^-$, it is seen that (335) implies

$$\begin{aligned} \sum_\alpha \|g_\alpha^+\|^2 &= \sum_\alpha \Big\| F_\alpha^+ \sum_\beta \Omega_\beta^- g_\beta^-\Big\|^2 \\ &= \Big\|F^+ \sum_\beta \Omega_\beta^- g_\beta^-\Big\|^2 = \Big\|\sum_\beta \Omega_\beta^- g_\beta^-\Big\|^2 \\ &= \sum_\beta \|\Omega_\beta^- g_\beta^-\|^2 = \sum_\beta \|g_\beta^-\|^2 . \end{aligned} \tag{337}$$

This completes the analysis of Problems I and II (modulo the assumption $R^+ = R^-$). The information has been written down "channel by channel" in terms of the "partial S-matrices" $S_{\alpha\beta}$. This is an acceptable solution. However, the results can be put into an elegant and economical form by defining a "big S-matrix" which serves the same function as the set of partial S-matrices. Two proposals for the "big S-matrix" will be considered, that of Ekstein [8] and that of Jauch [18]. The Ekstein proposal is more natural in the framework of this exposition while the Jauch proposal gives an S-matrix more closely analogous to the S-matrix defined in relativistic field theories.

Proposal of Ekstein: The problem of scattering theory is this: Given the initial set $\{g_\alpha\}$ of (the first version of) Problem I, with $g_\alpha \in D_\alpha$ and $\sum_\alpha \|g_\alpha^-\|^2 < \infty$, find the final set $\{g_\alpha^+\}$ of Problem II, with $g_\alpha^+ \in D_\alpha$ and $\sum_\alpha \|g_\alpha^+\|^2 < \infty$. I.e., given $\{g_\alpha^-\}$ and defining ψ_0 by

(338) $$\lim_{t\to-\infty} \left\| e^{-iHt}\psi_0 - \sum_\alpha e^{-iH_\alpha t} g_\alpha^- \right\| = 0 ,$$

find $\{g_\alpha^+\}$ such that

(339) $$\lim_{t\to+\infty} \left\| e^{-iHt}\psi_0 - \sum_\alpha e^{-iH_\alpha t} g_\alpha^+ \right\| = 0 .$$

Introduce a Hilbert space $\mathcal{H}$ defined as the *direct* sum of all the Hilbert spaces D_α. (*Note*: the D_α are not orthogonal as subspaces of $L^2(R^6)$, and $\mathcal{H}$ is not a subspace of $L^2(R^6)$.) An element g of $\mathcal{H}$ is a set $\{g_\alpha\}$, $g_\alpha \in D_\alpha$,

(340) $$g = \{g_\alpha\}$$

with addition of two elements or multiplication of an element by a complex number defined componentwise, and the inner product inherited from the inner products on the D_α (i.e., on $L^2(R^6)$):

(341) $$(f, g) = \sum_\alpha (f_\alpha, g_\alpha)_{D_\alpha} = \sum_\alpha (f_\alpha, g_\alpha)_{L^2} .$$

A set $\{f_\alpha\}$ belongs to $\mathcal{H}$ if and only if

(342) $$\sum_\alpha \|f_\alpha\|^2 < \infty .$$

One can write in the usual notation

(343) $$\mathcal{H} = \bigoplus_\alpha D_\alpha .$$

The sets $\{g_\alpha^-\}$ and $\{g_\alpha^+\}$, discussed a moment ago, may now be viewed as elements of $\mathcal{H}$:

(344) $$g^- = \{g_\alpha^-\}, \qquad g^+ = \{g_\alpha^+\} .$$

The problem of scattering theory is then this: Given any $g^- = \{g_\alpha^-\} \in \mathcal{H}$, find $g^+ = \{g_\alpha^+\} \in \mathcal{H}$ such that the components of g^- and g^+ satisfy (338) and (339). (I.e., defining ψ_0 by (338), the components of g^+ satisfy (339).) Thus a map S of $\mathcal{H}$ into itself is sought, such that

(345) $$g^+ = Sg^- \ .$$

In the following, the components of g^+ are sometimes denoted by $\{(Sg^-)_\alpha\}$:

(346) $$g^+ = \{g_\alpha^+\} = \{(Sg^-)_\alpha\}\,.$$

$\{g_\alpha^+\}$ is computed by using equations (333) and (334), which give

(347) $$g_\alpha^+ = (Sg^-)_\alpha = \sum_\beta \Omega_\alpha^{+*}\Omega_\beta^- g_\beta^- = \sum_\beta S_{\alpha\beta} g_\beta^- \ .$$

From (347) it is seen that if g^+ and g^- are considered as column vectors (entries in D_α) and S as a matrix whose (α, β)th entry is the operator $S_{\alpha\beta}$ defined before, then the column vector g^+ is obtained from g^- by acting on it with the matrix S.

If $R^+ = R^-$, then the operator S is a unitary map of $\mathcal{H}$ into itself. It can be seen that S is isometric from (337), which states in the new notation, that

(348) $$\|Sg^-\|^2 = \|g^-\|^2 \ .$$

That S is actually a map *onto* $\mathcal{H}$, and hence unitary, can be seen as follows: It is easy to verify that if $k = \{k_\alpha\}$ and $h = \{h_\alpha\}$ belong to $\mathcal{H}$, then the orthogonal sums (in $L^2(R^6)$) $\sum_\alpha \Omega_\alpha^+ k_\alpha$ and $\sum_\alpha \Omega_\alpha^- g_\alpha$ converge, and

(349) $$k = Sh \iff \sum_\alpha \Omega_\alpha^+ k_\alpha = \sum_\alpha \Omega_\alpha^- h_\alpha \ .$$

This equivalence is of some interest in itself. It shows the relation between the expressions of a central concept in scattering theory in $\mathcal{H}$ and in $L^2(R^6)$. Physically, the statement $\{k_\alpha\} = S\{h_\alpha\}$ means "If the wave function $e^{-iHt}\,\psi_0$ behaves like $\sum_\alpha e^{-iH_\alpha t}\,h_\alpha$ as $t \to -\infty$, then it will behave like $\sum_\alpha e^{-iH_\alpha t}\,k_\alpha$ as $t \to +\infty$". The equation $\sum_\alpha \Omega_\alpha^+ k_\alpha$ $\sum_\alpha \Omega_\alpha h_\alpha$ gives, in two different forms, the value of the state ψ_0 such that $e^{-iHt}\psi_0$ will have these asymptotic behaviors. One form is appropriate to the analysis of $e^{-iHt}\psi_0$ as $t \to +\infty$, and the other to the analysis as $t \to -\infty$. To return to the proof that S is onto: Let $k \in \mathcal{H}$. Then $\sum_\alpha \Omega_\alpha^+ k_\alpha \in R^+$. Since $R^+ = R^-$, there exist $h_\alpha \in D_\alpha$ such that $\sum_\alpha \Omega_\alpha^+ k_\alpha = \sum_\alpha \Omega_\alpha^- h_\alpha$ and

(350) $$\sum_\alpha \|h_\alpha\|^2 = \sum_\alpha \|\Omega_\alpha h_\alpha\|^2 < \infty \ ,$$

i.e., $\{h_\alpha\} \in \mathcal{H}$. But then by (349) $\{k_\alpha\} = S\{h_\alpha\}$, showing that S is onto. Clearly all the information needed to solve the problem of scattering theory is contained in a simple way in the unitary operator S acting on $\mathcal{H}$.

Proposal of Jauch: In relativistic scattering theory one normally introduces the S-matrix as a map between the set of "out" states and the set of "in" states. In the present nonrelativistic situation, "in" and "out" states are defined as follows: the subspace of "in" states consists of the values at time $t = 0$ of states which exhibit free motion of at least one simple or composite particle as $t \to -\infty$, and the subspace of "out" states consists of the values at time $t = 0$ of states exhibiting such motion as $t \to +\infty$. In our terminology these subspaces are R^- and R^+ respectively. Of course, one hopes that these two subspaces are identical, and in this case the distinction between an "in" state and an "out" state is entirely one of labelling. An "in" state is then an element of $R^+ = R^-$ which has been labelled in such a way as to draw attention to its behavior as $t \to -\infty$, and an "out" state is one labelled in such a way as to draw attention to its behavior as $t \to +\infty$. One frequently takes as a basis for $R^\pm$ a basis consisting of "in" ("out") states with especially simple behavior as $t \to -\infty$ ($t \to +\infty$). As an example of the difference between "in" and "out" labellings, consider the state $\Omega^- f_\alpha \in R^-$. We know that as $t \to -\infty$ the state $e^{-iHt}\Omega_\alpha^- f_\alpha$ approaches $e^{-iH_\alpha t} f_\alpha$; in a natural terminology we can refer to $\Omega_\alpha^- f_\alpha$ as "f_α in"—that is, the state of $t = 0$ which results from sending in f_α at large negative times. If $R^- = R^+$ so that $\Omega_\alpha^- f_\alpha \in R^+$, we could also label $\Omega_\alpha^- f_\alpha$ by its behavior as $t \to +\infty$, but this labelling would be more complicated since the behavior of this state is not so simple as $t \to +\infty$. A general state $\psi_0 \in R^-$, given by

$$\psi_0 = \sum_\alpha \Omega_\alpha^- f_\alpha \tag{351}$$

could be labelled by the set $\{f_\alpha\}$ of all its f_α's. It could be called "$\{f_\alpha\}$ in". Likewise one could label the state

$$\Phi_0 = \sum_\alpha \Omega_\alpha^+ g_\alpha \tag{352}$$

by the set $\{g_\alpha\}$ of all its g_α's and call it "$\{g_\alpha\}$ out". Thus to rewrite things as is done in the relativistic theory, one needs these labels:

$$\sum_\alpha \Omega_\alpha^+ g_\alpha = [\{g_\alpha\} \text{ out}] \ , \qquad \sum_\alpha \Omega_\alpha^- f_\alpha = [\{f_\alpha\} \text{ in}] \ . \tag{353}$$

Now in the relativistic theory, as stated above, the S-matrix is defined as an operator from out states to in states—in the present case that means from R^+ to R^-—and is defined as the operator which converts a state having a certain form as $t \to +\infty$ into the state which has the *same* form as $t \to -\infty$. In the present case, denoting the operator which has this effect by S', the action of S' is

(354) $$S'[\{f_\alpha\}\text{ out}] = [\{f_\alpha\}\text{ in}] \quad ,$$

that is to say,

(355) $$S'\Big(\sum_\alpha \Omega^+_\alpha f_\alpha\Big) = \sum_\alpha \Omega^-_\alpha f_\alpha \ .$$

The physical relevance of the operator S' will be discussed presently. At present a few of its mathematical properties are derived. In the first place consider operators S'_α defined by

(356) $$S'_\alpha = \Omega^-_\alpha \Omega^{+*}_\alpha \ .$$

They satisfy

(357) $$S'_\alpha \Omega^+_\alpha = \Omega^-_\alpha(\Omega^{+*}_\alpha \Omega^+_\alpha) = \Omega^-_\alpha E_\alpha$$

so that if $f_\alpha \in D_\alpha$ then

(358) $$S'_\alpha \Omega^+_\alpha f_\alpha = \Omega^-_\alpha f_\alpha \, .$$

Next note that from equation (314)

(359) $$S'_\alpha = S'_\alpha F^+_\alpha \, ,$$

and since the subspaces R^+_α are pairwise orthogonal

(360) $$S'_\alpha \Omega^+_\beta = 0 \quad \text{for } \alpha \neq \beta \ .$$

Because of (359) and because Ω^{+*}_α and Ω^-_α are partial isometries and therefore cannot increase the norm, for any $f \in L^2(R^6)$,

(361) $$\|S'_\alpha f\| = \|\Omega^-_\alpha \Omega^{+*}_\alpha F^+_\alpha f\| \leqq \|F^+_\alpha f\| \, ,$$

and because the F_α are othogonal projections, (361) implies that

(362) $$\sum_\alpha \|S'_\alpha f\|^2 < \infty \ .$$

Combining (362) with the fact that $S'_\alpha f$ clearly belongs to R^-_α and is hence orthogonal to $S'_\beta f$ for $\alpha \neq \beta$, it is clear that the sum (our notation anticipates our result slightly)

(363) $$\sum_\alpha S'_\alpha f \equiv S'f$$

converges strongly for any $f \in L^2$. It is easy to see that S' satisfies equation (355), so that S' is indeed the operator we are looking for:

(364) $$S' = \sum_\alpha S'_\alpha \, , \qquad \text{(strong convergence)},$$

although it is defined somewhat more generally than was expected. Because of (359), however, S' annihilates anything orthogonal to R^+, so one might as well speak only of its action on R^+. It should be clear from the definition that S$'$ maps R^+ *onto* R^-. For if one wants to get the state $\sum_\alpha \Omega_\alpha^- f_\alpha$ in R^-, he need only apply S$'$ to the state $\sum_\alpha \Omega_\alpha^+ f_\alpha$ in R^+. Using (313) and (315), it is found that

$$
\begin{aligned}
\mathrm{S}'^*\mathrm{S}' &= \sum_{\alpha,\beta} \Omega_\beta^+ \Omega_\beta^{-*} \Omega_\alpha^- \Omega_\alpha^{+*} \\
&= \sum_{\alpha,\beta} \Omega_\beta^+ (\delta_{\alpha\beta} E_\alpha)\, \Omega_\alpha^{+*} = \sum_\alpha \Omega_\alpha^+ \Omega_\alpha^{+*} = \sum_\alpha F_\alpha^+ = P_{R^+}
\end{aligned}
\tag{365}
$$

where $P_{R^\pm}$ is the projection on $R^\pm$. Similarly,

$$\mathrm{S}'\mathrm{S}'^* = P_{R^-} \ . \tag{366}$$

Equations (365) and (366) state that S$'$ is a unitary map of the Hilbert space R^+ onto the Hilbert space R^-, whether or not R^+ and R^- are equal as subspaces of $L^2(R^6)$.

Although the "meaning" of the operator S$'$ has been discussed—i.e., the fact that it converts a wave-function with a certain behavior as $t \to +\infty$ to a wave-function with the *same* behavior as $t \to -\infty$, it is still unclear how to use S$'$ to calculate interesting quantities related to scattering problems. To understand this, one more concept from quantum mechanics is needed. Suppose that ψ_t and ϕ_t are two wave-functions which can describe a quantum mechanical system (the "system" consists of two particles in our present discussion). The following question is asked: Suppose that at time t the system is in the state described by ψ_t. What is the probability that a measurement made at this time will reveal that it is in the state described by ϕ_t? (To get some feeling for what this rather peculiar-sounding question means, an analogy can be used: Suppose a coin has been thrown, and describe the "state" of the coin after throwing but before looking at it by a two-component vector with entries signifying the probabilities for heads or tails. Suppose that the state of the coin is correctly described by the vector $(\frac{1}{2}, \frac{1}{2})$. Then one may ask for the probability that a "measurement"—i.e., looking at the coin—will reveal the coin to be in the state $(1, 0)$. This is the sort of thing that is meant by the quantum-mechanical question above, although the notion of "measurement" in quantum mechanics is somewhat different.) The quantum-mechanical prescription for the required probability, which will be denoted by $P(\phi_t, \psi_t)$, is

$$P(\phi_t, \psi_t) = |(\phi_t, \psi_t)|^2 \ . \tag{367}$$

$P(\phi_t, \psi_t)$ is also called "the transition probability from ψ_t to ϕ_t." Because ψ_t and ϕ_t satisfy the Schrödinger equation, we have

$$P(\phi_t, \psi_t) = |(e^{-iHt}\phi_0, e^{-iHt}\psi_0)|^2 = |(\phi_0, \psi_0)|^2 = P(\phi_0, \psi_0), \tag{368}$$

so that the transition probability does not depend on time and may in particular be computed at $t = 0$. Now suppose that $\psi_t = e^{-iHt}\psi_0$ is a wave-function with specified behavior as $t \to -\infty$, and $\phi_t = e^{-iHt}\phi_0$ is a wave-function with specified behavior as $t \to +\infty$. Then $P(\phi_0, \psi_0)$ is the probability that a measurement (at any time) on a state with a specified behavior as $t \to -\infty$ will reveal that it has a specified behavior as $t \to +\infty$. To take a concrete example, let

$$\begin{aligned} \psi_0 &= \Omega_\alpha^- g_\alpha\,, & g_\alpha &\in D_\alpha\,, \\ \phi_0 &= \Omega_\beta^+ g_\beta\,, & g_\beta &\in D_\beta\,. \end{aligned} \tag{369}$$

$e^{-iHt}\psi_0$ behaves like $e^{-iH_\alpha t} g_\alpha$ as $t \to -\infty$. $e^{-iHt}\phi_0$ behaves like $e^{-iH_\beta t} g_\beta$ as $t \to +\infty$. $P(\phi_0, \psi_0)$ is given by

$$P(\phi_0, \psi_0) = |(\Omega_\beta^+ g_\beta, \Omega_\alpha^- g_\alpha)|^2 = |(g_\beta, S_{\beta\alpha} g_\alpha)|^2 \tag{370}$$

and is the probability that if the two-particle system originally ($t \to -\infty$) was described by $e^{-iH_\alpha t} g_\alpha$, it is finally ($t \to +\infty$) described by $e^{-iH_\beta t} g_\beta$. To economize in notation write

$$P(\Omega_\beta^+ g_\beta, \Omega_\alpha^- g_\alpha) \equiv P(g_\beta \leftarrow g_\alpha)\,. \tag{371}$$

The operator S' can be used to rewrite $P(g_\beta \leftarrow g_\alpha)$ as follows:

$$P(g_\beta \leftarrow g_\alpha) = |(\Omega_\beta^+ g_\beta, S'\Omega_\alpha^+ g_\alpha)|^2\,. \tag{372}$$

Of course, equation (372) is no advantage over equation (370)—in fact it looks clumsier, and one is tempted to think that there is no advantage in introducing S', except, perhaps, that the "matrix elements" $(f, S'g)$ with $f, g \in R^+$, yield all the $P(g_\beta \leftarrow g_\alpha)$. (This is of considerable interest in the relativistic theory.) However, S' is a map which arises naturally when one fixes one's attention on the time-dependence of the operators in a theory instead of the time-dependence of the states. Since this is frequently done in field theory, S' arises there in a natural way. A brief explanation follows. The "motion" of a system described by a wave-function governed by the Schrödinger equation

$$\psi_t = e^{-iHt}\psi_0 \tag{373}$$

induces a natural motion of a linear operator A_0 on L^2 by the prescription

$$(\phi_0, A_t\psi_0) \equiv (\phi_t, A_0\psi_t)\,. \tag{374}$$

Unravelling (374) gives

$$A_t = e^{iHt}A_0 e^{-iHt}\,. \tag{375}$$

Consider the simple case of potential scattering, with a Hamiltonian

H such that the Møller wave-matrices $\Omega^{\pm}$ exist and the theory is asymptotically complete. If it is also assumed that ϕ_0 and ψ_0 of (374) belong to $R = R^{\pm}$, then we know that

$$\lim_{t\to\pm\infty} \|\psi_t - e^{-iH_0t}\,\Omega^{\pm *}\psi_0\| = 0 \tag{376}$$

and

$$\lim_{t\to\pm\infty} \|\phi_t - e^{-iH_0t}\,\Omega^{\pm *}\phi_0\| = 0\,. \tag{377}$$

Hence, (if A_0 is bounded, say)

$$\lim_{t\to\pm\infty} |(\phi_0, A_t\psi_0) - (\phi_0, \Omega^{\pm}\, e^{iH_0t}\, A_0\, e^{-iH_0t}\, \Omega^{\pm *}\psi_0| = 0\,. \tag{378}$$

Equation (378) says that the operator

$$\Delta_t = A_t - \Omega^{\pm} e^{iH_0t} A_0\, e^{-iH_0t}\, \Omega^{\pm *} \tag{379}$$

converges weakly to zero on R. If the intertwining relations in (379) are used it is seen that

$$\Omega^{\pm} e^{iH_0t} A_0\, e^{-iH_0t}\;\; \Omega^{\pm *} = e^{iHt}(\Omega^{\pm} A_0 \Omega^{\pm *})e^{-iHt} = e^{iHt}\, \overset{\pm}{A_0}\, e^{-iHt} \tag{380}$$

with

$$\overset{\pm}{A_0} = \Omega^{\pm} A_0 \Omega^{\pm *}\;. \tag{381}$$

Now in terms of the notation (375),

$$e^{iHt}\, \overset{\pm}{A_0}\, e^{-iHt} = (\overset{\pm}{A_0})_t \tag{382}$$

so that

$$A_t - (\overset{\pm}{A_0})_t \xrightarrow[t\to\pm\infty]{} 0 \quad \text{on } R\;. \tag{383}$$

$(\overset{\pm}{A_0})_t$ can be viewed as the "asymptotic forms" of the operator A_t. Although the equation (382) defining the time-dependence of $(\overset{\pm}{A_0})_t$ is the same as that for A_t, it is clear from (380) that the time-dependence of $(\overset{\pm}{A_0})_t$ can actually be analyzed in terms of the free Hamiltonian H_0. Now consider the connection between the "asymptotic operators" $\overset{\pm}{A_0}$. This can be read off from (381)

$$\overset{+}{A_0} = \Omega^{+} A_0 \Omega^{+ *} = \Omega^{+}(\Omega^{+ *} \overset{-}{A_0}\, \Omega^{-})\Omega^{+ *} = S'^{*}\, \overset{-}{A_0}\, S'\;, \tag{384}$$

where

$$S' = \Omega^{-}\Omega^{+ *}\;. \tag{385}$$

Although for simplicity the case of potential scattering has been considered, it is hoped that the analogy between the operator S' in equation (385) and the two-body operator S' of equations (364) (see also (356)) is sufficiently clear, and that the reader is convinced that such operators can arise in a natural way if one decides to think in terms of the time-dependence of operators rather than the time-dependence of states in the theory. Why anyone would want to do this has not been discussed here, and will not be. Suffice it to say that the two ways of looking at the theory are essentially equivalent. As a conclusion to the discussion of S', note that S and S' are unitarily equivalent by the map U of $\mathcal{H}$ into $R = R^{\pm}$ which sends the element $\{f_\alpha\}$ into $\sum_\alpha \Omega_\alpha^+ f_\alpha$.[2]

To finish the discussion of the two-body problem for square-integrable potentials, it is necessary to note that there is a sort of asymptotic behavior of the two particles which has not yet been considered—namely, the case in which asymptotically *both* particles are trapped near the origin. This statement is interpreted to mean that the particles are in a "true bound state"—that is their wave-function is an eigenfunction of the full Hamiltonian H, (or a superposition of such eigenfunctions). Denote such an eigenfunction by ψ_n, $n = 1, 2, \cdots$. Thus

$$H\psi_n = E_n\psi_n \ . \tag{386}$$

Then of course

$$e^{-iHt}\,\psi_n = e^{-iE_nt}\,\psi_n \tag{387}$$

so the ppd and mpd of the particles will never change if they are described by $e^{-iHt}\,\psi_n$. The behavior of a superposition of such eigenfunctions is easy to analyze using (387).

Denote by B the subspace of $L^2(R^6)$ spanned by the ψ_n, $n = 1, 2, \cdots$. It is quite easy to see, just as in the case of potential scattering, that B is orthogonal to $R^{\pm}$. Just as in the case of potential scattering, it is hoped that states with at least one particle free as $t \to -\infty$ also have at least one particle free as $t \to +\infty$. This implies that the condition $R^+ = R^- \equiv R$ should hold. It is also hoped that all possible asymptotic behaviors for our two particles have been classified. This is expressed by the condition

$$L^2(R^6) = R \oplus B \ . \tag{388}$$

As before, (388) is called the requirement of *asymptotic completeness*, and is known to be satisfied under strong enough assumptions on the potentials because of the work of Faddeev [9]. (Faddeev studied three-body scattering, but asymptotic completeness in the present situation is an easy consequence of his work, granted his conditions on

[2]Thanks are due to Dr. J. Cannon for pointing out this important fact about which the author hadn't thought.

the potentials.) Professor Combes [2] also has some results on asymptotic completeness for the present case. Also see at this point the Appendix with the remarks of Professor Ekstein.

The existence of $\Omega_\alpha^\pm$ could also have been proved under other assumptions on the potentials. For instance if each $V_{ij}(\vec{x})$ is the sum of a square-integrable function and a function which is locally square-integrable and falls off like $1/|\vec{x}|^\beta$, $\beta > 1$, for large $|\vec{x}|$, then all goes as before. (This does not necessarily mean that asymptotic completeness holds. It means that everything proved before about square-integrable potentials remains true.) If Coulomb potentials are involved, however, the proofs break down just as in the case of potential scattering. The changes which should be made in the above discussion when Coulomb potentials are involved will now be indicated.

The principle for dealing with cases in which Coulomb potentials are involved is very simple. Namely, in any case in which one is dealing with charges which are asymptotically moving away from each other, one inserts in the relevant channel Hamiltonian an anomalous factor of the type described before, and in which the Laplacians occurring are differential operators in the separation coordinates of the two charges. To see how this is done, consider the example of two particles. Imagine that particles 1 and 2 carry charges e_1 and e_2 respectively, and that in addition to the fixed potentials V_{01} and V_{02} a charge $\mathcal{E}$ is fixed at the origin of three-space. In order to describe this situation one need only add to the Hamiltonian H of (256) the term

$$V_c = \frac{e_1 e_2}{|\vec{x}_1 - \vec{x}_2|} + \frac{e_1 \mathcal{E}}{|\vec{x}_1|} + \frac{e_2 \mathcal{E}}{|\vec{x}_2|}\,. \tag{389}$$

(This could be generalized slightly to the case in which particles 1 and 2 interact with different charges at the origin by replacing V_c of (389) by

$$\frac{e_1 e_2}{|\vec{x}_1 - \vec{x}_2|} + \frac{e_1 \mathcal{E}_1}{|\vec{x}_1|} + \frac{e_2 \mathcal{E}_2}{|\vec{x}_2|}\,,$$

say. This case is also easy to handle, but not very realistic— physically both particles should interact with the same charge at the origin. The charge at the origin could also be displaced to some other fixed point $\vec{a}$, etc.) The program of classification of asymptotic behaviors can now be carried through as before and the same essential features will be found. First there is the case in which both particles are asymptotically far from everything else. Instead of discovering that the asymptotic form of such a state behaves like $e^{-iH_0 t}f$, $f \in L^2$, it is found that it behaves like $e^{-iH_{0c}(t)} f$, where $H_{0c}(t)$ is given as before by

$$H_{0c}(t) = H_0 t + A(t) \ , \tag{390}$$

where $A(t)$ is a term representing the "anomalous" behavior due to the presence of Coulomb interactions. This time, however, $A(t)$ is the sum of three terms—one resulting from the interaction of e_1 with the charge $\mathcal{E}$ at the origin, one resulting from the interaction of e_2 with $\mathcal{E}$, and one from the interaction of e_1 with e_2. Explicitly,

$$\begin{aligned} A(t) = \epsilon(t) \Big\{ & \frac{m_1 e_1 \mathcal{E}}{(-\Delta_1)^{1/2}} \log \Big\{ \frac{-2|t|\Delta_1}{m_1} \Big\} \\ & + \frac{m_2 e_2 \mathcal{E}}{(-\Delta_2)^{1/2}} \log \Big\{ \frac{-2|t|\Delta_2}{m_2} \Big\} \\ & + \frac{\mu e_1 e_2}{(-D_{12})^{1/2}} \log \Big\{ \frac{-2|t| D_{12}}{\mu} \Big\} \Big\} . \end{aligned} \tag{391}$$

The only unfamiliar-looking term is the last one. Quantities appearing therein are defined as follows:

$$\mu = m_1 m_2/(m_1 + m_2) \ . \tag{392}$$

D_{12} is actually the operator $\Delta_{\vec{x}}$—the Laplacian with respect to the relative coordinate $\vec{x}$. It has been written in a new notation to call attention to its physical significance, which will now be explained. Write

$$-D_{12} = \vec{P}_{12} \cdot \vec{P}_{12} \qquad (= -\Delta_{\vec{x}}) \ , \tag{393}$$

where

$$\vec{P}_{12} = -i \left(\frac{m_2 \vec{\nabla}_1 - m_1 \vec{\nabla}_2}{m_1 + m_2} \right) \ . \tag{394}$$

$\vec{P}_{12}$ is the momentum operator associated with the relative coordinate $\vec{x}$, and μ is the reduced mass associated with this coordinate. $\vec{P}_{12}$ is μ times the relative velocity operator for particles 1 and 2. Recall that the velocity operator for a particle is $-i\vec{\nabla}/m$. Hence

$$\vec{P}_{12} = \mu \left(\frac{-i\vec{\nabla}_i}{m_i} - \left(\frac{-i\vec{\nabla}_2}{m_2} \right) \right) \tag{395}$$

is μ times the difference of the velocity operators for particles 1 and 2. The operator

$$-D_{12} = \vec{P}_{12} \cdot \vec{P}_{12} \tag{396}$$

is therefore the square of the relative momentum of particles 1 and 2. It corresponds to the operators $(-\Delta_1)$ and $(-\Delta_2)$ in the first and second terms of (391), which are, respectively, the squares of the momenta of particles 1 and 2.

At this point, the reader should review the convergence proof for the "free channel" with square-integrable potentials, and imagine how, with Coulomb potentials present, the "anomalous" terms resulting from differentiation of $e^{-iH_{0c}(t)}$ would tend to cancel against the sum of potentials in (389) at large times. The earlier discussion of velocity operators, and the origin of the anomalous factor, following equation (192), should also be reviewed to see the reasoning behind the form of the third term of (391).

To continue the analysis, consider the situation in which the two particles are asymptotically close together, but far from the fixed potentials and the origin. In this situation, the behavior of the relative coordinate $\vec{x}$ of the particles is described by a bound state $\psi_{c\delta}$ of the Hamiltonian

(397) $$H_c' = -\Delta_{\vec{x}}/2\mu + V_{12}(\vec{x}) + e_1e_2/|\vec{x}|$$

with energy $E_{c\delta}$, say ($\delta = 1, \cdots, \delta_{\max} \leqq \infty$).

(398) $$H_c' \psi_{c\delta} = E_{c\delta}\psi_{c\delta} .$$

The wave-function for the two particles has the form $e^{-iH_{c\delta}(t)}f(\vec{X})\psi_\delta(\vec{x})$ (or a linear combination of such functions) where $f \in L^2(R^3)$ and

(399) $$H_{c\delta}(t) = (-\Delta_{\vec{X}}/2M + E_\delta)t + A'(t) , \qquad M = m_1 + m_2 ,$$

$A'(t)$ as usual being the anomalous factor. $A'(t)$ should be the sum of two terms, due to the separation of the two particles from the fixed charge at the origin. In fact, an acceptable $A'(t)$ is the sum of the first two terms of (391). ("Acceptable" means the desired convergence proof can be obtained using this $A'(t)$.) However, it is physically plausible that the two particles bound together and a long way from the origin appear, to the charge at the origin, like a single particle of mass $m_1 + m_2$, whose distance from the origin is measured by the center-of-mass coordinate $\vec{X}$. It is therefore a plausible conjecture that one may use for $A'(t)$ the anomalous factor appropriate to a particle of mass $M = m_1 + m_2$, coordinate $\vec{X}$, and charge $e_1 + e_2$. Explicitly, this factor is

(400) $$A'(t) = \frac{\epsilon(t)M(e_1 + e_2)}{(-\Delta_{\vec{X}})^{1/2}} \log\left(\frac{-2|t|\Delta_{\vec{X}}}{M}\right) .$$

Physically, the operator $-\Delta_{\vec{X}}$ is the square of the momentum operator appropriate to the center-of-mass coordinate, and it occurs in the same way in (400) as other such operators do in (391). In line with our expectations, $A'(t)$ is an acceptable anomalous factor for the situation under discussion.

The remaining asymptotic forms of wave-functions describing the other scattering situations will now be written down with little discussion.

Particle 1 far from fixed center, particle 2 "trapped". In this case the asymptotic form is a linear combination of the following forms:

(401) $$e^{-iHt}\psi_0 \sim e^{-iH_{2c\beta}(t)} f(\vec{x}_1)\psi_{2c\beta}(\vec{x}_2)$$

where $\psi_{2c\beta}$ is an eigenfunction of the Hamiltonian

(402) $$H'_{2c} = \frac{-\Delta_{\vec{x}_2}}{2m_2} + V_{02}(\vec{x}_2) + \frac{e_2\mathcal{E}}{|\vec{x}_2|}$$

with eigenvalue $E_{2c\beta}$, $(\beta = 1, \cdots, \beta_{\max} \leqq \infty)$.

(403) $$H'_{2c}\psi_{2c\beta} = E_{2c\beta}\psi_{2c\beta}$$

and

(404) $$H_{2c\beta}(t) = -\Delta_{\vec{x}_1}/2m_1 + E_{2c\beta} + A_2(t)$$

with

(405) $$A_2(t) = \epsilon(t)\frac{m_1e_1(\mathcal{E}+e_2)}{(-\Delta_{\vec{x}_1})^{1/2}}\log\left(\frac{-2|t|\Delta_{\vec{x}_1}}{m_1}\right).$$

Note that particle 2 and the fixed charge at the origin combine to an effective charge $\mathcal{E} + e_2$ in computing the anomalous factor (405).

The distorted propagator can now be written down for the case in which particle 1 is trapped and 2 is far from the origin—it suffices to interchange 1 and 2 in $H_{2c\beta}(t)$, and change $E_{2c\beta}$ to the energy $E_{1c\delta}$ of a bound state of the operator H'_{1c} analogous to H'_{2c} of (402).

Having assembled these asymptotic forms, the discussion proceeds as before. The operators $H_{0c}(t)$, $H_{c\delta}(t)$, $H_{2c\beta}(t)$, $H_{1c\beta}(t)$ are called distorted "channel Hamiltonians". The subspaces $D_{c\delta}$, $D_{2c\beta}$, $D_{1c\gamma}$ are defined in exact analogy to the way in which D_{δ}, $D_{2\beta}$, $D_{1\gamma}$ were defined earlier. Then everything is relabelled as before with multi-indices $\alpha = (p, q)$ (but with a "c" in front of the α) where p is a distorted channel Hamiltonian and q is a bound state that goes with it. In analogy to what was done before, the "free channel" gets the index $(H_{0c}(t), 1)$ and the associated subspace $D_{c(H_{0c}(t),1)} = L^2(R^6)$. The same conventions about degenerate eigenvalues are also observed. With these conventions, a convergence theorem analogous to the earlier one can be proved. (Note, however, the additional hypothesis.) Namely:

THEOREM. *Let H_c be given by*

$$(406)\quad \begin{aligned} H_c = & -\frac{\Delta_1}{2m_1} - \frac{\Delta_2}{2m_2} + V_{01}(\vec{x}_1) + V_{02}(\vec{x}_2) \\ & + V_{12}(\vec{x}_1 - \vec{x}_2) + \frac{e_1 \mathcal{E}}{|\vec{x}_1|} + \frac{e_2 \mathcal{E}}{|\vec{x}_2|} + \frac{e_1 e_2}{|\vec{x}_1 - \vec{x}_2|}, \end{aligned}$$

where for each i, j $(0 \leqq i < j \leqq 2)$ $V_{ij}(\vec{r})$ is the sum of a square-integrable function and a locally square-integrable function which falls off like $1/|\vec{r}|^\beta$, $\beta > 1$, for large $|\vec{r}|$. Let $H_{c\alpha}(t)$ and $D_{c\alpha}$ be as above. Then if for each of the bound states $\psi_{c\delta}(\vec{r})$, $\psi_{c2\beta}(\vec{r})$, $\psi_{c1\gamma}(\vec{r})$, there exists an $\epsilon > 0$ such that $|\vec{r}|^\epsilon$ times the bound state belongs to $L^2(R^3)$, the strong limits

$$(407)\quad \operatorname*{s-lim}_{t \to \pm\infty} e^{iH_c t} e^{-iH_{c\alpha}(t)} \equiv \Omega_{c\alpha}^{\pm}$$

exist on D_α.

Note: For the "free" channel proof the hypothesis on the bound states is unnecessary. The hypothesis demands that each bound state be a bit more than square-integrable. This hypothesis occurs again in the corresponding theorem for n bodies, where the bound states are functions of several three-vector variables, and then the same thing is required with respect to each of the three-vector variables on which the bound state depends. Physicists are prone to think that the hypothesis is true (even "obviously true" to some). It is certainly true for the bound states of a particle interacting with a pure Coulomb potential, which die off exponentially. This gives an indication that the hypothesis is a "mild" one, but it is still rather annoying.

In any case, once this theorem is proved, all the later analysis of the $\Omega_{c\alpha}^{\pm}$ goes in a way directly parallel with what was done before, with the obvious changes—namely particles asymptotically propagate with distorted propagators, etc. It is important to note, moreover, that when the intertwining relations are proved the anomalous terms drop out of the distorted channel Hamiltonians, just as the anomalous term dropped out in the case of potential scattering. Thus

$$(408)\quad e^{iH_c t}\, \Omega_{c\alpha}^{\pm} = \Omega_{c\alpha}^{\pm}\, e^{iH_{c\alpha}^{x} t}$$

where $H_{c\alpha}^{x} t$ is $H_{c\alpha}(t)$ with all anomalous factors crossed out and has, as indicated, the form of a selfadjoint operator times t.

The definitions of S and S′ and the discussion of asymptotic completeness go much as before.

Now the definitions of channels, channel Hamiltonians, etc., are given briefly for a problem involving n particles, whose behavior is governed by the Hamiltonian H of the form

$$H = \sum_{i=1}^{n} \frac{-\Delta_i}{2m_i} + \sum_{j=1}^{n} V_{0j}(\vec{x}_j) + \sum_{1 \leqq i < j \leqq n} V_{ij}(\vec{x}_i - \vec{x}_j) \ . \tag{409}$$

The channels are found just as in the case of two particles, by classifying the possible asymptotic configurations of the particles. The classification is straightforward, consisting essentially in finding all the ways in which the n particles can break up into subsets which can travel together as "composite particles." In the case of two bodies, it was possible to have particles 1 and 2 asymptotically travelling together in any one of the bound states allowed by the Hamiltonian H'. When in such a configuration, the two particles could be thought of as bound together and constituting a "composite particle." The n-body situation is similar, but the list is longer. The general method for finding one channel, the associated channel Hamiltonian, etc., is indicated below. By carrying out all possible cases of this method, one finds all channels of the system.

Method for finding one channel: partition the n indices $1, \cdots, n$ into $m + 1 \leqq n$ subsets $\Gamma_1, \cdots, \Gamma_{m+1}$. (By abuse of language, one may speak of "the particles belonging to Γ_k" instead of "the particles whose indices belong to Γ_k." Γ_k is to be thought of as containing all indices of particles which together make up the kth "composite particle" in the channel.) Choose a subset Γ_k containing more than one particle, if there are any such Γ_k. It may happen that the particles in Γ_k, if they were alone in the universe, could form a bound state and travel together as a composite particle. Whether or not this is so can be judged by studying the Hamiltonian which governs the mutual interactions of these particles when they are alone, and this Hamiltonian, which is called H_{Γ_k}, can be read off from the Hamiltonian H of (409) by striking off all terms which do not correspond to the particles being "alone". The result is

$$H_{\Gamma_k} = \sum_{i \in \Gamma_k} \frac{-\Delta_i}{2m_i} + \sum_{i,j \in \Gamma_k; i<j} V_{ij}(\vec{x}_i - \vec{x}_j) \ . \tag{410}$$

The next step is to write the *total mass* M_{Γ_k} of the subset Γ_k

$$M_{\Gamma_k} = \sum_{i \in \Gamma_k} m_i \tag{411}$$

and the center-of-mass coordinate $\vec{X}_{\Gamma_k}$ of Γ_k:

$$\vec{X}_{\Gamma_k} = \sum_{i \in \Gamma_k} m_i \vec{x}_i / M_{\Gamma_k} \ . \tag{412}$$

Now H_{Γ_k} can be decomposed into permuting parts, one pertaining to the center-of-mass coordinates and one to "internal variables":

$$H_{\Gamma_k} = H_{0\Gamma_k} + H'_{\Gamma_k} \tag{413}$$

where

$$H_{0\Gamma_k} = \Delta_{\vec{X}_{\Gamma_k}}/2M_{\Gamma_k} \ . \tag{414}$$

The "internal variables" are a set of independent three-vector variables linearly related to the $\vec{x}_i$'s with $i \in \Gamma_k$. Together with $\vec{X}_{\Gamma_k}$, the internal variables span the same subspace of R^{3n} as the $\vec{x}_i$'s with $i \in \Gamma_k$.

An explicit definition of these internal variables could be given and H'_{Γ_k} could be written in terms of them, but this would be pointless. The reader should refer at this point to (227) and the following discussion. The internal variables are denoted collectively by $\vec{Z}_{\Gamma_k}$. The two terms of (413) permute because all the functions $V_{ij}(\vec{x}_i - \vec{x}_j)$ in H'_{Γ_k} are functions of the "internal variables"—this is because V_{ij} depends only on the difference $\vec{x}_i - \vec{x}_j$ in position between particles rather than on where the center of mass of the entire set is located—and $H_{0\Gamma_k}$ in addition permutes with the "leftover Laplacians" in H'_{Γ_k}. The operator $e^{-iH_{\Gamma_k}t}$ factors into a product

$$e^{-iH_{\Gamma_k}t} = e^{-iH_{0\Gamma_k}t}\, e^{-iH'_{\Gamma_k}} \tag{415}$$

of which the first factor describes free motion of the center of mass $\vec{X}_{\Gamma_k}$ and the second describes the "internal motion" of the particles in Γ_k. To find out how the particles in Γ_k can move bound together, one must find the eigenfunctions ψ_{Γ_k, n_k} $(n_k = 1, 2, \cdots, n_{k,\max} \leqq \infty)$ and corresponding eigenvalues E_{Γ_k, n_k} of H'_{Γ_k}:

$$H'_{\Gamma_k}\psi_{\Gamma_k, n_k}(\vec{Z}_{\Gamma_k}) = E_{\Gamma_k, n_k}\psi_{\Gamma_k, n_k}(\vec{Z}_{\Gamma_k}) \ . \tag{416}$$

The motion of the particles in Γ_k bound together in the bound state ψ_{Γ_k, n_k} is then described by a wave-function of this form:

$$\begin{aligned} &e^{-iH_{\Gamma_k}t} g_k(\vec{X}_{\Gamma_k})\psi_{\Gamma_k, n_k}(\vec{Z}_{\Gamma_k}) \\ &\quad = (e^{-iH_{0\Gamma_k}t} g_k)(\vec{X}_{\Gamma_k})\, e^{-iE_{\Gamma_k, n_k}t}\psi_{\Gamma_k, n_k}(\vec{Z}_{\Gamma_k}). \end{aligned} \tag{417}$$

The above discussion shows how to describe the particles in Γ_k, moving together in a bound state, when they are "alone in the universe," i.e., far from all other particles and the fixed centers of force. It was assumed that Γ_k contains more than one particle. If Γ_k contains only one particle, then much of the above discussion is irrelevant. In this case if the index of the single particle in Γ_k is i, then the Hamiltonian H_{Γ_k} when the particle in Γ_k is alone in the universe is

(418) $$H_{\Gamma_k} = -\Delta_i/2m_i\,.$$

The "center-of-mass coordinate" $\vec{X}_{\Gamma_k}$ is $\vec{x}_i$ and there are no "internal coordinates" and also, of course, no bound states to consider. For uniform notation, however, it is convenient to define

(419) $$\psi_{\Gamma_k,\,n_k}(\vec{Z}_k) \equiv 1; \qquad E_{\Gamma_k,\,n_k} = 0; \qquad n_k = 1 = n_{k,\,\max}$$

for the case when Γ_k consists of just one particle. The notation $-\Delta_{\vec{x}_{\Gamma_k}}/2M_{\Gamma_k}$ instead of $-\Delta_i/2m_i$ will also be used in this case.

As before, there is another conceivable behavior of the particles in a subset Γ_k which may be of interest asymptotically. This is the case in which they are all trapped near the origin by the static potentials which interact with them. Such a situation is described as follows: Consider the Hamiltonian

(420) $$H_{\Gamma_k,\,B} = \sum_{i\in\Gamma_k} \frac{-\Delta_i}{2m_i} + \sum_{i\in\Gamma_k} V_{0i}(\vec{x}_i) + \sum_{i,j\in\Gamma_k;\, i<j} V_{ij}(\vec{x}_i - \vec{x}_j)$$

which describes the particles in Γ_k along with their mutual interactions *and* their interactions with the fixed potentials. Then the situation under discussion is described by a wave-function

(421) $$e^{-iH_{\Gamma_k,\,B}t}\,\psi_{\Gamma_k,\,B,\ell_k} = e^{-iE_{\Gamma_k,\,B,\ell_k}t}\,\psi_{\Gamma_k,B,\ell_k}$$

where $\psi_{\Gamma_k,\,B,\ell_k}$ is a function of *all* the variables $\vec{x}_i$ with $i \in \Gamma_k$, which is an eigenfunction of $H_{\Gamma_k,\,B}$ with eigenvalue $E_{\Gamma_k,\,B,\ell_k}$: $\ell_k = 1, \cdots, \ell_{k,\,\max} \leqq \infty$

(422) $$H_{\Gamma_k,\,B}\,\psi_{\Gamma_k,\,B,\ell_k} = E_{\Gamma_k,\,B,\ell_k}\,\psi_{\Gamma_k,\,B,\ell_k}.$$

A channel of the n-particle system is now defined as follows: Partition the indices $1, \cdots, n$ into subsets $\Gamma_1, \cdots, \Gamma_{m+1}$. Select one of the subsets (it may be assumed that it is Γ_{m+1}) as the particles that are "trapped" near the origin (if a situation in which no particles are trapped is considered, then one may by special definition take Γ_{m+1} to be the empty set and set the corresponding bound state $\psi_{\Gamma_{m+1},\,B,\,\ell_{m+1}}$ equal to 1 and its eigenvalue $E_{\Gamma_{m+1},\,B,\,\ell_{m+1}}$ equal to zero. See equations (423) and (424)). Now for each $k = 1, \cdots, m$ select a bound state $\psi_{\Gamma_k,\,n_k}$ as in (416) or (419) (whichever is appropriate). For $k = m+1$ select a bound state $\psi_{\Gamma_{m+1},\,B,\,\ell_{m+1}}$ as in (422). The total wave-function describing the behavior of the n particles in the channel being defined is then

(423) $$e^{-iH_\alpha t}\,g(\vec{X}_{\Gamma_1}, \cdots, \vec{X}_{\Gamma_m})\Big(\prod_{k=1}^{m} \psi_{\Gamma_k,\,n_k}(\vec{Z}_{\Gamma_k})\Big)\psi_{\Gamma_{m+1},\,B,\,\ell_{m+1}}$$

where $g \in L^2(\vec{X}_{\Gamma_1}, \cdots, \vec{X}_{\Gamma_m})$ and

$$H_\alpha = \sum_{k=1}^{m} \frac{-\Delta_{\vec{x}_{\Gamma_k}}}{2M_{\Gamma_k}} + \sum_{k=1}^{m} E_{\Gamma_k, n_k} + E_{\Gamma_{m+1}, B, \ell_{m+1}} \; . \tag{424}$$

α as usual is a multi-index which lists the channel Hamiltonian H_α and all the bound states of (423). Equations (423) and (424) describe a situation in which the distributions of the centers of mass $\vec{X}_{\Gamma_k}$ are determined by the function g and the relevant propagators, the variable $\vec{X}_{\Gamma_k}$ being propagated freely with the appropriate operator $e^{-iH_{0\Gamma_k}t}$ $(H_{0\Gamma_k} = -\Delta_{x_{\Gamma_k}}/2M_{\Gamma_k})$ while the bound states ψ_{Γ_k, n_k} have their appropriate time-dependence $e^{-iE_{\Gamma_k, n_k}t}$, as does $\psi_{\Gamma_{m+1}, B, \ell_{m+1}}$. Physically the situation described is one in which the particles of Γ_{m+1} (if any—see earlier remark that Γ_{m+1} may be empty and how to modify (423) and (424) in this case) are trapped near the origin, while the particles of each set Γ_k, $k=1, \cdots, m$, combine in one of the bound states allowed by their mutual interactions to travel together as a "composite particle," with their center of mass moving freely. The situation described above is a possible asymptotic configuration of the n particles, when the subsets $\Gamma_1, \cdots, \Gamma_m$ are far from the fixed potentials and each other, and Γ_{m+1} contains "trapped" particles. This is one of the typical simple situations one may expect to find at the beginning or end of a scattering experiment. Of course, a linear combination of such things is also possible. If the channel described by (423) and (424) is called "α", then D_α denotes the closed subspace of all functions of the form of the product $g(\prod_{k=1}^{m} \psi_{\Gamma_k, n_k})\ \psi_{\Gamma_{m+1}, B, \ell_{m+1}}$ with $g \in L^2(\vec{X}_{\Gamma_1}, \cdots, \vec{X}_{\Gamma_m})$. If each Γ_k contains one particle, $n = m$, and $\Gamma_{m+1} = \phi$ then one is in the free channel and $D_\alpha = L^2(R^{3n})$ by definition. By special definition, one does not count as a channel the case $m = 0$, $\Gamma_{m+1} = (1, \cdots, n)$ which corresponds to *all* particles being bound by the fixed potentials. One does not expect such a situation to initiate or end a scattering process. With these conventions out of the way, one can now sweep the dirt under the carpet and announce a very clean-looking theorem due to Hack [13]:

THEOREM. *Let H, H_α, D_α be as above, and let all the V_{ij}, $0 \leqq i < j \leqq n$, be square-integrable over R^3. Then the strong limits*

$$\lim_{t \to \pm\infty} e^{iHt} e^{-iH_\alpha t} \equiv \Omega_\alpha^\pm \tag{425}$$

exist on D_α.

In the proof, of course, one has to lift up the carpet again, and do things channel by channel. Although the definitions of channels, etc., are more formidable than in the two-body case, the proof (Cook's method!) is no different in conception than it was in the two-body case, and will not be given here. (The hypotheses of the theorem can be weakened to the case in which each potential is the sum of a square-integrable function and a locally square-integrable function that falls off like $1/|\vec{x}|^{\beta}$, $\beta > 1$, for large $|\vec{x}|$.) The theorem provides operators $\Omega_\alpha^{\pm}$ for each channel α, and the entire discussion of Problems I and II, $R^{\pm}$, S and S′ now goes just as in the two-body case—all assertions on the $\Omega_\alpha^{\pm}$ made in the two-body case remain true, although sometimes the proofs require more writing. The space B is again spanned by the eigenfunctions of the full Hamiltonian. (This corresponds physically to the case in which *all* particles are bound by the fixed potentials—this is the case discarded above in the enumeration of scattering states.) The requirement of asymptotic completeness is $R^{+} = R^{-} \equiv R$ and

$$R \oplus B = L^2(R^{3n}) \ . \tag{426}$$

Asymptotic completeness has been discussed for n-body problems by Klaus Hepp [14], who has actually proved it under certain conditions on the theory; although some of the hypotheses are stronger than one would wish, this work is a major contribution to nonrelativistic quantum mechanics.

The modifications of n-body scattering theory that are necessary when Coulomb potentials are present should be clear—all goes much as in the two-body case. Details can be found in Dollard [4].

Regretfully, there was insufficient time for a discussion of adiabatic switching or screening techniques in the theories presented, or for a discussion of scattering into cones in the n-body theory (see Dollard [5], [6], and [7]). The interested reader should also consult the forthcoming paper [29] of Zinnes and Muhlerin on Coulomb scattering and the papers [24], [25] of Lavine and a forthcoming work of Combes on long-range potentials, as well as the interesting new work of Buslaev and Matveev [30], who consider potentials falling off like $|\vec{x}|^{-\alpha}$ for any $\alpha > 0$.

In conclusion I wish to thank Mrs. Nancy Kirk for her superb typing and her inexhaustible patience concerning revisions.

Appendix

Professor Ekstein has pointed out the following to me. One can obtain convergence of $e^{iHt}\, e^{-iH_\alpha t}$ to $\Omega_\alpha^{\pm}$ on all of L^2 if one is willing to modify the sense in which the convergence takes place. It is in fact almost correct to say that $e^{iHt}\, e^{-iH_\alpha t}$ converges weakly to $\Omega_\alpha^{\pm}$ everywhere, the idea being that on the orthogonal complement $D_\alpha^\perp$ of D_α, $e^{iHt}e^{-iH_\alpha t}$ converges weakly to zero. The reason this is not quite correct can be seen from an example in two-particle scattering. Suppose $\alpha = (H_{2\beta}, \psi_{2\beta})$ is a channel in which particle 2 is asymptotically "trapped" and particle 1 is asymptotically free. Let $\alpha' = (H_{2\beta'}, \psi_{2\beta'})$ be another such channel, where the eigenvalue $E_{2\beta'}$ of $\psi_{2\beta'}$ is *not* equal to the eigenvalue $E_{2\beta}$ of $\psi_{2\beta}$. Then

$$H_{2\beta} = H_{2\beta'} + (E_{2\beta} - E_{2\beta'}) \ . \tag{A1}$$

Thus letting $f(\vec{x}_1)\psi_{2\beta'}(\vec{x}_2) \in D_{2\beta'}$ gives

$$\begin{aligned} & e^{iHt}e^{-iH_{2\beta}t}f(\vec{x}_1)\psi_{2\beta'}(\vec{x}_2) \\ & \qquad = e^{-i(E_{2\beta}-E_{2\beta'})t}e^{iHt}e^{-iH_{2\beta'}t}f(\vec{x}_1)\psi_{2\beta'}(\vec{x}_2) \ . \end{aligned} \tag{A2}$$

Now the expression on the right is $e^{-i(E_{2\beta}-E_{2\beta'})t}$ times something that converges strongly as $t \to \pm\infty$ to $\Omega_{\beta'}^{\pm}f\psi_{2\beta'}$. Thus the right-hand side of (A2) does not converge weakly to zero as $t \to \pm\infty$, i.e., $e^{iHt}e^{-iH_\beta t}$ does not converge weakly to zero on $D_{2\beta'}$. However it should be clear from the explicit form of the right-hand side of (A2) that the Abel limit of this right-hand side is zero, since asymptotically it behaves like $e^{-i(E_{2\beta}-E_{2\beta'})t}$ times a constant. In fact even the strong Abel limit is zero in this case. On other portions of $D_{2\beta}^\perp$, however, convergence is weak and the following theorem holds for n particle scattering:

THEOREM. *Suppose the potentials V_{ij}, $0 \leqq i < j \leqq n$, are square-integrable over R^3. Suppose also that the requirement of asymptotic completeness is satisfied. Then*

$$\operatorname*{w-lim}_{\epsilon \downarrow 0} \epsilon \int_0^\infty e^{-\epsilon s}e^{\pm iHs}e^{\mp iH_\alpha s}\,ds = \Omega_\alpha^\pm \tag{A3}$$

the limit existing on all of L^2.

This theorem is of great interest because it shows, in the case considered at least, that one does not have to know the domains D_α to find $\Omega_\alpha^\pm$. One can just use (A3). In practice this could be very advantageous, since it is most difficult to find bound states explicitly, while not so hard to find eigenvalues and hence the channel Hamiltonians H_α. One can, of course, weaken the hypotheses on the potentials as in the n-body theorem stated earlier. Whether one can weaken them still further because one only wishes to conclude weak Abel convergence is not known to the author, but for Coulomb potentials (A3) does not give the desired operators.

Another observation of Professor Ekstein is that, in an asymptotically complete scattering theory, the equations

(A4) $$H\Omega_\alpha^{\pm} = \Omega_\alpha^{\pm} H_\alpha E_\alpha$$

and

(A5) $$\lim_{t \to \pm\infty} e^{-iH_\alpha t}\Omega_\alpha^{\pm} e^{-iH_\alpha t} = E_\alpha$$

characterize the operators $\Omega^{\pm}$ uniquely, without ever mentioning the "complicated" unitary group e^{-iHt}. Such a characterization might be useful if one wanted to try to use computer techniques on these problems, since one would not have to give the computer the disheartening task of calculating e^{-iHt}.

I thank Professor Ekstein for pointing out these interesting facts.

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