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ABSTRACT WEIGHTED PSEUDO ALMOST AUTOMORPHIC FUNCTIONS, CONVOLUTION INVARIANCE AND NEUTRAL INTEGRAL EQUATIONS WITH APPLICATIONS

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ABSTRACT. This paper deals with a systematic study of the convolution operator defined on the weighted pseudo almost automorphic functions space ($PAA(\mathbb{X},\rho)$). The purpose of this, is to ensure the existence and uniqueness of solutions in $PAA(\mathbb{X},\rho)$ for general abstract neutral integral equations of convolution type. Upon making different assumptions on the kernel k of the convolution operator and the weight ρ we obtain results about the convolution invariance of the operator on $PAA(\mathbb{X},\rho)$. Essentially the assumptions are of two type, one is a new condition on ρ valid for every kernel k and the other is a $(k,\ \rho)$-type condition, in which the kernel k helps the weight ρ. These conditions are not known in the literature. Explicit examples show the utility of these two different assumptions. Taking advantage of the rich properties of the convolution we have obtained new results about composition which permits study the existence and uniqueness of mild solutions in $PAA(\mathbb{X},\rho)$ for general abstract neutral integral equations. The results obtained are directly applied to integro differential equations, partial differential equations, logistic equations and differential equations of first and fractional order, among other. Several examples and concrete classes of differential equations illustrate our results.

1. Introduction. Neutral integral equations have gained importance due to their applications in physics, mechanics, chemistry, engineering and other fields. Significant development has been made in ordinary and partial differential equations involving integral neutral equations.

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In this work, we consider the abstract neutral integral equation on a Banach space $\mathbb{X}$:

$$u(t) = f_0(t, u(t), u(h_0(t))) + \int_{-\infty}^{t} R_1(t,s) f_1(s, u(s), u(h_1(s)))\, ds + \int_{t}^{\infty} R_2(t,s) f_2(s, u(s), u(h_2(s)))\, ds, \tag{1.1}$$

$t \in \mathbb{R}$, where the $h_i : \mathbb{R} \to \mathbb{R}$ are continuous functions with $h_i(\mathbb{R}) = \mathbb{R}$, $f_i : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, for $i = 0, 1, 2$ and $R_i : \mathbb{R} \times \mathbb{R} \to \mathcal{B}(\mathbb{X})$, for $i = 1, 2$.

Equations of the form (1.1) have associated operators that include general operators of Fredholm, Volterra, advanced and delayed type [**6, 12, 17, 28, 46**] and appear in the study of solutions of many evolution equations as partial differential equations, neutral differential equations, fractional order differential equations and integro-differential equations, among others.

The integral equation (1.1) represents a typical dichotomic situation:

$$u(t) = f_0(t, u(t), u(h_0(t))) + \int_{\mathbb{R}} R(t,s) f(s, u(s), u(h(s)))\, ds, \quad t \in \mathbb{R}, \tag{1.2}$$

where R is the Green function of the involved linear part $u'(t) = A(t)u(t)$, of the associated differential system; see Section 4. Particularly, (1.2) could come from a general neutral differential equation

$$u'(t) = A(t)u(t) + \frac{d}{dt}[f_0(t, u(t), u(h_0(t)))] + f(t, u(t), u(h(t))), \quad t \in \mathbb{R},$$

or from an integro-differential equation

$$u'(t) = A(t)u(t) + \int_{\mathbb{R}} R(t,s) f(s, u(s), u(h(s)))\, ds + f_0(t, u(t), u(h_0(t)), \quad t \in \mathbb{R},$$

among others; see [**41, 42, 43, 45, 50**].

Among all of them, we have distinguished classes of integral equations (1.1) with special R_i as $R_i(t,s) = R_i(t-s)$, for $i = 1, 2$, of convolution type or more generally integral equations of type "sub-convolutive" when $\|R_i(t,s)\| \le |k_i(t-s)|$, for $i = 1, 2$. In this case R_i must be bi-almost automorphic (see Definition 2.5). Actually, we are looking for weighted pseudo almost automorphic solutions of (1.1).

The set of almost automorphic functions was introduced in 1962 by S. Bochner [**4**], see also [**18**], which is a significant development of the well-known set of almost periodic functions introduced by H. Bohr. Since then many researchers studied these type of solutions for abstract or/and functional differential equations, we refer to [**13, 22, 24, 27, 39, 40, 48**].

In [**47**], J. Liang et al. introduced the pseudo almost automorphic function theory as an extension of the almost automorphic function theory (see also [**16, 23, 32, 34**]). Afterwards, in [**3**], Blot et al. introduced the weighted pseudo automorphic function theory, which extends the concept of the pseudo almost automorphic function. The concept of weighted pseudo almost periodic function was introduced by Diagana [**16**] as a generalization of the set of pseudo almost periodic maps, introduced by Zhang [**49**].

The central idea of [**3**] was the enlargement of the ergodic component space with the help of a so-called *weighted measure* $d\mu(t)=\rho(t)\,dt$, with $\rho:\mathbb{R}\to\mathbb{R}_+$ a locally integrable function commonly called *weight.* More precisely, a continuous function f defined from $\mathbb{R}$ to the Banach space $\mathbb{X}$ is called a weighted pseudo-almost automorphic function if it can be written as $f=g+\phi$, with g an almost automorphic function and ϕ a weighted ergodic function in the sense that: $\phi:\mathbb{R}\to\mathbb{X}$ is a bounded continuous function satisfying

$$\lim_{r\to\infty}\frac{1}{\int_{-r}^{r}\rho(t)\,dt}\int_{-r}^{r}\|\phi(t)\|_{\mathbb{X}}\rho(t)\,dt=0 \quad \text{with} \lim_{r\to\infty}\int_{-r}^{r}\rho(t)\,dt=\infty.$$

The space of all weighted ergodic functions from $\mathbb{R}$ to $\mathbb{X}$ is denoted by $PAA_0(\mathbb{X},\rho)$.

The space of weighted pseudo almost automorphic functions, which is denoted by $PAA(\mathbb{X},\rho)$, has been investigated by several authors and the interest in this topic is still increasing and far to be complete, see [**51, 8, 20, 7, 37, 11, 38, 5, 35, 10**]. For example, Ding et al. [**20**] revealed several basic properties about nonlinear weighted pseudo almost automorphic functions including equivalence, completeness, translation invariance and composition results. As another example Zheng and Ding [**51**] proved that for every $\rho\in\mathbb{U}_\infty$, the space of weighted pseudo almost automorphic functions is complete under the supremum norm. In the especial case $\rho=1$, these types of results involving the pseudo almost automorphic functions have been studied by Liang et al. [**32, 34**].

A concrete example is given by the neutral logistic differential equation:

$$u'(t) = \alpha u(t) + a u'(t-p) - q(t, u(t), u(t-p)), \quad t \in \mathbb{R}, \tag{1.3}$$

where $\alpha > 0$, $0 \leq |a| < 1$ and $p > 0$. This differential equation has several applications in physical sciences (see [**25, 28, 31**]) and has been considered by several authors. For example, Zitane and Bensouda [**52**] studied some sufficient conditions to ensure the existence and uniqueness of weighted pseudo almost automorphic solutions of the equation (1.3), using the equivalent integral equation of advanced type:

$$u(t) = f_0(t, u(t), u(t-p)) + \int_t^{\infty} R_2(t-s) f_2(s, u(s), u(s-p))\, ds, \tag{1.4}$$

which is a particular convolution version of (1.1). The so called neutral delay integral equation of advanced type (1.4) was introduced in the literature by Burton [**6**]. Both, delayed and advanced integral equations have been studied in Pinto [**43**].

Setting R_2 as the null operator, $f_0 \equiv 0$ and $R_1(t,s) = S(t-s)$, $t \geq s$, we obtain the integral equation:

$$u(t) = \int_{-\infty}^{t} S(t-s) f_2(s, u(s), u(h_2(s)))\, ds.$$

Then if $\{S(t)\}_{t \geq 0}$ represents an integrable resolvent in the sense of Prüss (see [**46,** Definition 1.3]), associated to the kernel $a \in L^1_{loc}(\mathbb{R}_+)$ and generated by the closed operator A, the preceding equation is a natural intermediate step while studying the existence and uniqueness of mild bounded solutions of the deviated semilinear abstract integral equation of fractional order:

$$D_t^{\alpha} u(t) = A u(t) + \int_{-\infty}^{t} a(t-s) A u(s)\, ds + f_2(t, u(t), u(h_2(t))), \quad t \in \mathbb{R}. \tag{1.5}$$

Here the derivative of fractional order is understood in the sense of Weyl. The equation (1.5) was treated, in the non deviated case, by Ponce [**45**] and Chang et al. [**8**]. In recent years, differential equations of fractional order have gained considerable importance for the mathematical community due to their applications in various fields of the science. Significant development has been made in ordinary and partial differential equations involving derivatives of fractional order, we refer to the monographs **[30]**, **[19]**, and **[36]**. In particular Chang et al. **[8]** investigated some existence results of weighted pseudo almost automorphic mild solutions of the equation (1.5). Agarwal et al. **[2]** studied the existence and uniqueness of a weighted pseudo-almost periodic (mild) solution to the semilinear fractional equation:

$$D_t^{\alpha} u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad 1 < \alpha < 2, \tag{1.6}$$

where A is a linear operator of sectorial negative type and D_t^{α} is the standard Riemann-Liouville fractional derivative. Cao et al. **[7]**, studied some sufficient conditions for the existence and uniqueness of weighted pseudo-almost automorphic classical solutions for the following reaction-diffusion differential equations of fractional order:

$$D_t^{\alpha} u(t) = Au(t) + f(t, u(t)), \quad t \in I := [t_0, T], \quad u(t_0) = x_0,$$

where D_t^{α} is also the Riemann-Liouville fractional derivative with $0 < \alpha \leq 1$ and A is the infinitesimal generator of an analytic semigroup. Blot et al. **[3]**, has investigated the existence and uniqueness of weighted pseudo almost automorphic mild solution to the class of abstract semilinear equation:

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.7}$$

where A generates an exponentially stable semigroup, which corresponds to a particular situation of the equation (1.5) without delayed term, when $\alpha = 1$ and $a \equiv 0$.

When $\sigma(A)$, the spectrum of A, does not satisfy $\Re(\sigma(A)) \subseteq (-\infty, 0)$ but A is hyperbolic, then the variation of constant formula carry out to the equation (1.2). In general, (1.2) represents a dichotomic situation where R is the Green's function with "delayed" part R_1 and "advanced" part R_2 (see **[17, 20, 43, 44]**). The reaction-diffusion equation

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \frac{\partial^2 u(t,x)}{\partial x^2} + g(t)u(t,x) + f(t, u(t,x), u(h(t),x)), \\ u(t,0) &= u(t,\pi) = 0, \end{aligned} \tag{1.8}$$

$t \in \mathbb{R}$, $x \in [0, \pi]$, with $g : \mathbb{R} \to \mathbb{C}$ an ergodic function, is a concrete example of a dichotomic case (see Example 4.21 and Theorem 4.22).

Our main goal is to study weighted pseudo-almost automorphic functions and obtain new sufficient conditions for the existence and uniqueness of weighted pseudo almost automorphic solution of the abstract neutral integral (1.1), to be applied to concrete problems as (1.3), (1.5), (1.6) or (1.8). To achieve our goal we apply the Banach Fixed Point Theorem to the respective integral operator associated to (1.1), under the assumption that there exist $k_1 \in L^1(\mathbb{R}_+)$ and $k_2 \in L^1(\mathbb{R}_-)$ such that $\|R_1(t,s)\| \leq |k_1(t-s)|$ for $t \geq s$ and $\|R_2(t,s)\| \leq |k_2(t-s)|$ for $t \leq s$, respectively. For these reasons, as a first step, is natural to look for conditions that guarantee the invariance of the space $PAA(\mathbb{X}, \rho)$ under the convolution operators given by

$$(1.9)\qquad \begin{aligned} (\mathcal{K}_1 f)(t) &= \int_{-\infty}^{t} k_1(t-s) f(s)\, ds, \quad t \in \mathbb{R}, \\ (\mathcal{K}_2 f)(t) &= \int_{t}^{\infty} k_2(t-s) f(s)\, ds, \quad t \in \mathbb{R}. \end{aligned}$$

Convolution operators appear in problems for which the invariance, in some spaces, is fundamental. On $PAA(\mathbb{X}, \rho)$ is necessary to establish the convolution invariance of the ergodic space $PAA_0(\mathbb{X}, \rho)$ which is not know. The absence of a convolution invariance result for a class of kernels $k_1 \in L^1(\mathbb{R}_+)$ or $k_2 \in L^1(\mathbb{R}_-)$, associated to $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, does not allow to treat several important cases without imposing an exponential domination on the kernels. In the general case, $PAA_0(\mathbb{X}, \rho)$ is not necessarily invariant by $\mathcal{K}_1$ or $\mathcal{K}_2$. We build explicit functions $f(\cdot)$, $\rho(\cdot)$ such that $f \in PAA_0(\mathbb{X}, \rho)$ but $\mathcal{K}_1 f \notin PAA_0(\mathbb{X}, \rho)$. It seems that our example is not known in the literature. For this reason we will focus on findings sufficient conditions over k_i $(i = 1, 2)$ and ρ to ensure that $\mathcal{K}_i \phi \in PAA_0(\mathbb{X}, \rho)$, whenever $\phi \in PAA_0(\mathbb{X}, \rho)$.

Two sufficient conditions are found, one of ρ-type which depends only on the weight ρ and the other of (k_i, ρ)-type which depends on the kernel k_i and ρ. Thus for example, in the very important case $k(t) = e^{-t}$, $t \in \mathbb{R}$, is possible to obtain new conditions for ρ which give validity of the results obtained. Both are new conditions, however the (k_i, ρ)-type condition is more especial because permits to treat cases where the weight ρ does not satisfy conditions which are frequently used in the context of results involving weighted pseudo almost automorphic functions, as for example, for each $s \in \mathbb{R}$, $\limsup_{|t|\to\infty} \frac{\rho(t+s)}{\rho(t)} < \infty$ (see **[45, 20]**). In this context we are found an explicit example of k_1 and ρ such that satisfies the second type condition but the first type does not, see Example 3.6. The (k_i, ρ)-type condition is obtained thanks to the rich properties of convolutions, allowing to treat new weights that never can be considered in previous works. Also this rich properties of convolutions allows us to give precise conditions to fundamental questions as translation invariance, convolution invariance, composition theorems and contractivity conditions, among others.

All of the mild solutions considered in (1.3), (1.7) (1.5), (1.6) and (1.8) can be described as a solution of (1.1) with especial selected $\mathcal{K}_i$, $i=1,2$. Thus the next natural second step, is the study of sufficient conditions on f to ensure $\mathcal{K}_i f(\cdot, u(\cdot)) \in PAA(\mathbb{X},\rho)$ when $f \in PAA(\mathbb{X},\mathbb{X},\rho)$ and $u \in PAA(\mathbb{X},\rho)$. The most common is the Lipschitz condition

$$\|f(t,x)-f(t,y)\| \leq L\|x-y\| \quad x,y \in \mathbb{X}, \tag{1.10}$$

for some $L \in \mathbb{R}$ (see **[52]**, **[3]**, **[45]** and **[8]**). However if for example we set $f(t,x)=a(t)x+l(t)x$, $x\in\mathbb{X}$, $t\in\mathbb{R}$ where a is a real almost automorphic function and $l \in PAA(\mathbb{R},\rho)$, the constant L in the inequality (1.10) is replaced by the real function described by $L(t)=a(t)+l(t)$, $t\in\mathbb{R}$, in such case $L \in PAA(\mathbb{R},\rho)$. Even more, as will be shown in the Example 4.5 there is a function f such that L does not belongs to $PAA(\mathbb{R},\rho)$ and for this case the convolution composition results will be crucial. In this direction, thanks to the conditions of ρ-type and (k_i,ρ)-type, we have developed a new way of finding sufficient conditions, making use of the convolution on the Lipschitz functions associated, to ensure that $\mathcal{K}_i f(\cdot, u(\cdot)) \in PAA(\mathbb{X},\rho)$ when $f \in PAA(\mathbb{X},\mathbb{X},\rho)$ and $u \in PAA(\mathbb{X},\rho)$, see Theorem 4.3. Particularly in this procedure we do not consider an intermediate step to ensure $f(\cdot, u(\cdot)) \in PAA(\mathbb{X},\rho)$ when $u \in PAA(\mathbb{X},\rho)$, as has traditionally been done. The most important consequence of this is given by the Example 4.5, which shows how the convolution composition results permits to treat functions f that can not addresses by previous works (see **[52]**, **[3]**, **[45]** and **[8]**).

We apply our main theorems in the cases described in (1.3), (1.7) (1.5), (1.6) and (1.8) to obtain new results on existence and uniqueness of solution in $PAA(\mathbb{X},\rho)$. In all of these cases we extend and improve existing results.

The paper is organized as follows. In Section 2 we introduce the notation and definitions involved in the weighted almost automorphic functions and recall some concepts and previous results. In Section 3, we state and prove results about the invariance of $PAA(\mathbb{X},\rho)$ by the convolution operator $\mathcal{K}_i$ $(i=1,2)$ and some consequences. To close the article, in Section 4 we state sufficient conditions, including compositions theorem, in order to ensure that the abstract neutral integral equation (1.1) has a unique weighted pseudo almost automorphic solution. In addition we use these results to ensure the existence and uniqueness of mild solution in $PAA(\mathbb{X},\rho)$ for different abstract differential equations and partial differential equations.

2. Preliminaries. In this section we present the concept of a weighted pseudo almost automorphic function and related concepts like the weighted mean of a function and the weighted ergodic function. Moreover, we also recall some useful results.

Most of the notation used throughout this paper are standard. Hence, we will denote by $\mathbb{N},\mathbb{Z},\mathbb{R},\mathbb{R}_+$, $\mathbb{R}_-$ and $\mathbb{C}$ the sets of natural, integer, real, $[0,\infty)$, $(-\infty,0]$ and complex numbers respectively. For the rest of the paper, $\mathbb{X}$ and $\mathbb{Y}$ always are Banach spaces with norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$, the subscripts will be dropped when there is no danger of confusion. We denote by $C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$, $BC(\mathbb{R},\mathbb{X})$ and $\mathcal{B}(\mathbb{X})$ for the jointly continuous function from $\mathbb{R}\times\mathbb{Y}$ to $\mathbb{X}$, the Banach space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$ and the space of all bounded linear operators from $\mathbb{X}$ into itself, respectively.

Weight notion, related notation and results. We say that a function ρ is a weight if it has the following properties. (i) ρ is defined from $\mathbb{R}\to\mathbb{R}_+$, (ii) ρ is locally integrable over $\mathbb{R}$ and (iii) ρ is strictly positive almost everywhere on $\mathbb{R}$. The set of such functions is denoted by $\mathbb{U}$. Now, in relation to the weight notion, we introduce the function μ defined by follows

$$\mu(r,\rho) := \int_{-r}^{r} \rho(t)\,dt$$

as well as the sets

$$\mathbb{U}_\infty = \{\rho\in\mathbb{U} : \lim_{r\to\infty}\mu(r,\rho)=\infty\},$$
$$\mathbb{U}_B = \{\rho\in\mathbb{U}_\infty : \rho \text{ is bounded with } \inf_{x\in\mathbb{R}}\rho(x)>0\}.$$

Note that $\mathbb{U}$, $\mathbb{U}_\infty$ and $\mathbb{U}_B$ are the collections of all possible weight functions, the weights such that belong $L^1_{loc}(\mathbb{R})$-$L^1(\mathbb{R})$, and the positive bounded weights, respectively. Clearly, the sets $\mathbb{U}$, $\mathbb{U}_\infty$ and $\mathbb{U}_B$ are not empty. Two examples of weight functions are given by ρ_0 , $\rho_1:\mathbb{R}\to\mathbb{R}_+$

defined by

$$\rho_0(t) = \frac{a+b|t|}{1+b|t|}, \qquad \text{with } a \geq 1,\ b > 0,$$
$$\rho_1(t) = e^{(1-t)};$$

we observe that $\rho_0 \in \mathbb{U}_B$ and $\rho_1 \in \mathbb{U}_\infty - \mathbb{U}_B$.

Definition 2.1. Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. Then, we say that ρ_1 is equivalent to ρ_2 if $\rho_1/\rho_2 \in \mathbb{U}_B$. The equivalence of ρ_1 and ρ_2 is denoted by $\rho_1 \sim \rho_2$.

We note that $\rho_1 \sim \rho_2$ if and only if there exist $a_i > 0$ for $i = 1, 2$ such that $a_1\rho_2 \leq \rho_1 \leq a_2\rho_2$ and it implies $a_1\mu(r, \rho_2) \leq \mu(r, \rho_1) \leq a_2\mu(r, \rho_2)$. Hence $\sim$ is a binary equivalence relation on $\mathbb{U}_\infty$. The equivalence class of a given weight $\rho \in \mathbb{U}_\infty$ will be denoted by

$$cl(\rho) = \{\bar{\rho} \in \mathbb{U}_\infty : \bar{\rho} \sim \rho\}.$$

It is clear that $\mathbb{U}_\infty = \cup_{\rho \in \mathbb{U}_\infty} cl(\rho)$.

In order to introduce the weighted pseudo almost automorphic functions, we firstly need to define the "weighted ergodic" space $PAA_0(\mathbb{X}, \rho)$. Then, the weighted pseudo almost automorphic functions appear as perturbations of almost automorphic functions by elements of $PAA_0(\mathbb{X}, \rho)$ (see Definition 2.6). Indeed, we shall introduce notation and then we precise the definition of $PAA_0(\mathbb{X}, \rho)$. Let us recall that, given $\rho \in \mathbb{U}$, the weighted mean of $g : \mathbb{R} \to \mathbb{X}$, which will be used, is denoted by $\mathcal{M}(\|g\|)$ and is defined by the real limit

$$\mathcal{M}(\|g\|) = \lim_{r \to \infty} \frac{1}{\mu(r, \rho)} \int_{-r}^{r} \|g(s)\|\rho(s)\, ds, \tag{2.1}$$

whenever this limit exists. Now assume that $\rho \in \mathbb{U}_\infty$, and define the weighted ergodic space associated to ρ by

$$PAA_0(\mathbb{X}, \rho) := \{f \in BC(\mathbb{R}, \mathbb{X}) : \mathcal{M}(\|f\|) = 0\}.$$

We define $PAA_0(\mathbb{Y}, \mathbb{X}, \rho)$ as the collection of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ such that $F(\cdot, y)$ is bounded for each $y \in \mathbb{Y}$ and $\mathcal{M}(\|F(\cdot, y)\|) = 0$ uniformly in $y \in \mathbb{Y}$.

The restriction, $\rho \in \mathbb{U}_B$ reduces $PAA_0(\mathbb{X}, \rho)$ to $PAA_0(\mathbb{X}, 1)$ then we can ask. What is the relationship between $PAA_0(\mathbb{X}, \rho_1)$ and $PAA_0(\mathbb{X}, \rho_2)$

for $\rho_1, \rho_2 \in \mathbb{U}_\infty$? This question was studied by Ding et al. [**20**] establishing different results; see Theorems 2.1, 2.2 and 2.3.

An interesting example, which is a consequence of the above mentioned theorems given in [**20,** Example 2.7], shows that if $\rho_1(t) = |t|^n$, $n \in \mathbb{N}$ and $\rho_2(t) = 1$ for $t \in \mathbb{R}$, then $PAA_0(\mathbb{X}, \rho_1) = PAA_0(\mathbb{X}, \rho_2)$ while ρ_1, ρ_2 are not equivalent.

To finish this subsection, we give a useful criterion for establishing when a function $f \in BC(\mathbb{R}, \mathbb{X})$ belong to $PAA_0(\mathbb{X}, \rho)$.

Lemma 2.2. [**21,** Lemma 1.1] *Let $\rho \in \mathbb{U}_\infty$. Then $f \in PAA_0(\mathbb{X}, \rho)$ if and only if for any $\varepsilon > 0$*

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{M_{r,\varepsilon}} \rho(t)\, dt = 0,$$

where $M_{r,\varepsilon}(f) = \{t \in [-r, r] : \|f(t)\| \geq \varepsilon\}$.

Definition of weighted pseudo almost automorphic functions.

Definition 2.3. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of such functions will be denote by $AA(\mathbb{X})$.

We recall that every almost periodic function is almost automorphic, but the class of almost automorphic functions is larger than the class of almost periodic solutions. For example, the function

$$f(t) = \cos \frac{1}{2 + \sin(\sqrt{2}t) + \sin t}, \quad t \in \mathbb{R},$$

is almost automorphic but not almost periodic. Next we recall the definition of weighted pseudo almost automorphic functions.

Definition 2.4. A continuous function $F:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ is said to be almost automorphic if $F(t,x)$ is almost automorphic w.r.t. $t\in\mathbb{R}$ uniformly for all $x\in K$, where K is any bounded subset of $\mathbb{Y}$. The collection of such functions will be denote by $AA(\mathbb{Y},\mathbb{X})$.

Definition 2.5. A continuous function $R:\mathbb{R}\times\mathbb{R}\to\mathcal{B}(\mathbb{X})$ is said to be bi-almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$R^*(t,s)x := \lim_{n\to\infty} R(t+s_n, s+s_n)x$$

is well defined for each $x\in\mathbb{X}$, $t,s\in\mathbb{R}$ and

$$\lim_{n\to\infty} R^*(t-s_n, s-s_n)x = R(t,s)x$$

for each $x\in\mathbb{X}$, $t,s\in\mathbb{R}$.

Definition 2.6. Let $\rho\in\mathbb{U}_\infty$.

(a) A function $f\in BC(\mathbb{R},\mathbb{X})$ is called a weighted pseudo almost automorphic if it can be expressed as follows $f=g+\phi$, where $g\in AA(\mathbb{X})$ and $\phi\in PAA_0(\mathbb{X},\rho)$. The collection of such kind of functions will be denoted by $PAA(\mathbb{X},\rho)$.

(b) A function $F\in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is called a weighted pseudo almost automorphic in $t\in\mathbb{R}$ uniformly in K, where K is any bounded subset of $\mathbb{Y}$, if it can be expressed as follows $f=g+\phi$, where $g\in AA(\mathbb{Y},\mathbb{X})$ and $\phi\in PAA_0(\mathbb{Y},\mathbb{X},\rho)$. The collection of such kind of functions will be denoted by $PAA(\mathbb{Y},\mathbb{X},\rho)$.

Now we summarize some useful results about the spaces $AA(\mathbb{X})$ and $PAA(\mathbb{X},\rho)$.

Lemma 2.7. **[39]** *$AA(\mathbb{X})$ is a Banach space endowed with the supremum norm given by*

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|.$$

We say that $PAA_0(\mathbb{X},\rho)$ is translation invariant if and only if $f\in PAA_0(\mathbb{X},\rho)$ implies $f((\cdot)+s)\in PAA_0(\mathbb{X},\rho)$ for all $s\in\mathbb{R}$.

Lemma 2.8. [**33**] *Let $\rho \in \mathbb{U}_\infty$. Suppose that $PAA_0(\mathbb{X},\rho)$ is translation invariant. Then the decomposition of weighted almost automorphic functions is unique.*

We note that when $\rho \in \mathbb{U}_\infty$ the space $PAA(\mathbb{X},\rho)$ cannot be always decomposed as a direct sum of $AA(\mathbb{X})$ and $PAA_0(\mathbb{X},\rho)$. In [**33**] exhibit an almost periodic function f and $\rho \in \mathbb{U}_\infty$ such that $f \in PAA_0(\mathbb{X},\rho)$. Since the almost periodic functions are contained in $AA(\mathbb{X})$, it follows that the decomposition of f is not unique in $PAA(\mathbb{X},\rho)$.

Mophou in [**38,** Theorem 2.15] shows that $PAA(\mathbb{X},\rho)$ is a Banach space endowed with the supremum norm, under the assumption that $PAA_0(\mathbb{X},\rho)$ is translation invariant. Recently Zheng and Ding [**51**] extended the mentioned Mophou result and prove that the completeness of $PAA(\mathbb{X},\rho)$ holds for all $\rho \in \mathbb{U}_\infty$.

Theorem 2.9. [**51**] *For every $\rho \in \mathbb{U}_\infty$, $PAA(\mathbb{X},\rho)$ is a Banach space under the supremum norm.*

3. Convolution and some consequences. In this section we find sufficient conditions for invariance of the space $PAA_0(\mathbb{X},\rho)$ under the convolution operators $\mathcal{K}_i$ for $i = 1, 2$, defined in (1.9).

The authors of [**14**] considered the problem of invariance for the space of weighted pseudo almost periodic functions and found three sufficient conditions over k and ρ which imply a positive answer to the question of convolution invariance of $PAA_0(\mathbb{X},\rho)$. They, for a fixed $k \in L^1(\mathbb{R})$ and $f \in BC(\mathbb{R},\mathbb{X})$, considered the convolution operator

$$\mathcal{K}f(t) = \int_{-\infty}^{\infty} k(t-s)f(s)\,ds, \quad t \in \mathbb{R}.$$

The results and the proofs about $\mathcal{K}$-convolution invariance of [**14**] allow us to establish theorems to ensure the invariance of $PAA(\mathbb{X},\rho)$ by $\mathcal{K}_i$, for $i = 1, 2$.

Convolution invariance.

Proposition 3.1. [**17,** pp. 17–19] *Fix $\rho \in \mathbb{U}_B$. Let $f \in PAA_0(\mathbb{X},\rho)$ and $k_1 \in L^1(\mathbb{R}_+)$. Then $\mathcal{K}_1 f$ belongs to $PAA_0(\mathbb{X},\rho)$.*

The assumption $\rho \in \mathbb{U}_B$ implies that $PAA_0(\mathbb{X},\rho) = PAA_0(\mathbb{X},1)$. However, if the weight $\rho_1 \in \mathbb{U}_\infty$ satisfies Theorems (2.2) and (2.3) of **[20]** with $\rho_2 = 1$, then $PAA_0(\mathbb{X},\rho_1) = PAA_0(\mathbb{X},1)$ also. In order to overcome this situation, throughout this work we assume that

$$\rho \in \mathbb{U}_\infty \quad \text{and} \quad k_1 \in L^1(\mathbb{R}_+). \tag{3.1}$$

Also we denote $|\mathcal{K}|_1\|f\| : \mathbb{R} \to \mathbb{R}_+$ by

$$(|\mathcal{K}|_1\|f\|)(t) = \int_{-\infty}^{t} |k_1(t-s)|\|f(s)\|\, ds, \tag{3.2}$$

for $f \in BC(\mathbb{R},\mathbb{X})$. Naturally we say that $PAA_0(\mathbb{X},\rho)$ is invariant by $|\mathcal{K}|_1$ if for all $f \in PAA_0(\mathbb{X},\rho)$ we have $|\mathcal{K}|_1\|f\| \in PAA_0(\mathbb{R},\rho)$. Note that the $PAA_0(\mathbb{X},\rho)$ invariance by $|\mathcal{K}|_1$ is sufficient condition for the $PAA_0(\mathbb{X},\rho)$ invariance by $\mathcal{K}_1$, since $\|(\mathcal{K}_1 f)(t)\| \le (|\mathcal{K}|_1\|f\|)(t)$, for all $t \in \mathbb{R}$.

Theorems 3.2, 3.3 and 3.5 given in **[14]** will help us to find sufficient conditions on k_1 and ρ to ensure that the $\mathcal{K}_1$-convolution operator, defined in (1.9), maps $PAA_0(\mathbb{X},\rho)$ into itself.

The next theorem is a result of (k_1,ρ)-type. It is original.

Theorem 3.2. *Let ρ and k_1 satisfy* (3.1) *and*

$$\sup_{|s|\le r, r\in\mathbb{R}} \frac{1}{\rho(s)} \int_s^r |k_1(t-s)|\rho(t)\, dt < \infty, \tag{3.3}$$

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-\infty}^{-r} \left(\int_{-r}^{r} |k_1(t-s)|\rho(t)\, dt \right) ds = 0. \tag{3.4}$$

Then $f \in PAA_0(\mathbb{X},\rho)$ implies $\mathcal{K}_1 f \in PAA_0(\mathbb{X},\rho)$.

Proof. Let $f \in PAA_0(\mathbb{X},\rho)$. Since $k_1 \in L^1(\mathbb{R}_+)$ and $f \in BC(\mathbb{R},\mathbb{X})$, we have $\mathcal{K}_1 f \in BC(\mathbb{R},\mathbb{X})$. In order to prove that $\mathcal{K}_1 f \in PAA_0(\mathbb{X},\rho)$ we must deduce that $\mathcal{M}(\|\mathcal{K}_1 f\|) = 0$. In fact, from (3.2) and applying Fubini's theorem, we obtain

$$\mathcal{M}(\|\mathcal{K}_1 f\|) = \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \|(\mathcal{K}_1 f)(t)\|\rho(t)\, dt$$

$$\le \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} (|\mathcal{K}|_1 \|f\|)(t)\rho(t)\, dt$$
$$= \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t) \int_{-\infty}^{t} |k_1(t-s)| \|f(s)\| \, ds\, dt$$
$$= \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t) \left(\int_{-\infty}^{-r} |k_1(t-s)| \|f(s)\| \, ds \right) dt$$
$$+ \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t) \left(\int_{-r}^{t} |k_1(t-s)| \|f(s)\| \, ds \right) dt$$
$$= \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-\infty}^{-r} \|f(s)\| \int_{-r}^{r} |k_1(t-s)| \rho(t)\, dt\, ds$$
$$+ \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \|f(s)\| \int_{s}^{r} |k_1(t-s)| \rho(t)\, dt\, ds$$
$$= \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-\infty}^{-r} \|f(s)\| \int_{-r}^{r} |k_1(t-s)| \rho(t)\, dt\, ds \tag{3.5}$$
$$+ \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \|f(s)\| \rho(s) \left[\frac{1}{\rho(s)} \int_{s}^{r} |k_1(t-s)| \rho(t)\, dt \right] ds.$$

These two limits are zero: the first, because $f \in BC(\mathbb{R}, \mathbb{X})$ and (3.4) is satisfied; the second, from (3.3) and because $f \in PAA_0(\mathbb{X}, \rho)$. It follows that $\mathcal{M}(\|\mathcal{K}_1 f\|) = 0$. □

The next theorem is a result of ρ-type.

Theorem 3.3. *Assume that* (3.1) *holds and that for each* $s \ge 0$ *there exists* $r_s \ge 0$ *such that*

$$C^+(s) := \sup_{r \ge r_s} \sup_{t \in \Omega^+_{r,s}} \frac{\rho(t+s)}{\rho(t)} < \infty, \tag{3.6}$$

where

$$\Omega^+_{r,s} = \{t \in \mathbb{R} : r_s + s \le |t| \le r + s\} \cap \{t \in \mathbb{R} : \rho(t) \ne 0\}.$$

Then $f \in PAA_0(\mathbb{X}, \rho)$ *implies* $\mathcal{K}_1 f \in PAA_0(\mathbb{X}, \rho)$.

Proof. For $f \in PAA_0(\mathbb{X}, \rho)$, from $k_1 \in L^1(\mathbb{R}_+)$ and $f \in BC(\mathbb{R}, \mathbb{X})$ it follows that $\mathcal{K}_1 f \in BC(\mathbb{R}, \mathbb{X})$. To prove that $\mathcal{M}(\|\mathcal{K}_1 f\|) = 0$, let $r \ge 0$;

then

$$\begin{aligned}\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\|\mathcal{K}_1 f(t)\|\rho(t)\,dt &\leq \frac{1}{\mu(r,\rho)}\int_{-r}^{r}(|\mathcal{K}|_1\|f\|)(t)\rho(t)\,dt\\ &= \frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_{-\infty}^{t}|k_1(t-s)|\|f(s)\|\,ds\,\rho(t)\,dt\\ &= \frac{1}{\mu(r,\rho)}\int_{-r}^{r}\int_{0}^{\infty}|k_1(s)|\|f(t-s)\|\rho(t)\,ds\,dt.\end{aligned}$$

Using Fubini's theorem, we obtain

$$\begin{aligned}\frac{1}{\mu(r,\rho)}&\int_{-r}^{r}\|\mathcal{K}_1 f(t)\|\rho(t)\,dt\\ &\leq \int_0^\infty |k_1(s)|\left[\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\|f(t-s)\|\rho(t)\,dt\right]ds\\ &= \int_0^\infty |k_1(s)| f_{r,\rho}(s)\,ds,\end{aligned}\tag{3.7}$$

where $f_{r,\rho}$ is defined by

$$f_{r,\rho}(s) = \frac{1}{\mu(r,\rho)}\int_{-r}^{r}\|f(t-s)\|\rho(t)\,dt.\tag{3.8}$$

By the definition of $f_{r,\rho}$ we can deduce that $|f_{r,\rho}(s)| \leq \|f\|_\infty$, which immediately implies, for each $s \geq 0$, that

$$\|k_1(s)f_{r,\rho}(s)\| \leq |k_1(s)|\|f\|_\infty, \ \text{ for all } r \in \mathbb{R}_+.\tag{3.9}$$

We claim that for all $s \geq 0$, $f_{r,\rho}(s) \to 0$ as $r \to \infty$. Indeed, let $s \geq 0$ fixed and $r \geq r_s$ big enough. Since $\{t \in \mathbb{R} : r_s \leq |t| \leq r\} \subseteq \Omega^+_{r,s}$ we have

$$\begin{aligned}\mu(r+s,\rho) &= \mu(r_s+s,\rho) + \int_{r_s+s\leq|t|\leq r+s}\rho(t)\,dt\\ &= \mu(r_s+s,\rho) + \int_{r_s\leq|t|\leq r}\rho(t+s)\,dt\\ &\leq \mu(r_s+s,\rho) + C^+(s)\int_{r_s\leq|t|\leq r}\rho(t)\,dt\\ &\leq \mu(r_s+s,\rho) + C^+(s)\int_{-r}^{r}\rho(t)\,dt.\end{aligned}\tag{3.10}$$

Since $\lim\limits_{r\to\infty} \frac{\mu(r_s+s,\rho)}{\mu(r,\rho)} = 0$ there exists $K>0$ such that $\sup\limits_{r\in\mathbb{R}_+} \frac{\mu(r_s+s,\rho)}{\mu(r,\rho)} \le K$. Thus, (3.10) become into

$$\mu(r+s,\rho) \le (K+C^+(s))\mu(r,\rho), \quad r>r_s. \tag{3.11}$$

It follows from (3.11) and $\{t\in\mathbb{R} : r_s+s\le |t|\le r+s\}\subseteq \Omega^+_{r,s}$ that

$$\begin{aligned}
0 \le f_{r,\rho}(s) &= \frac{1}{\mu(r,\rho)}\int_{-r}^{r} \|f(t-s)\|\rho(t)\,dt \\
&= \frac{1}{\mu(r,\rho)}\int_{-r-s}^{r-s} \|f(t)\|\rho(t+s)\,dt \\
&\le \frac{1}{\mu(r,\rho)}\int_{-(r+s)}^{r+s} \|f(t)\|\rho(t+s)\,dt \\
&= \frac{1}{\mu(r,\rho)}\left[\int_{-(r_s+s)}^{r_s+s} \|f(t)\|\rho(t+s)\,dt + \int_{r_s+s\le|t|\le r+s} \|f(t)\|\rho(t+s)\,dt\right] \\
&\le \frac{1}{\mu(r,\rho)}\left[\int_{-(r_s+s)}^{r_s+s} \|f(t)\|\rho(t+s)\,dt + C^+(s)\int_{-(r+s)}^{r+s} \|f(t)\|\rho(t)\,dt\right] \\
&\le \frac{\|f\|_\infty}{\mu(r,\rho)}\int_{-(r_s+s)}^{r_s+s} \rho(t+s)\,dt + \frac{\mu(r+s,\rho)}{\mu(r,\rho)}\frac{C^+(s)}{\mu(r+s,\rho)}\int_{-(r+s)}^{r+s} \|f(t)\|\rho(t)\,dt \\
&\le \frac{\|f\|_\infty}{\mu(r,\rho)}\int_{-r_s}^{r_s+2s} \rho(t)\,dt + (K+C^+(s))\frac{C^+(s)}{\mu(r+s,\rho)}\int_{-(r+s)}^{r+s} \|f(t)\|\rho(t)\,dt.
\end{aligned} \tag{3.12}$$

The first term of the preceding inequality tends to zero, since r_s, s are fixed and $\mu(r,\rho)\to\infty$ as $r\to\infty$. Now, the condition $f\in PAA_0(\mathbb{X},\rho)$ implies that the second term tends to zero as $r\to\infty$. Consequently $f_{r,\rho}(s)\to 0$ as $r\to\infty$ for all $s\ge 0$, showing the claimed. Thus, naturally $|k_1(s)|f_{r,\rho}(s)\to 0$ as $r\to\infty$ almost everywhere. Then, taking into account the inequality (3.9) and that the function $k_1\|f\|_\infty\in L^1(\mathbb{R}_+)$, we can apply the Lebesgue dominated convergence theorem to (3.7) to conclude that $\mathcal{M}(\|\mathcal{K}_1 f\|)=0$. Hence $\mathcal{K}_1 f\in PAA_0(\mathbb{X},\rho)$. □

Remark 3.4. In the proof of Theorems 3.2 or 3.3, the conditions are sufficient to ensure that $PAA_0(\mathbb{X},\rho)$ is invariant by $|\mathcal{K}|_1$.

Remark 3.5. We emphasize that the condition of Theorem 3.3 are satisfied by a wide range of weight functions. For instance, if we consider the weight $\rho\in\mathbb{U}_\infty$ defined by $\rho(t)=|t|^\beta$, $\beta>0$, we can note that for $s\ge 0$, $r_s=0$ and $r\ge r_s$ fixed $\Omega^+_{r,s}=\{t\in\mathbb{R} : |t|\le r+s\}\setminus\{0\}$ and for all

$t \in \Omega^+_{r,s}$

$$\frac{\rho(t+s)}{\rho(t)} = \frac{|t+s|^\beta}{|t|^\beta} \le \frac{||t|+s|^\beta}{|t|^\beta} \le \left(\frac{r+2s}{r+s}\right)^\beta \le 2^\beta.$$

Then ρ satisfies the condition of Theorem 3.3. In a similar way we can show the same, setting the weight ρ by $\rho(t) = e^{\alpha t}$, $\alpha \in \mathbb{R}$. We can observe that, if the weights ρ_1 and ρ_2 satisfy the condition of Theorem 3.3, also $\rho_1 + \rho_2$ and $\rho_1\rho_2$ they do. In this manner the weight defined by $\rho(t) = \sum_{i=0}^N c_i |t|^{\beta_i} e^{\alpha_i t}$, where $N \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ and $c_i, \beta_i > 0$ for all $i \in \{1, 2, \dots N\}$, satisfies the condition of Theorem 3.3, which implies that $\mathcal{K}_1 f \in PAA_0(\mathbb{X}, \rho)$ provided $f \in PAA_0(\mathbb{X}, \rho)$. Also, if the weight ρ satisfy that $\rho(t+s) \le C\rho(t)\rho(s)$ for some $C > 0$ and for all $t, s \in \mathbb{R}$ we obtain that the condition of Theorem 3.3 is satisfied.

In contrast to the foregoing, there are examples of weights that do not satisfy the condition of Theorem 3.3. In fact, let the weight $\rho \in \mathbb{U}_\infty$ defined by $\rho(t) = e^{\alpha t^2}$, $\alpha > 0$. For $r, s \ge 0$ given and $t_r \in \Omega^+_{r,s}$ defined by $t_r = r + s$ we have

$$\frac{\rho(t_r+s)}{\rho(t_r)} = \frac{e^{\alpha(t_r+s)^2}}{e^{\alpha t_r^2}} = e^{2\alpha t_r s} e^{\alpha s^2} = e^{2\alpha r s} e^{2\alpha s^2 + \alpha s^2}.$$

Since the set $\{e^{2\alpha r s} : r > 0\}$ is not bounded, the weight ρ can not satisfy the condition of Theorem 3.3. Moreover we can see that for each $k_1 \in L^1(\mathbb{R}_+)$ the condition (3.3) of Theorem 3.2 is not satisfied. In fact, if $r > 2$ and $s_r = \frac{r}{2}$ we have

$$\begin{aligned}
\frac{1}{\rho(s_r)} \int_{s_r}^r |k_1(t-s_r)|\rho(t)\,dt &= \frac{1}{\rho(s_r)} \int_0^{\frac{r}{2}} |k_1(t)|\rho(t+s_r)\,dt \\
&= \int_0^{\frac{r}{2}} |k_1(t)| e^{t^2} e^{t\cdot r} dt \\
&\ge \int_1^{\frac{r}{2}} |k_1(t)| e^{tr} dt \ge e^r \int_1^{\frac{r}{2}} |k_1(t)|\,dt.
\end{aligned}$$

Thus we cannot take into account Theorem 3.2 to impose conditions on k_1 to ensure that $\mathcal{K}_1 f \in PAA_0(\mathbb{X}, \rho)$ provided $f \in PAA_0(\mathbb{X}, \rho)$.

Hence we can ask, is there some $k_1 \in L^1(\mathbb{R}_+)$ and a weight ρ such that condition (3.6) of Theorem 3.3 is not satisfied but conditions (3.3) and (3.4) are satisfied? The answer is yes:

Example 3.6. Take $k_1(t) = e^{-t}$ for $t \geq 0$ and $\rho = 1 + \sum_{n=0}^{\infty} \mathcal{X}_n$, where

$$\mathcal{X}_n(t) = \begin{cases} 2^{2n} & \text{if } t \in [n, n + \frac{1}{2^{2n}}], \\ 0 & \text{otherwise.} \end{cases}$$

Let $r \geq 0$, $s \in \mathbb{R}$ and let $x_s \in \mathbb{Z}$ be the integer such that $s \in [x_s, x_s + 1)$. Observe that

$$\begin{aligned} \int_s^r e^{-t}\rho(t)\,dt &= \int_s^r e^{-t}(1 + \sum_{n=0}^{\infty} \mathcal{X}_n(t))\,dt \\ &\leq e^{-s} - e^{-r} + \int_{x_s}^r e^{-t} \sum_{n=0}^{\infty} \mathcal{X}_n(t)\,dt \\ &\leq e^{-s} - e^{-r} + \sum_{n=\max\{0,x_s\}}^{\infty} \int_n^{n+\frac{1}{2^{2n}}} e^{-t} 2^{2n} dt \\ &= e^{-s} - e^{-r} + \sum_{n=\max\{0,x_s\}}^{\infty} 2^{2n} e^{-n}(1 - e^{\frac{-1}{2^{2n}}}) \\ &\leq e^{-s} - e^{-r} + \sum_{n=\max\{0,x_s\}}^{\infty} e^{-n} \\ &= e^{-s} - e^{-r} + e^{-\max\{0,x_s\}} \left(\frac{1}{1 - \frac{1}{e}} \right) \\ &\leq e^{-s} - e^{-r} + e^{-(s-1)} \left(\frac{1}{1 - \frac{1}{e}} \right). \end{aligned} \tag{3.13}$$

Thus,

$$\begin{aligned} \sup_{|s| \leq r, r \in \mathbb{R}} \frac{1}{\rho(s)} \int_s^r |k_1(t-s)|\rho(t)\,dt &= \sup_{|s| \leq r, r \in \mathbb{R}} \frac{e^s}{\rho(s)} \int_s^r e^{-t}\rho(t)\,dt \\ &\leq \sup_{|s| \leq r, r \in \mathbb{R}} \left(1 - e^{-(r-s)} + \left(\frac{e}{1 - \frac{1}{e}} \right) \right) \\ &< \infty. \end{aligned}$$

In view of (3.13) we can write

$$\begin{aligned} \frac{1}{\mu(r,\rho)} \int_{-\infty}^{-r} \left(\int_{-r}^r |k(t-s)|\rho(t)\,dt \right) ds &= \frac{e^{-r}}{\mu(r,\rho)} \int_{-r}^r e^{-t}\rho(t)\,dt \\ &\leq \frac{e^{-r}}{\mu(r,\rho)} \left(e^r - e^{-r} + \left(\frac{e}{1 - \frac{1}{e}} \right) \right), \end{aligned}$$

which tends to 0 as $r \to 0$. Hence the conditions expressed in (3.3) and (3.4) of Theorem 3.2 are satisfied.

However if we set $s = \frac{1}{4}$ and $t_r = r - \frac{1}{4} + \frac{1}{2 \cdot 2^{2r}}$, $r \in \mathbb{N} \setminus \{1\}$, we can see that $t_r \in (r - 1 + \frac{1}{2^{2(r-1)}}, r)$ and $t_r + s \in [r, r + \frac{1}{2^{2r}}]$. Since $t_r \in \Omega^+_{r,s}$, where $\Omega^+_{r,s}$ is defined in (3.6) with $r_s = 0$, and

$$\frac{\rho(t_r + s)}{\rho(t_r)} = 1 + 2^{2r},$$

we conclude that the condition (3.6) is not satisfied by ρ.

Remark 3.7. In most of works, that consider pseudo almost automorphic functions, the weight ρ satisfies a ρ-type condition. The most common is that for each $s \in \mathbb{R}$, $\limsup_{|t| \to \infty} \frac{\rho(t+s)}{\rho(t)} < \infty$; see for example [**45, 20**]. In this context Theorem 3.2 is a new tool to consider another class of weights which do not satisfies conditions of ρ-type. We remark the importance of the convolution that permits to obtain (k_1, ρ)-type conditions. In other words, in this case, the kernel of convolution helps the weight to obtain the convolution invariance of $PAA_0(\mathbb{X}, \rho)$.

Despite this new tool, provided by the conditions of (k_1, ρ)-type, it has been pending the creation of a tool to treat weights that do not satisfy our conditions of ρ-type and (k_1, ρ)-type, as for example $\rho(t) = e^{t^2}$, $t \in \mathbb{R}$ (see Remark 3.5).

As a simple example, it is possible to obtain a $(e^{-\alpha(\,\cdot\,)}, \rho)$-condition.

Example 3.8. We assume that (3.1) holds, that $k_1(t) = e^{-\alpha t}$, for $\alpha > 0$ and $\in \mathbb{R}_+$, and that there exist $i_0 > 0$ and $c \geq 1$ such that

$$\int_s^r e^{-\alpha t} \rho(t)\, dt < c e^{-\alpha s} - e^{-\alpha r},\ |s| \leq r,\ r \in \mathbb{R}_+ \textit{ and } i_0 \leq \inf_{s \in \mathbb{R}} \rho(s). \tag{3.14}$$

Then $f \in PAA_0(\mathbb{X}, \rho)$ implies $\mathcal{K}_1 f \in PAA_0(\mathbb{X}, \rho)$.

In fact, observe that

$$\sup_{|s|\le r, r\in\mathbb{R}} \frac{1}{\rho(s)}\int_s^r |k_1(t-s)|\rho(t)\,dt = \sup_{|s|\le r, r\in\mathbb{R}} \frac{e^{\alpha s}}{\rho(s)}\int_s^r e^{-\alpha t}\rho(t)\,dt$$
$$\le \frac{1}{i_0}\cdot \sup_{|s|\le r, r\in\mathbb{R}} \left(c - e^{-\alpha(r-s)}\right) < \infty.$$

On the other hand,

$$\frac{1}{\mu(r,\rho)}\int_{-\infty}^{-r}\left(\int_{-r}^{r}|k(t-s)|\rho(t)\,dt\right)ds = \frac{e^{-\alpha r}}{\alpha\cdot\mu(r,\rho)}\int_{-r}^{r} e^{-\alpha t}\rho(t)\,dt$$
$$\le \frac{e^{-\alpha r}}{\alpha\cdot\mu(r,\rho)}\left(ce^{\alpha r} - e^{-\alpha r}\right),$$

which tends to 0 as $r\to\infty$. By Theorem 3.2, then, $\mathcal{K}_1 f\in PAA_0(\mathbb{X},\rho)$ if $f\in PAA_0(\mathbb{X},\rho)$.

Remark 3.9. Let $f\in PAA_0(\mathbb{X},\rho)$, $\{T(t)\}_{t\ge 0}\subseteq \mathcal{B}(\mathbb{X})$ and $F:\mathbb{R}\to\mathbb{X}$ defined by

$$F(t) = \int_{-\infty}^{t} T(t-s)f(s)\,ds.$$

Suppose that, there exist $M,\alpha>0$ such that $\|T(t)\|\le e^{-\alpha t}$ for all $t\ge 0$ and the weight ρ satisfies at least one of the two following conditions:

ρ is such that (3.6) holds or

$$\sup_{|s|\le r, r\in\mathbb{R}} \frac{e^{\alpha s}}{\rho(s)}\int_s^r e^{-\alpha t}\rho(t)\,dt < \infty, \quad \lim_{r\to\infty}\frac{e^{-\alpha r}}{\mu(r,\rho)}\int_{-r}^{r} e^{-\alpha t}\rho(t)\,dt = 0.$$

Then, $F\in PAA_0(\mathbb{X},\rho)$.

Now, we will establish the respective results of the Theorems 3.2 and 3.3 considering the $\mathcal{K}_2$-convolution operator, for a fixed

$$\rho\in\mathbb{U}_\infty \text{ and } k_2\in L^1(\mathbb{R}_-), \tag{3.15}$$

which is defined by

$$(\mathcal{K}_2 f)(t) = \int_t^{\infty} k_2(t-s)f(s)\,ds, \quad t\in\mathbb{R}, \quad k_2\in L^1(\mathbb{R}_-).$$

Since the proofs are analogous to those of the earlier results they will be omitted. Also, as in (3.2), for $f \in BC(\mathbb{R},\mathbb{X})$ we denote $|\mathcal{K}|_2\|f\| : \mathbb{R} \to \mathbb{R}_+$ by

$$(|\mathcal{K}|_2\|f\|)(t) = \int_t^\infty |k_2(t-s)|\|f(s)\|\, ds.$$

We say that $PAA_0(\mathbb{X},\rho)$ is invariant by $|\mathcal{K}|_2$ if for all $f \in PAA_0(\mathbb{X},\rho)$ we have $|\mathcal{K}|_2\|f\| \in PAA_0(\mathbb{R},\rho)$. Observe that the $PAA_0(\mathbb{X},\rho)$ invariance by $|\mathcal{K}|_2$ is sufficient condition for $PAA_0(\mathbb{X},\rho)$ invariance by $\mathcal{K}_2$, since $\|(\mathcal{K}_2 f)(t)\| \le (|\mathcal{K}|_2\|f\|)(t)$, for all $t \in \mathbb{R}$.

Theorem 3.10. *Assume that* (3.15) *is satisfied and that ρ and k_2 satisfy the conditions*

$$\sup_{|s|\le r, r\in\mathbb{R}} \frac{1}{\rho(s)} \int_{-r}^{s} |k_2(t-s)|\rho(t)\, dt < \infty,$$
$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_r^\infty \left(\int_{-r}^{r} |k_2(t-s)|\rho(t)\, dt \right) ds = 0. \tag{3.16}$$

Then $f \in PAA_0(\mathbb{X},\rho)$ implies $\mathcal{K}_2 f \in PAA_0(\mathbb{X},\rho)$.

Theorem 3.11. *Assume that* (3.15) *is satisfied and that ρ satisfies the following condition: For each $s \ge 0$, there exists $r_s \ge 0$ such that*

$$C^-(s) := \sup_{r\ge r_s} \sup_{t\in\Omega^-_{r,s}} \frac{\rho(t-s)}{\rho(t)} < \infty, \tag{3.17}$$

where $\Omega^-_{r,s} = \{t \in \mathbb{R} : r_s + s \le |t| \le r + s\} \cap \{t \in \mathbb{R} : \rho(t) \ne 0\}$. Then $f \in PAA_0(\mathbb{X},\rho)$ implies $\mathcal{K}_2 f \in PAA_0(\mathbb{X},\rho)$.

Remark 3.12. The conditions of Theorems 3.10 or 3.11 are sufficient to ensure that $PAA_0(\mathbb{X},\rho)$ is invariant by $|\mathcal{K}|_2$.

Remark 3.13. By an analogous argument given in Remark 3.5 we can show that the weight defined by $\rho(t) = \sum_{i=0}^N c_i|t|^{\beta_i}e^{\alpha_i t}$, where $N \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ and $c_i, \beta_i > 0$ for all $i \in \{1,2,\dots N\}$, satisfies the condition of Theorem 3.11, which implies that $\mathcal{K}_2 f \in PAA_0(\mathbb{X},\rho)$ provided $f \in PAA_0(\mathbb{X},\rho)$. Also we can show that the weight $\rho \in \mathbb{U}_\infty$ defined by $\rho(t) = e^{\alpha t^2}$, $\alpha > 0$, do not satisfy the conditions of Theorem

3.11. In contrast the weight defined by

$$\rho_1(t) = \begin{cases} e^{t^2}, & t < 0 \\ e^{-t^2}, & t \geq 0, \end{cases} \tag{3.18}$$

has the property that ρ_1 satisfies the condition of Theorem 3.3 but not of Theorem 3.11. Indeed, let $r, s \geq 0$ fixed, $A_1 = \{t \in \mathbb{R} : t < -s\}$, $A_2 = \{t \in \mathbb{R} : -s \leq t \leq 0\}$, $A_3 = \{t \in \mathbb{R} : 0 < t\}$ and $\Omega^+_{r,s}$ defined as in (3.6). Then we have

$$\begin{aligned} \sup_{t\in\Omega^+_{r,s}} \frac{\rho(t+s)}{\rho(t)} &\leq \sup_{t\in\Omega^+_{r,s}\cap A_1} \frac{\rho(t+s)}{\rho(t)} + \sup_{t\in\Omega^+_{r,s}\cap A_2} \frac{\rho(t+s)}{\rho(t)} + \sup_{t\in\Omega^+_{r,s}\cap A_3} \frac{\rho(t+s)}{\rho(t)} \\ &= \sup_{t\in\Omega^+_{r,s}\cap A_1} \frac{e^{t^2+2ts+s^2}}{e^{t^2}} + \sup_{t\in\Omega^+_{r,s}\cap A_2} \frac{e^{-t^2-2ts-s^2}}{e^{t^2}} \\ &\quad + \sup_{t\in\Omega^+_{r,s}\cap A_3} \frac{e^{-t^2-2ts-s^2}}{e^{-t^2}} \\ &\leq e^{-2s^2+s^2} + e^{s^2} + 1, \end{aligned}$$

which allows to show that ρ_1 satisfies the condition of Theorem 3.3. In contrast, let $s \geq 0$ and $t_r \in \Omega^-_{r,s}$ given by $t_r = -r - s$, where $\Omega^-_{r,s}$ is defined as in (3.17). Then we have

$$\frac{\rho(t_r - s)}{\rho(t_r)} = \frac{e^{(t_r-s)^2}}{e^{t_r^2}} = e^{-2t_r s} e^{s^2} = e^{2rs} e^{2s^2+s^2}.$$

Since for $s > 0$ the set $\{e^{2rs} : r > 0\}$ is not bounded, the weight ρ can not satisfy the condition of Theorem 3.11. Analogously we can see that, the weight ρ_2 defined by $\rho_2(t) = e^{t^2}$ for $t \geq 0$ and $\rho_2(t) = e^{-t^2}$ for $t < 0$, satisfies the condition of Theorem 3.11 but not of Theorem 3.3.

Convolution, translation invariance and unique decomposition. This section begins with the study about the invariance of the space $PAA(\mathbb{X}, \rho)$ by the convolution operator $\mathcal{K}_1$. We note that in general case the statement: if $f \in PAA_0(\mathbb{X}, \rho)$ then $\mathcal{K}_1 f \in PAA_0(\mathbb{X}, \rho)$ is false. Indeed we give the following original counterexample.

Let $k_1(t) = e^{-\alpha t}$, $t \geq 0$, where $\alpha > 0$ and $\rho = \rho_1 + \rho_2 \in \mathbb{U}_\infty$ where

$\rho_1(t) = e^{-|t|}$ and ρ_2 is defined as a even function, where

$$\rho_2(t) = \begin{cases} 0, & t \in [2k, 2k+1], \quad k \in \mathbb{Z}_+, \\ e^{\alpha t}, & t \in (2k+1, 2k+2), \quad k \in \mathbb{Z}_+, \end{cases} \tag{3.19}$$

for $t \geq 0$. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}_+$ the even real-valued functions defined by

$$f_1(t) = \begin{cases} 1, & t \in [2k, 2k+1],\ k \in \mathbb{Z}_+, \\ 0, & t \in (2k+1, 2k+2),\ k \in \mathbb{Z}_+, \end{cases}$$

$$f_2(t) = \begin{cases} \frac{1}{\varepsilon_k}(t - 2k + \varepsilon_k), & t \in [2k - \varepsilon_k, 2k),\ k \in \mathbb{Z}_+, \\ -\frac{1}{\varepsilon_k}(t - 2k - 1 - \varepsilon_k), & t \in (2k+1, 2k+1+\varepsilon_k], \\ 0 & \text{otherwise}, \end{cases} \tag{3.20}$$

for $t \geq 0$. Here the ε_k are positive real numbers less than $1/2$.

Let $x_0 \in \mathbb{X}$ fixed and $f : \mathbb{R} \to \mathbb{X}$ defined by $f(t) = f_1(t)x_0 + f_2(t)x_0$. We observe that $\rho \in \mathbb{U}_\infty - \mathbb{U}_B$, $f \in BC(\mathbb{R}, \mathbb{X})$ and, from the definitions of f and ρ, $(f \cdot \rho)(t) = f_1(t)e^{-|t|}x_0 + f_2(t) \cdot \rho_1(t)x_0 + f_2(t) \cdot \rho_2(t)x_0$ for all $t \in \mathbb{R}$. Since $f_1(\cdot)e^{-|\cdot|}x_0$ is integrable and we can choose the sequence (ε_k) to ensure that $f_2(\cdot)\rho_1(\cdot)x_0$, $f_2(\cdot)\rho_2(\cdot)x_0$ are integrable. In view of (2.1) and the definition of $PAA_0(\mathbb{X}, \rho)$, we have that $f \in PAA_0(\mathbb{X}, \rho)$.

Now, let $t > 3$ and let k_t be the integer such that $t - 2 < 2k_t + 1 \leq t$. Then $k_t > 0$ and

$$\begin{aligned} \|\mathcal{K}_1 f(t)\| &\geq \|x_0\| \int_{-\infty}^{t} e^{-\alpha(t-s)} f_1(s)\, ds \\ &\geq \|x_0\| \sum_{k=0}^{k_t} \int_{2k}^{2k+1} e^{-\alpha(t-s)} ds \\ &\geq \|x_0\| e^{-\alpha t} \frac{e^\alpha - 1}{\alpha} \left(\sum_{k=0}^{k_t} e^{\alpha 2k} \right) \\ &= \|x_0\| e^{-\alpha t} M_\alpha \left(\frac{e^{2\alpha(k_t+1)} - 1}{e^{2\alpha} - 1} \right), \end{aligned}$$

where $M_\alpha = \frac{e^\alpha - 1}{\alpha}$. Since $t - 2 < 2k_t + 1 \leq t$, we have

$$\|\mathcal{K}_1 f(t)\| \geq \|x_0\| M_\alpha e^{-\alpha t} \frac{e^{\alpha(t-1)} - 1}{e^{2\alpha} - 1}, \quad \text{for all } t > 3. \tag{3.21}$$

Now let $r>3$ and let n_r be the integer such that $r-2<2n_r\le r$. By definition of ρ and the inequality (3.21), we obtain

$$\begin{aligned}\int_{-r}^{r}\|\mathcal{K}_1 f(t)\|\rho(t)dt &\ge \int_3^r \|\mathcal{K}_1 f_1(t)\|\rho(t)dt\\ &\ge \sum_{k=0}^{n_r}\int_{2k+1}^{2k+2}\|\mathcal{K}_1 f_1(t)\|e^{\alpha t}dt+\int_3^r\|\mathcal{K}_1 f_1(t)\|e^{-|t|}dt\\ &\ge \|x_0\|\sum_{k=0}^{n_r}\int_{2k+1}^{2k+2} M_\alpha\frac{e^{\alpha(t-1)}-1}{e^{2\alpha}-1}dt\\ &=\|x_0\|\frac{M_\alpha}{(e^{2\alpha}-1)}\sum_{k=0}^{n_r}\left[e^{\alpha(2k)}\frac{e^\alpha-1}{\alpha}-1\right]\\ &=\|x_0\|\frac{M_\alpha}{(e^{2\alpha}-1)}\left[\frac{e^\alpha-1}{\alpha}\frac{e^{2\alpha(n_r+1)}-1}{e^{2\alpha}-1}-(n_r+1)\right].\end{aligned}$$

Since the real-valued function $g:[1,\infty)\to\mathbb{R}$ defined by

$$g(x)=\frac{e^\alpha-1}{\alpha}\frac{e^{2\alpha x}-1}{e^{2\alpha}-1}-x$$

is nondecreasing, we obtain

$$\int_{-r}^{r}\|\mathcal{K}_1 f(t)\|\rho(t)dt\ge\|x_0\|\left[M_{2,\alpha}(e^{\alpha r}-1)-M_{1,\alpha}\frac{r}{2}\right]\ge 0 \quad \text{for all } r>3,$$

where, $M_{1,\alpha}=M_\alpha/(e^{2\alpha}-1)$ and $M_{2,\alpha}=(M_{1,\alpha})^2$.

Now we will estimate $\mathcal{M}(\|\mathcal{K}_1 f\|)$. Indeed let $r>3$, then taking into account the above inequality we obtain

$$\begin{aligned}\frac{1}{\int_{-r}^r\rho(s)\,ds}\int_{-r}^{r}\|(\mathcal{K}_1 f)(s)\|\rho(s)\,ds &\ge \frac{\|x_0\|\left[M_{2,\alpha}(e^{\alpha r}-1)-M_{1,\alpha}\frac{r}{2}\right]}{\int_{-\infty}^{\infty}e^{-|s|}ds+\int_{-r}^{r}e^{\alpha s}ds}\\ &=\frac{\alpha\|x_0\|\left[M_{2,\alpha}(e^{\alpha r}-1)-M_{1,\alpha}\frac{r}{2}\right]}{2\alpha+2(e^{\alpha r}-e^{-\alpha r})}.\end{aligned}$$

Since the right side of the preceding inequality has positive limit as $r\to\infty$ we obtain that $\mathcal{M}(\|\mathcal{K}_1 f\|)\ne 0$. Hence $\mathcal{K}_1 f\notin PAA_0(\mathbb{R},\rho)$ even when $f\in PAA_0(\mathbb{R},\rho)$ holds.

Remark 3.14. This construction has the following consequences:

(1) The weight ρ and $k_1 \in L^1(\mathbb{R}_+)$ defined above does not satisfy the condition (3.3) of Theorem 3.2. Indeed, let $s = 0$, $n \in \mathbb{N}$ and $r_n = 2n+2$, then we have

$$\frac{1}{\rho(s)}\int_s^{r_n} |k_1(t-s)|\rho(t)\,dt = \frac{1}{\rho(0)}\int_0^{r_n} |k_1(t)|(\rho_1(t)+\rho_2(t))\,dt \geq \sum_{k=0}^{n}\int_{2k+1}^{2k+2} e^{-\alpha t}e^{\alpha t}dt = n,$$

which shows that the condition (3.3) of Theorem 3.2 is not satisfied.

(2) The weight ρ does not satisfy the condition (3.6) of Theorem 3.3. In fact let, $r_n = 2n+2$, $n \in \mathbb{N}$, $s = 1$ and $t_n = 2n+1/2 \in \Omega^+_{r_n,s}$, then we obtain that

$$\frac{\rho(t_n+s)}{\rho(t_n)} = \frac{\rho(2n+3/2)}{\rho(2n+1/2)} = \frac{e^{\alpha(2n+3/2)} + e^{-(2n+3/2)}}{e^{-(2n+1/2)}} = e^{2n(\alpha+1)+2} + e^{-1},$$

which shows that the condition (3.6) of Theorem 3.3 is not satisfied.

(3) The function f is not translation invariant, since $f((\cdot)+1) \notin PAA_0(\mathbb{X},\rho)$.

Ji and Zhang [**29,** Th. 2.4], assuming that $PAA_0(\mathbb{X},\rho)$ is translation invariant for $\rho \in \mathbb{U}_\infty$, prove that for $k \in L^1(\mathbb{R})$ $PAA_0(\mathbb{X},\rho)$ is $\mathcal{K}$-convolution invariant, in particular for $k_1 \in L^1(\mathbb{R}_+)$ and $k_2 \in L^1(\mathbb{R}_-)$ then $PAA_0(\mathbb{X},\rho)$ is $\mathcal{K}_i$-convolution invariant, for $i = 1,2$. In general is not easy to verify the $PAA_0(\mathbb{X},\rho)$ translation invariance. However this property is implied for different conditions see for example Ding et al. [**20,** Theorem 4.1]. The following proposition gives sufficient condition to ensure that $PAA_0(\mathbb{X},\rho)$ is translation invariant.

Proposition 3.15. *Assume that $\rho \in \mathbb{U}_\infty$ satisfies the following condition: For each $s \in \mathbb{R}$, there exists $r_s \geq 0$ such that*

$$C(s) := \sup_{r \geq r_s}\sup_{t\in\Omega_{r,s}} \frac{\rho(t+s)}{\rho(t)} < \infty, \tag{3.22}$$

where

$$\Omega_{r,s} = \{t \in \mathbb{R} : r_s + |s| \leq |t| \leq r + |s|\} \cap \{t \in \mathbb{R} : \rho(t) \neq 0\}.$$

Then, $PAA_0(\mathbb{X},\rho)$ is translation invariant.

Proof. Let $s \in \mathbb{R}$ fixed, $C^*(s) = \max\{C(s), C(-s)\}$ and $r_s^* = \max\{r_s, r_{-s}\}$. If $r \geq r_s^*$, it follows from (3.22) that

$$\rho(t+|s|) \leq C^*(s)\rho(t) \tag{3.23}$$

for all t satisfying $r_s^* + |s| \leq |t| \leq r + |s|$. Following a similar argument given in (3.10) and (3.12) we can conclude that

$$\mu(r+|s|, \rho) \leq (K + C^*(s))\mu(r, \rho), \quad r > r_s^*, \tag{3.24}$$

for some $K > 0$ and for each $f \in PAA_0(\mathbb{X}, \rho)$

$$\begin{aligned} f_{r,\rho}(s) &:= \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \|f(t-s)\|\rho(t)\,dt \\ &\leq \frac{\|f\|_\infty}{\mu(r,\rho)} \int_{-(r_s^*+|s|)}^{r_s^*+|s|} \rho(t+s)\,dt + \frac{(K+C^*(s))C(s)}{\mu(r+|s|,\rho)} \int_{-(r+|s|)}^{r+|s|} \|f(t)\|\rho(t)\,dt \\ &\leq \frac{\|f\|_\infty}{\mu(r,\rho)} \int_{-r_s^*}^{r_s^*+2|s|} \rho(t)\,dt + \frac{(K+C^*(s))C(s)}{\mu(r+|s|,\rho)} \int_{-(r+|s|)}^{r+|s|} \|f(t)\|\rho(t)\,dt. \end{aligned}$$

The first term on the last line tends to zero since r_s^*, s are fixed and $\mu(r, \rho) \to \infty$ as $r \to \infty$. Now, the condition $f \in PAA_0(\mathbb{X}, \rho)$ implies that the second term tends to zero as $r \to \infty$. Consequently $f_{r,\rho}(s) \to 0$ as $r \to \infty$ for all $s \geq 0$. Thus $\mathcal{M}(\|f((\cdot) - s)\|) = 0$ for each $s \in \mathbb{R}$. Hence $PAA_0(\mathbb{X}, \rho)$ is translation invariant. □

The counterexample given in (3.19), (3.20) shows that if (3.22) does not hold then translation invariance does not necessarily hold.

Remark 3.16. As in the proof of Proposition 3.15, under the condition (3.6) we can show that $PAA_0(\mathbb{X}, \rho)$ is right translation invariant, that is, $f \in PAA_0(\mathbb{X}, \rho)$ implies $f((\cdot) - s) \in PAA_0(\mathbb{X}, \rho)$ for all $s \geq 0$. Analogously we can see that the condition (3.17) implies that $PAA_0(\mathbb{X}, \rho)$ is left translation invariant i.e., $f \in PAA_0(\mathbb{X}, \rho)$ implies $f((\cdot) + s) \in PAA_0(\mathbb{X}, \rho)$ for all $s \geq 0$.

Corollary 1. *Assume that* $\rho \in \mathbb{U}_\infty$ *satisfies the condition* (3.22). *Then the following assertion are satisfied:*

1. $PAA_0(\mathbb{X}, \rho)$ *is translation invariant.*
2. *The decomposition of weighted almost automorphic functions is unique.*

(3) *Let* $k_1 \in L^1(\mathbb{R}_+)$*. Then* $PAA_0(\mathbb{X},\rho)$ *is* $\mathcal{K}_1$*-convolution invariant.*
(4) *Let* $k_2 \in L^1(\mathbb{R}_-)$*. Then* $PAA_0(\mathbb{X},\rho)$ *is* $\mathcal{K}_2$*-convolution invariant.*

Remark 3.17. As mentioned above, there exist different conditions on ρ to ensure the $PAA_0(\mathbb{X},\rho)$ translation invariance. The most common, is that for all $s \in \mathbb{R}$

$$\limsup_{|t|\to\infty} \frac{\rho(t+s)}{\rho(t)} < \infty, \tag{3.25}$$

or variations of this; see, e.g., [**45, 20**]. It is not difficult to see that when $\rho(t) > 0$ for all $t \in \mathbb{R}$ the conditions (3.22) and (3.25) are equivalent. However, the condition (3.22) considers the cases when ρ can be made to vanish at different points. On the other hand, we mention that condition (3.6) or (3.17) is not equivalent with (3.25); to see this consider the weight ρ_1 and ρ_2 defined in Remark 3.13. This particularity corresponds to the fact that the conditions (3.6) or (3.17) are focused on the $\mathcal{K}_1$-, $\mathcal{K}_2$-convolution invariance of $PAA_0(\mathbb{X},\rho)$ respectively, which is weaker than $\mathcal{K}$-convolution invariance.

4. Abstract neutral integral equations.

Now we are ready to state some sufficient conditions to ensure the existence and uniqueness of weighted pseudo almost automorphic to the abstract neutral integral equation of the form

$$u(t) = f_0(t,u(t),u(h_0(t))) + \int_{\mathbb{R}} R(t,s) f(s,u(s),u(h(s)))\, ds, \quad t \in \mathbb{R},$$

or, more specifically, its advanced and delayed decomposition:

$$\begin{aligned} u(t) = f_0(t,u(t),u(h_0(t))) &+ \int_{-\infty}^{t} R_1(t,s) f_1(s,u(s),u(h_1(s)))\, ds \\ &+ \int_{t}^{\infty} R_2(t,s) f_2(s,u(s),u(h_2(s)))\, ds, \end{aligned} \tag{4.1}$$

where $h_i : \mathbb{R} \to \mathbb{R}$ are continuous function with $h_i(\mathbb{R}) = \mathbb{R}$, $f_i : \mathbb{R}\times\mathbb{X}\times\mathbb{X} \to \mathbb{X}$, for $i = 0, 1, 2$, and $R_i : \mathbb{R}\times\mathbb{R} \to \mathcal{B}(\mathbb{X})$, for $i = 1, 2$. Due to the character of pseudo-almost automorphic functions in the two variables t and s of kernels R_i we consider in the next hypotheses the concept of bi-almost automorphic function (see [**44**], [**13**] and Definition 2.5), which be useful to apply the Theorems 3.2 and 3.3, in order to obtain the existence and uniqueness of a weighted pseudo-almost automorphic result for the neutral integral equation (4.1).

(H1) For $i=0,1,2$, the function $h_i:\mathbb{R}\to\mathbb{R}$ satisfies, $h_i(\mathbb{R})=\mathbb{R}$ and $x(h_i)\in AA(\mathbb{X})$, $z(h_i)\in PAA_0(\mathbb{X},\rho)$ provided that $x\in AA(\mathbb{X})$, $z\in PAA_0(\mathbb{X},\rho)$ respectively.

(H2) The functions f_i are the form $g_i+\phi_i$ where $g_i\in AA(\mathbb{X}\times\mathbb{X},\mathbb{X})$, $\phi_i\in PAA_0(\mathbb{X}\times\mathbb{X},\mathbb{X},\rho)$ for $i=0,1,2$ and f_i satisfies the Lipschitz type condition

$$\|f_i(t,x_1,y_1)-f_i(t,x_2,y_2)\|\le L_{f_i,1}(t)\|x_1-x_2\|+L_{f_i,2}(t)\|y_1-y_2\|,$$

where $t\in\mathbb{R}$ and $x_i,y_i\in\mathbb{X}$. In addition for all compact subsets $K_1,K_2\subseteq\mathbb{X}$ for any given $\varepsilon>0$ there exists $\delta>0$ such that $\|g_i(t,u_1,v_1)-g_i(t,u_2,v_2)\|\le\varepsilon$ for all $\|(u_1,v_1)-(u_2,v_2)\|\le\delta$ provided that $(u_1,v_1),(u_2,v_2)\in K_1\times K_2$ and $t\in\mathbb{R}$.

In this case we say that $g_i(t,\cdot,\cdot)$ is uniformly continuous in $K_1\times K_2$ uniformly for $t\in\mathbb{R}$.

(H3) For the kernels R_1, R_2 there exist functions $k_1\in L^1(\mathbb{R}_+)$, $k_2\in L_2(\mathbb{R}_-)$ such that

$$\|R_1(t,s)\|\le|k_1(t-s)|,\ \text{for all } t\ge s,$$
$$\|R_2(t,s)\|\le|k_2(t-s)|,\ \text{for all } t\le s.$$

(H4) For $i=1,2$, the kernel $R_i\in\mathcal{B}(\mathbb{X})$ is bi-almost automorphic, that is, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$R_i^*(t,s)x:=\lim_{n\to\infty}R_i(t+s_n,s+s_n)x$$

is well defined for each $t,s\in\mathbb{R}$, $x\in\mathbb{X}$ and

$$\lim_{n\to\infty}R_i^*(t-s_n,s-s_n)x=R_i(t,s)x$$

for each $t,s\in\mathbb{R}$, $x\in\mathbb{X}$.

Remark 4.1. By Remark 3.16 we have, for $p>0$, that $h(t)=t-p$ satisfies (H1) provided that the weight ρ satisfies the condition (3.6). Analogously $h(t)=t+p$ satisfies (H1) provided that the weight ρ satisfies the condition (3.17)

Condition (H3) is not enough to have a bi-almost automorphic condition (H4), which is fundamental to have almost automorphic solutions. In fact, consider $\mathbb{X} = \mathbb{R}$ and the non homogenous differential equation

$$u'(t) = (-1 + a(t))u(t) + 1, \quad t \in \mathbb{R}. \tag{4.2}$$

If $a \leq 0$ the kernel $R_1(t,s) = e^{\int_s^t [-1+a(\omega)]d\omega}$, $t, s \in \mathbb{R}$, satisfies the condition (H3) for $k_1(t) = e^{-t}$ and the bounded solution of the above equation is described by

$$u(t) = \int_{-\infty}^{t} R_1(t,s)\, ds.$$

However, for $a(t) = -1/(1+t^2)$ the equation (4.2) does not have almost automorphic solutions. If we impose that $a \in AA(\mathbb{R})$ we obtain that R_1 is a bi-almost automorphic function and consequently the bounded solution of (4.2) belongs to $AA(\mathbb{R})$.

Composition theorems. Before stating the main result of this section we need the following two new composition results, which take into account the previous theorems.

Theorem 4.2. *Let $h : \mathbb{R} \to \mathbb{R}$ and $f \in PAA(\mathbb{X} \times \mathbb{X}, \mathbb{X}, \rho)$ satisfying (H1), (H2) respectively. Suppose that the functions $L_{f,1}$ and $L_{f,2}$ considered in (H2) satisfy*

$$S_j := \sup_{r \in \mathbb{R}} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} L_{f,j}(t)\rho(t)\, dt < \infty, \quad j = 1, 2. \tag{4.3}$$

Then, $u \in PAA(\mathbb{X}, \rho)$ implies $f(\,\cdot\,, u(\,\cdot\,), u(h(\,\cdot\,))) \in PAA(\mathbb{X}, \rho)$.

Proof. It follows from Definition 2.6 that the function u can be written as $u = y + z$ with $y \in AA(\mathbb{X})$ and $z \in PAA_0(\mathbb{X}, \rho)$, similarly $f = g + \phi$ with $g \in AA(\mathbb{X} \times \mathbb{X}, \mathbb{X})$ and $\phi \in PAA_0(\mathbb{X} \times \mathbb{X}, \mathbb{X}, \rho)$. We observe that $f(\,\cdot\,, u(\,\cdot\,), u(h(\,\cdot\,))) = G(\,\cdot\,) + F(\,\cdot\,) + \Phi(\,\cdot\,)$ where the functions G, F, $\Phi \in BC(\mathbb{R}, \mathbb{X})$ are defined, for each $t \in \mathbb{R}$, by $G(t) = g(t, y(t), y(h(t)))$, $F(t) = f(t, u(t), u(h(t))) - f(t, y(t), y(h(t)))$ and $\Phi(t) = \phi(t, y(t), y(h(t)))$ respectively.

To ensure that $f(\,\cdot\,, u(\,\cdot\,), u(h(\,\cdot\,))) \in PAA(\mathbb{X}, \rho)$ is sufficient to show that $G \in AA(\mathbb{X})$ and $F, \Phi \in PAA_0(\mathbb{X}, \rho)$. Indeed, $G \in AA(\mathbb{X})$ by Lemma [**34,** Lemma 2.2]. Now, let $\varepsilon, r > 0$ and $A_{r,\varepsilon} = M_{r,\varepsilon}(z) \cup M_{r,\varepsilon}(z(h))$, where $M_{r,\varepsilon}$ is defined in Lemma 2.2. It follows from (H2) and the definition of $A_{r,\varepsilon}$ that

$$\begin{aligned}\int_{[-r,r]\setminus A_{r,\varepsilon}} &\|F(t)\|\rho(t)\,dt\\ &= \int_{[-r,r]\setminus A_{r,\varepsilon}} \|f(t,u(t),u(h(t))) - f(t,y(t),y(h(t)))\|\rho(t)\,dt\\ &\le \int_{[-r,r]\setminus A_{r,\varepsilon}} (L_{f,1}(t)\|z(t)\| + L_{f,2}(t)\|z(h(t))\|)\,\rho(t)\,dt\\ &\le \varepsilon \int_{[-r,r]\setminus A_{r,\varepsilon}} (L_{f,1}(t) + L_{f,2}(t))\rho(t)\,dt\\ &\le \varepsilon \int_{-r}^{r} (L_{f,1}(t) + L_{f,2}(t))\rho(t)\,dt.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{\mu(r,\rho)} \int_{[-r,r]\setminus A_{r,\varepsilon}} &\|F(t)\|\rho(t)\,dt\\ &\le \frac{\varepsilon}{\mu(r,\rho)} \int_{-r}^{r} (L_{f,1}(t) + L_{f,2}(t))\rho(t)\,dt.\end{aligned} \tag{4.4}$$

Furthermore, since $F \in BC(\mathbb{R},\mathbb{X})$ we have

$$\begin{aligned}\frac{1}{\mu(r,\rho)} \int_{A_{r,\varepsilon}} &\|F(t)\|\rho(t)\,dt\\ &\le \frac{\|F\|_\infty}{\mu(r,\rho)} \int_{M_{r,\varepsilon}(z)} \rho(t)\,dt + \frac{\|F\|_\infty}{\mu(r,\rho)} \int_{M_{r,\varepsilon}(z(h))} \rho(t)\,dt.\end{aligned} \tag{4.5}$$

Since (H1) is satisfied, $z, z(h) \in PAA_0(\mathbb{X},\rho)$; thus, by Lemma 2.2,

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{M_{r,\varepsilon}(z)} \rho(t)\,dt = 0, \quad \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{M_{r,\varepsilon}(z(h))} \rho(t)\,dt = 0.$$

Combining the inequalities (4.4) and (4.5) with (4.3) we deduce that

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \|F(t)\|\rho(t)\,dt = 0,$$

i.e., $F \in PAA_0(\mathbb{X},\rho)$.

We shall prove that $\Phi \in PAA_0(\mathbb{X},\rho)$. Consider the compact subsets of $\mathbb{X}$ given by $U_1 = \overline{\{y(t) : t \in \mathbb{R}\}}$ and $U_2 = \overline{\{y(h(t)) : t \in \mathbb{R}\}}$, and

$$\Phi^+(t) = \sup_{(u,v)\in U_1\times U_2} \|\phi(t,u,v)\|,\ t\in\mathbb{R}. \tag{4.6}$$

Since $\|\Phi(t)\| \leq \Phi^+(t)$ for all $t \in \mathbb{R}$, it is sufficient to show that $\Phi^+ \in PAA_0(\mathbb{R},\rho)$. In fact, since g is uniformly continuous on the compact $C = U_1 \times U_2$, uniformly in $t\in\mathbb{R}$, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that $\|g(t,u_1,v_1) - g(t,u_2,v_2)\| \leq \varepsilon$ for all $\|(u_1,v_1) - (u_2,v_2)\| \leq \delta$ provided that $(u_1,v_1),(u_2,v_2) \in C$ and $t \in \mathbb{R}$. Let $\varepsilon' = \min\{\varepsilon,\delta\}$, since C is compact there exist $(a_1,b_1),\ldots,(a_k,b_k) \in C$ such that

$$C \subseteq \bigcup_{i=1}^{k} B((a_i,b_i),\varepsilon').$$

Let $(u,v) \in Clet$ and $(a,b) \in \{(a_1,b_1),\ldots,(a_k,b_k)\}$ be such that $\|(u,v)-(a,b)\| \leq \varepsilon'$. By (H2) we have, for each $t\in\mathbb{R}$,

$$\begin{aligned}\|\phi(t,u,v) - \phi(t,a,b)\| &\leq \|f(t,u,v) - f(t,a,b)\| + \|g(t,u,v) - g(t,a,b)\| \\ &\leq L_{f,1}(t)\epsilon' + L_{f,2}(t)\epsilon' + \epsilon.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{-r}^{r} \|\phi(t,u,v)\|\rho(t)\,dt \\ &\quad\leq \int_{-r}^{r} \|\phi(t,u,v) - \phi(t,a,b)\|\rho(t)\,dt + \int_{-r}^{r} \|\phi(t,a,b)\|\rho(t)\,dt \\ &\quad\leq \int_{-r}^{r} (L_{f,1}(t) + L_{f,2}(t) + 1)\varepsilon\rho(t)\,dt + \sum_{i=1}^{k}\int_{-r}^{r} \|\phi(t,a_i,b_i)\|\rho(t)\,dt \\ &\quad\leq \mu(r,\rho)(S_1+S_2+1)\varepsilon + \sum_{i=1}^{k}\int_{-r}^{r} \|\phi(t,a_i,b_i)\|\rho(t)\,dt,\end{aligned}$$

which implies

$$\begin{aligned}&\frac{1}{\mu(r,\rho)}\int_{-r}^{r} \|\Phi^+(t)\|\rho(t)\,dt \\ &\qquad\leq (S_1+S_2+1)\varepsilon + \frac{1}{\mu(r,\rho)}\sum_{i=1}^{k}\int_{-r}^{r} \|\phi(t,a_i,b_i)\|\rho(t)\,dt.\end{aligned} \tag{4.7}$$

Since $\phi \in PAA_0(\mathbb{X}\times\mathbb{X},\mathbb{X},\rho)$ and (4.7) is satisfied, we conclude that $\Phi^+ \in PAA_0(\mathbb{R},\rho)$, which completes the proof. □

Corollary 2. *Let* $h:\mathbb{R}\to\mathbb{R}$ *and* $f\in PAA_0(\mathbb{X}\times\mathbb{X},\mathbb{X},\rho)$ *satisfying (H1), (H2) respectively. Suppose that the functions* $L_{f,1}$ *and* $L_{f,2}$ *considered in (H2) are bounded. Then,* $u\in PAA(\mathbb{X},\rho)$ *implies* $f(\,\cdot\,,u(\,\cdot\,),u(h(\,\cdot\,)))\in PAA(\mathbb{X},\rho)$*.*

Let h, f, and R_i satisfying (H1), (H2) and (H3) respectively. We consider the operators $\mathcal{N}_1$, $\mathcal{N}_2$ defined by

$$(\mathcal{N}_1u)(t) := \int_{-\infty}^{t} R_1(t,s)f(s,u(s),u(h(s)))\,ds,$$
$$(\mathcal{N}_2u)(t) := \int_{t}^{\infty} R_2(t,s)f(s,u(s),u(h(s)))\,ds.$$

Theorem 4.3. *Let* $i\in\{1,2\}$ *given,* $h:\mathbb{R}\to\mathbb{R}$*,* $f\in PAA(\mathbb{X}\times\mathbb{X},\mathbb{X},\rho)$ *and* R_i *satisfying (H1), (H2) and* $(H3)$*-*$(H4)$ *respectively. Suppose that* $PAA_0(\mathbb{X},\rho)$ *is invariant by* $|\mathcal{K}|_i$ *and the functions* $L_{f,1}$ *and* $L_{f,2}$*, considered in (H2), satisfy*

$$\| |\mathcal{K}|_i L_{f,j}\|_\infty := \sup_{t\in\mathbb{R}}(|\mathcal{K}|_i L_{f,j})(t) < \infty, \quad j=1,2. \tag{4.8}$$

Then, $u\in PAA(\mathbb{X},\rho)$ *implies* $\mathcal{N}_i u\in PAA(\mathbb{X},\rho)$*.*

Proof. The proof follows the same steps and notation given in Theorem 4.2. It follows from Definition 2.6 that the function u can be written as $u=y+z$ with $y\in AA(\mathbb{X})$ and $z\in PAA_0(\mathbb{X},\rho)$, similarly $f=g+\phi$ with $g\in AA(\mathbb{X}\times\mathbb{X},\mathbb{X})$ and $\phi\in PAA_0(\mathbb{X}\times\mathbb{X},\mathbb{X},\rho)$. We observe that $f(\,\cdot\,,u(\,\cdot\,),u(h(\,\cdot\,))) = G(\,\cdot\,)+F(\,\cdot\,)+\Phi(\,\cdot\,)$ where the functions G, F, $\Phi\in BC(\mathbb{R},\mathbb{X})$ are defined, for each $t\in\mathbb{R}$, by $G(t)=g(t,y(t),y(h(t)))$, $F(t)=f(t,u(t),u(h(t)))-f(t,y(t),y(h(t)))$, $\Phi(t)=\phi(t,y(t),y(h(t)))$.

Let $i=1$. To ensure that $\mathcal{N}_1u\in PAA(\mathbb{X},\rho)$ is sufficient to show that $\mathfrak{N}_1\in AA(\mathbb{X})$, where $\mathfrak{N}_1(t):=\int_{-\infty}^{t}R(t,s)G(s)\,ds$, and $|\mathcal{K}|_1\|F\|$, $|\mathcal{K}|_1\|\Phi\|\in PAA_0(\mathbb{R},\rho)$. Indeed, it follows from **[34,** Lemma 2.2] that $G\in AA(\mathbb{X})$. By the definition of the space $AA(\mathbb{X})$ and (H4), for a sequence of real numbers $(s_n')_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$R_1^*(t,s)x := \lim_{n\to\infty} R_1(t+s_n,s+s_n)x \text{ and } G^*(s) := \lim_{n\to\infty} G(s+s_n)$$

is well defined for each $t, s \in \mathbb{R}$, $x \in \mathbb{X}$ and

$$\lim_{n\to\infty} R_1(t,s)x = R_1^*(t-s_n, s-s_n)x \text{ and } G(s) = \lim_{n\to\infty} G^*(s-s_n).$$

Observe that

$$\begin{aligned} \mathfrak{N}_1(t+s_n) &= \int_{-\infty}^{t+s_n} R_1(t+s_n, s)G(s)\, ds \\ &= \int_{-\infty}^{t} R_1(t+s_n, s+s_n)G(s+s_n)\, ds \\ &= \int_{-\infty}^{t} R_1(t+s_n, s+s_n)G^*(s)\, ds \\ &\quad + \int_{-\infty}^{t} R_1(t+s_n, s+s_n)(G(s+s_n) - G^*(s))\, ds \end{aligned} \tag{4.9}$$

and

$$\left\| \int_{-\infty}^{t} R_1(t+s_n, s+s_n)\big(G(s+s_n) - G^*(s)\big)\, ds \right\| \leq \int_{-\infty}^{t} |k(t-s)| \|G(s+s_n) - G^*(s)\|\, ds.$$

Since (H4) is satisfied and G is bounded we obtain, for $t \geq s$,

$$\|R_1(t+s_n, s+s_n)G(s+s_n)\| \leq |k_1(t-s)| \|G\|_\infty,$$
$$\|R_1(t+s_n, s+s_n)(G(s+s_n) - G^*(s))\| \leq 2\, |k_1(t-s)| \|G\|_\infty.$$

Also, $|k_1(t-\cdot)| \|G\|_\infty$ is integrable in $(-\infty, t]$; thus, by the Lebesgue dominated convergence theorem applied to (4.9), we obtain that

$$\mathfrak{N}_1^*(t) := \lim_{n\to\infty} \mathfrak{N}_1(t+s_n) = \int_{-\infty}^{t} R^*(t,s)G^*(s)\, ds$$

is well defined for each $t \in \mathbb{R}$. Using an argument as above we get

$$\begin{aligned} \lim_{n\to\infty} \mathfrak{N}_1^*(t-s_n) &= \lim_{n\to\infty} \int_{-\infty}^{t-s_n} R^*(t,s)G^*(s)\, ds \\ &= \lim_{n\to\infty} \int_{-\infty}^{t} R^*(t-s_n, s-s_n)G^*(s-s_n)\, ds \\ &= \int_{-\infty}^{t} R_1(t,s)G(s)\, ds = \mathfrak{N}_1(t). \end{aligned}$$

Thus $\mathfrak{N}_1 \in AA(\mathbb{X})$.

In order to show $|\mathcal{K}|_1\|F\| \in PAA_0(\mathbb{R},\rho)$, let $\varepsilon > 0$ and $A_{r,\varepsilon} = M_{r,\varepsilon}(z) \cup M_{r,\varepsilon}(z(h))$, where $M_{r,\varepsilon}$ is defined in Lemma 2.2. As in (4.4) considering (H2), (4.8) we have

$$\begin{aligned}(4.10)\quad \int_{-r}^{r} (|\mathcal{K}|_1\|F\|)(t)\rho(t)\,dt &\\ \leq \int_{A_{r,\varepsilon}} (|\mathcal{K}|_1\|F\|)(t)\rho(t)\,dt &+ \int_{[-r,r]\setminus A_{r,\varepsilon}} (|\mathcal{K}|_1\|F\|)(t)\rho(t)\,dt \\ \leq \|F\|_\infty \|k_1\|_{L^1(\mathbb{R}_+)} \bigg(\int_{M_{r,\varepsilon}(z)} \rho(t)\,dt &+ \int_{M_{r,\varepsilon}(z(h))} \rho(t)\,dt \bigg) \\ &+ \varepsilon\mu(r,\rho)(\||\mathcal{K}|_1 L_{f,1}\|_\infty + \||\mathcal{K}|_1 L_{f,2}\|_\infty).\end{aligned}$$

Since (H1) is satisfied, $z, z(h)$ lie in $PAA_0(\mathbb{X},\rho)$; thus, by Lemma 2.2,

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{M_{r,\varepsilon}(z)} \rho(t)\,dt = 0, \qquad \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{M_{r,\varepsilon}(z(h))} \rho(t)\,dt = 0.$$

Combining this with (4.10) we obtain that $|\mathcal{K}|_1\|F\| \in PAA_0(\mathbb{R},\rho)$.

Now we will show that $|\mathcal{K}|_1\|\Phi\| \in PAA_0(\mathbb{R},\rho)$. For this, since $(|\mathcal{K}|_1\|\Phi\|)(t) \leq (|\mathcal{K}|_1\Phi^+)(t)$ for all $t \in \mathbb{R}$, where Φ^+ is defined in (4.6), is sufficient to show that $|\mathcal{K}|_1\Phi^+ \in PAA_0(\mathbb{R},\rho)$. Given $\varepsilon > 0$ let $\delta > 0$, $\varepsilon' = \min\{\varepsilon, \delta\}$ and $(a_1,b_1),\dots,(a_k,b_k) \in C$ (which was defined in (4.6)) such that $\|g(t,u_1,v_1) - g(t,u_2,v_2)\| \leq \varepsilon$ for all $\|(u_1,v_1) - (u_2,v_2)\| \leq \delta$, $(u_1,v_1),(u_2,v_2) \in C$, $t \in \mathbb{R}$, and

$$C \subseteq \bigcup_{i=1}^{k} B((a_i,b_i),\varepsilon').$$

Let $(u,v) \in C$ and take $(a,b) \in \{(a_1,b_1),\dots,(a_k,b_k)\}$ such that $\|(u,v) - (a,b)\| \leq \varepsilon'$. By (H2) we have, for each $t \in \mathbb{R}$,

$$\begin{aligned}\|\phi(t,u,v) - \phi(t,a,b)\| &\leq \|f(t,u,v) - f(t,a,b)\| + \|g(t,u,v) - g(t,a,b)\| \\ &\leq L_{f,1}(t)\epsilon' + L_{f,2}(t)\epsilon' + \epsilon.\end{aligned}$$

Thus, denoting $\phi(\,\cdot\,,u,v)$ by $\phi_{u,v}$, we obtain

$$(|\mathcal{K}|_1\|\phi_{u,v} - \phi_{a,b}\|)(t) \leq \big((|\mathcal{K}|_1 L_{f,1})(t) + (|\mathcal{K}|_1 L_{f,2})(t) + \|k_1\|_{L^1_1}\big)\varepsilon$$

for $t \in \mathbb{R}$.

Considering the previous inequality and setting $M := \||\mathcal{K}|_1 L_{f,1}\|_\infty + \||\mathcal{K}|_1 L_{f,2}\|_\infty + \|k_1\|_{L^1_1}$, we have

$$\begin{aligned}
&\int_{-r}^{r} (|\mathcal{K}|_1\|\phi_{u,v}\|)(t)\rho(t)\,dt \\
&\quad\le \left(\int_{-r}^{r} (|\mathcal{K}|_1\|\phi_{u,v}-\phi_{a,b}\|)(t)\rho(t)\,dt + \int_{-r}^{r} (|\mathcal{K}|_1\|\phi_{a,b}\|)(t)\rho(t)\,dt\right) \\
&\quad\le \int_{-r}^{r} \left((|\mathcal{K}|_1 L_{f,1})(t) + (|\mathcal{K}|_1 L_{f,2})(t) + \|k_1\|_{L^1_1}\right)\varepsilon\rho(t)\,dt \\
&\qquad\qquad + \sum_{i=1}^{k}\int_{-r}^{r} (|\mathcal{K}|_1\|\phi_{a_i,b_i}\|)(t)\rho(t)\,dt \\
&\quad\le M\mu(r,\rho)\varepsilon + \sum_{i=1}^{k}\int_{-r}^{r} (|\mathcal{K}|_1\|\phi_{a_i,b_i}\|)(t)\rho(t)\,dt,
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{1}{\mu(r,\rho)}&\int_{-r}^{r} (|\mathcal{K}|_1\Phi^+)(t)\rho(t)\,dt \\
&\le M\varepsilon + \frac{1}{\mu(r,\rho)}\sum_{j=1}^{k}\int_{-r}^{r} (|\mathcal{K}|_1\|\phi_{a_j,b_j}\|)(t)\rho(t)\,dt.
\end{aligned} \tag{4.11}$$

Since $\phi \in PAA_0(\mathbb{X}\times\mathbb{X},\mathbb{X},\rho)$, $PAA_0(\mathbb{X},\rho)$ is invariant by $|\mathcal{K}|_1$ and (4.11) is satisfied we conclude that $\Phi^+ \in PAA_0(\mathbb{R},\rho)$, showing the claimed. The case $i=2$ proceeds analogously. □

Remark 4.4. Theorem 4.3 is also valid if we replace the condition (4.8) by (4.3). In this situation the $PAA(\mathbb{X},\rho)$ invariance by $\mathcal{N}_i$ is a direct consequence of Theorem 4.2. However, in this procedure we have not included the regulating effect of the convolution $\mathcal{K}_i$ $(i=1,2)$. This allows that, the condition (4.8) to be satisfied in those cases in which the condition (4.3) is not satisfied.

To illustrate this, let $i=1$, $k_1(t)=e^{-\alpha t}$, $t\in\mathbb{R}$, $\alpha>0$ and consider the real functions $L_f = 1+\sum_{n=0}^{\infty}\mathcal{D}_n$ and $\rho = 1+\sum_{n=0}^{\infty}\mathcal{X}_n$, where

$$\mathcal{D}_n(t) = \begin{cases} 2^{3n+1}(t-n) & \text{if } t\in[n, n+\frac{1}{2^{2n+1}}],\\ -2^{3n+1}(t-(n+\frac{1}{2^{2n}})) & \text{if } t\in[n+\frac{1}{2^{2n+1}}, n+\frac{1}{2^{2n}}],\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\mathcal{X}_n(t) = \begin{cases} 2^{2n} & \text{if } t \in [n, n + \frac{1}{2^{2n}}], \\ 0 & \text{otherwise.} \end{cases}$$

We observe that

$$(|\mathcal{K}|_1 L_f)(t) = \int_{-\infty}^{t} |k_1(t-s)| L_f(s)\, ds \leq \|k_1\|_{L^1(\mathbb{R}_+)} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n},$$

which implies $\||\mathcal{K}|_1 L_f\|_\infty < \infty$. Hence L_f satisfies (4.8). On the other hand, for $r \in \mathbb{N}$ we have

$$\frac{1}{\mu(r,\rho)} \int_{-r}^{r} L_f(t)\rho(t)\, dt$$
$$= \frac{\displaystyle\int_{-r}^{r} \left(1 + \sum_{n=0}^{\infty} \mathcal{D}_n(t) + \sum_{n=0}^{\infty} \mathcal{X}_n(t) + \sum_{n=0}^{\infty} \mathcal{D}_n(t)\mathcal{X}_n(t)\right) dt}{\displaystyle\int_{-r}^{r} \left(1 + \sum_{n=0}^{\infty} 2^{2n} \mathcal{X}_n(t)\right) dt}$$
$$= \frac{\displaystyle 2r + \sum_{n=0}^{r-1} \frac{1}{2^{n+1}} + \sum_{n=0}^{r-1} 1 + \sum_{n=0}^{r-1} 2^{n-1}}{\displaystyle 2r + \sum_{n=0}^{r-1} 1} \geq \frac{(2^{r-1} - 1)/(2-1)}{2r + r},$$

which implies that (4.3) is not satisfied by L_f.

Example 4.5. Now we exhibit a function $f_1 \in C(\mathbb{R}\times\mathbb{X}\times\mathbb{X}, \mathbb{X})$ satisfying the conditions of Theorem 4.3 but not of Theorem 4.2, with k_1, ρ satisfying the conditions of Theorem 3.2 but not of Theorem 3.3. Let $\mathbb{X} = \mathbb{R}$ and $\mathcal{D}$, $\mathcal{X}$, k_1, ρ as in Remark 4.4. Consider the function $f_1 \in C(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$ described by

$$f_1(t,x,u) = x + \sum_{n=0}^{[|x|]} \mathcal{D}_n(t)x \quad t \in \mathbb{R}, x \in \mathbb{X}. \tag{4.12}$$

where $[\cdot]$ is the integer part function. Observe that f_1 is described as $f_1 = g + \phi$ where $g(t,x,u) = x$ and $\phi(t,x,u) = \sum_{n=0}^{[|x|]} \mathcal{D}_n(t)x$ for each $t \in \mathbb{R}$, $x \in \mathbb{X}$. Is not difficult to see that g is pseudo almost automorphic in t uniformly on bounded sets. Since for each bounded set $K = K_1 \times K_2$

of $\mathbb{X}\times\mathbb{X}$ we have

$$\frac{1}{\mu(r,\rho)}\int_{-r}^{r}\|\phi(t,x,u)\|\rho(t)\,dt \le \frac{\sup_{x\in K_1}\|x\|}{\mu(r,\rho)}\int_{-r}^{r}\sum_{n=0}^{[\sup_{x\in K_1}\|x\|]}\mathcal{D}_n(t)\rho(t)\,dt \to 0$$

as $r\to\infty$ uniformly in $x\in K$ we have $\phi\in PAA_0(\mathbb{X}\times\mathbb{X},\mathbb{X},\rho)$, and hence $f\in PAA(\mathbb{X}\times\mathbb{X},\mathbb{X},\rho)$. Furthermore we see that for each $t\in\mathbb{R}$ and $|x|\ge|y|$

$$\begin{aligned}\|f_1(t,x,u)-f_1(t,y,v)\| &\le \left(1+\sum_{n=[|y|]}^{[|x|]}\mathcal{D}_n(t)\right)\|x-y\|\\ &\le L_{f_{1,1}}(t)\|x-y\|+L_{f_{1,2}}(t)\|u-v\|,\end{aligned}$$

where $L_{f_{1,1}}(t)=1+\sum_{n=0}^{\infty}\mathcal{D}_n(t)$, $t\in\mathbb{R}$ and $L_{f_{1,2}}\equiv 0$. Moreover the Lipschitz functions $L_{f_{1,1}}$, $L_{f_{1,2}}$ are optimal. Hence f_1 satisfies (H2).

Suppose that h satisfies (H1), $R_1:\mathbb{R}\times\mathbb{R}\to\mathcal{B}(\mathbb{X})$ is bi-almost automorphic and $\|R_1(t,s)\|\le e^{-(t-s)}$. Since $k_1(t)=e^{-t}$, $t\ge 0$ and ρ satisfy the conditions of Theorem 3.2 (see Example 3.6), $PAA_0(\mathbb{X},\rho)$ is invariant by $|\mathcal{K}|_1$. It follows from Theorem 4.3 that $PAA(\mathbb{X},\rho)$ is invariant by $\mathcal{N}_1$.

Remark 4.6. The previous example shows the importance of Theorems 4.3 and 3.2 with respect to a more classical ρ-type result as Theorem 3.3 and 4.2 in which their conditions are not satisfied. The key of Theorems 4.3 and 3.2 is the use of the regulating effect of convolution, which had not been used until now.

Abstract neutral integral equation (4.1)**.** Next we state the main result of this work.

Theorem 4.7. *Consider $\rho\in\mathbb{U}_\infty$, $k_1\in L^1(\mathbb{R}_+)$ satisfying the conditions given in Theorems* 3.2 *or* 3.3 *and $k_2\in L^1(\mathbb{R}_-)$ satisfying the conditions given in Theorems* 3.10 *or* 3.11*. Assume that* (H1)–(H4) *hold. Then, if*

$$1>\|L_{f_0,1}\|_\infty+\|L_{f_0,2}\|_\infty+\||\mathcal{K}|_1L_{f_1,1}\|_\infty+\||\mathcal{K}|_1L_{f_1,2}\|_\infty + \||\mathcal{K}|_2L_{f_2,1}\|_\infty+\||\mathcal{K}_2|L_{f_2,2}\|_\infty, \tag{4.13}$$

the integral equation (4.1) *has a unique cl(ρ)-pseudo almost automorphic solution.*

Proof. Let us consider the operators defined by

$$(\mathcal{N}_1 u)(t) := \int_{-\infty}^{t} R_1(t,s) f_1(s, u(s), u(h_1(s)))\, ds,$$
$$(\mathcal{N}_2 u)(t) := \int_{t}^{\infty} R_2(t,s) f_2(s, u(s), u(h_2(s)))\, ds,$$
$$(\mathcal{N}_0 u)(t) := f_0(t, u(t), u(h_0(t))).$$

Then the integral equation (4.1) can be rewritten equivalently as the operator equation of second kind $\mathcal{N}u = u$, where $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1 + \mathcal{N}_2$. Then the proof of the theorem follows by application of fixed point arguments. Indeed, we firstly obtain some estimates for $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2$ and then we specify the application of fixed point argument.

Let us consider $u \in PAA(\mathbb{X}, \rho)$ and the operator $\mathcal{N}_i$, for $i = 1, 2$. The hypotheses (H1)–(H2), Remarks 3.4, 3.12 and the inequality (4.13) allows us to apply the composition Theorem 4.3 to ensure that $\mathcal{N}_i f_i(\,\cdot\,, u(\,\cdot\,), u(h_i(\,\cdot\,))) \in PAA(\mathbb{X}, \rho)$. In a similar way using (H1), (H2), by Theorem 4.2, we can ensure that $\mathcal{N}_0 u \in PAA(\mathbb{X}, \rho)$, obtaining that $\mathcal{N}u = \mathcal{N}_0 u + \mathcal{N}_1 u + \mathcal{N}_2 u \in PAA(\mathbb{X}, \rho)$ provided $u \in PAA(\mathbb{X}, \rho)$.

To complete the proof, we apply the Banach fixed-point principle to the nonlinear operator $\mathcal{N}$ on the Banach space $PAA(\mathbb{X}, \rho)$ (see Theorem 2.9). Based on the above, it is clear that $\mathcal{N}$ maps $PAA(\mathbb{X}, \rho)$ into itself. Also, for all $u, v \in PAA(\mathbb{X}, \rho)$, using $h_i(\mathbb{R}) = \mathbb{R}$ $(i = 1, 2)$ we have that

$$\begin{aligned}
\|\mathcal{N}u - \mathcal{N}v\|_\infty &\le (\|L_{f_0,1}\|_\infty + \||\mathcal{K}|_1 L_{f_1,1}\|_\infty + \||\mathcal{K}|_2 L_{f_2,1}\|_\infty)\, \|u - v\|_\infty \\
&\quad + \|L_{f_0,2}\|_\infty \|u(h_0) - v(h_0)\|_\infty \\
&\quad + \||\mathcal{K}|_1 L_{f_1,2}\|_\infty \|u(h_1) - v(h_1)\|_\infty \\
&\quad + \||\mathcal{K}|_2 L_{f_2,2}\|_\infty \|u(h_2) - v(h_2)\|_\infty \\
&\le (\|L_{f_0,1}\|_\infty + \|L_{f_0,2}\|_\infty + \||\mathcal{K}|_1 L_{f_1,1}\|_\infty + \||\mathcal{K}|_1 L_{f_1,2}\|_\infty \\
&\quad + \||\mathcal{K}|_2 L_{f_2,1}\|_\infty + \||\mathcal{K}|_2 L_{f_2,2}\|_\infty) \|u - v\|_\infty,
\end{aligned} \tag{4.14}$$

and hence $\mathcal{N}$ has a unique fixed point, which is the unique $cl(\rho)$-pseudo almost automorphic solution to (4.1). □

One of the deviations most commonly used to represent advances and delays are given by $h_i(t) = t + p_i$, $p_i \in \mathbb{R}$, for $i = 1, 2$.

Corollary 3. *Consider $\rho \in \mathbb{U}_\infty$, $p_0, p_1, p_2 \in \mathbb{R}$, $k_1 \in L^1(\mathbb{R}_+)$ satisfying the conditions given in Theorems* 3.2 *or* 3.3 *and $k_2 \in L^1(\mathbb{R}_-)$ satisfying the conditions given in Theorems* 3.10 *or* 3.11. *Assume that $h_i(t) = t + p_i$ ($i = 1, 2, 3$) satisfy* (H1) *and* (H2)–(H4) *hold. Then, if*

$$1 > (\|L_{f_0,1}\|_\infty + \|L_{f_0,2}\|_\infty) + (\||\mathcal{K}|_1 L_{f_1,1}\|_\infty + \||\mathcal{K}|_1 L_{f_1,2}\|_\infty) + (\||\mathcal{K}|_2 L_{f_2,1}\|_\infty + \||\mathcal{K}_2| L_{f_2,2}\|_\infty), \tag{4.15}$$

the integral equation

$$u(t) = f_0(t, u(t), u(t+p_0)) + \int_{-\infty}^{t} R_1(t,s) f_1(s, u(s), u(s+p_1))\, ds + \int_{t}^{\infty} R_2(t,s) f_2(s, u(s), u(s+p_2))\, ds$$

has a unique $cl(\rho)$-pseudo almost automorphic solution.

In the non deviated case we can state the following result.

Theorem 4.8. *Consider $\rho \in \mathbb{U}_\infty$, $k_1 \in L^1(\mathbb{R}_+)$ satisfying the conditions given in Theorems 3.2 or 3.3 and $k_2 \in L^1(\mathbb{R}_-)$ satisfying the conditions given in Theorems 3.10 or 3.11. Assume that* (H3)–(H4) *are satisfied and the following condition holds: For $i = 0, 1, 2$, the functions $f_i = g_i + \phi_i$, where $g_i \in AA(\mathbb{X}, \mathbb{X})$, $\phi_i \in PAA_0(\mathbb{X}, \mathbb{X}, \rho)$ and f_i satisfies the condition of Lipschitz type*

$$\|f_i(t, x_1) - f_i(t, x_2)\| \le L_{f_i}(t) \|x_1 - x_2\|, \quad t \in \mathbb{R},\ x_i \in \mathbb{X}.$$

In addition for all compact subsets $K \subseteq \mathbb{X}$, we have that $g_i(t, \cdot)$ is uniformly continuous in $K \subseteq \mathbb{X}$ uniformly for $t \in \mathbb{R}$. Then, if

$$(\|L_{f_0}\|_\infty + \||\mathcal{K}|_1 f_1\|_\infty + \||\mathcal{K}|_2 f_2\|_\infty) < 1, \tag{4.16}$$

the integral equation

$$u(t) = f_0(t, u(t)) + \int_{-\infty}^{t} R_1(t,s) f_1(s, u(s))\, ds + \int_{t}^{\infty} R_2(t,s) f_2(s, u(s))\, ds, \quad t \in \mathbb{R},$$

has a unique $cl(\rho)$-pseudo almost automorphic solution.

Remark 4.9. The treated equation (1.1) does not have, in general, the null solution unless for $i = 0, 1, 2$ we have $f_i(t, 0, 0) = 0$ for each $t \in \mathbb{R}$.

Special interest attaches to the particular cases of Theorem 4.7, Corollary 3 and Theorem 4.8 when $f_0 \equiv 0$, due to the efficient use of the regulating effect of the k_i-convolution in the contractive conditions (4.13), (4.15) and (4.16) respectively. There are no similar results in the literature. The contractive convolution conditions are better than the uniform ones; see Remark 4.23.

Applications. As a simple application of Theorem 4.7 we obtain a ρ-pseudo almost automorphic mild solution result for some abstract differential equations.

Example 4.10. Let $\mathbb{X} = \mathbb{R}$ and consider the logistic equation

$$u'(t) = \alpha u(t) + au'(t-p) - q(t, u(t), u(t-p)), \quad t \in \mathbb{R}, \tag{4.17}$$

where $\alpha > 0$, $0 \le |a| < 1$ and $p > 0$. Suppose that $\widetilde{q} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\widetilde{q}(t, x, y) := q(t, x, y) - \alpha a y$ satisfies (H2) with constants $L_{f_2,1} = \alpha a$ and $L_{f_2,2} = |1 - \alpha a|$, that is

$$|\widetilde{q}(t, x_1, y_1) - \widetilde{q}(t, x_2, y_2)| \le \alpha a |x_1 - x_2| + |(1 - \alpha a)||y_1 - y_2|,$$

for all $t, x_1, x_2, y_1, y_2 \in \mathbb{R}$. We are interested in bounded solutions u of the equation (4.17), then

$$\frac{d}{dt}[(u(t) - au(t-p))e^{-\alpha t}] = [\alpha a u(t-p) - q(t, u(t), u(t-p))]e^{-\alpha t}.$$

Since $\lim_{t \to \infty} (u(t) - au(t-p))e^{-\alpha t} = 0$, by integration, we have

$$u(t) = au(t-p) + \int_t^{\infty} [q(s, u(s), u(s-p)) - \alpha a u(s-p)]e^{\alpha(t-s)} ds.$$

Setting $f_0(t, x, y) = ay$; $h_0(t) = h_2(t) = t - p$, for all $t \in \mathbb{R}$; $f_1 \equiv 0$; $f_2 = \widetilde{q}$; $R_2(t, s) = e^{\alpha(t-s)}$ for all $s \ge t$; $k_2(t) = e^{\alpha t}$ for all $t \in \mathbb{R}_-$ and R_1 as the null operator, we have that (H2)–(H4) are satisfied. In view of Theorem 4.7 we have the following result:

Theorem 4.11. *Suppose that (H1) is satisfied. If the weight ρ satisfies at least one of the following conditions:*

$$\rho \textit{ is such that } (3.17) \textit{ holds or}$$

$$\sup_{|s| \le r, r \in \mathbb{R}} \frac{e^{-\alpha s}}{\rho(s)} \int_s^r e^{\alpha t} \rho(t)\, dt < \infty, \quad \lim_{r \to \infty} \frac{e^{-\alpha r}}{\mu(r, \rho)} \int_{-r}^r e^{\alpha t} \rho(t)\, dt = 0.$$

Then the logistic equation (4.17) *has a unique cl(ρ)-pseudo almost automorphic mild solution whenever*

$$\|L_{f_0,1}\|_\infty + \||\mathcal{K}|_2 L_{f_2,1}\|_\infty + \||\mathcal{K}|_2 L_{f_2,2}\|_\infty = 2a + \frac{1}{\alpha}|1-\alpha a| < 1.$$

Remark 4.12. The existence and uniqueness of cl(ρ)-pseudo almost automorphic mild solution of the equation (4.17) was studied by Zitane and Bensouda [**52**]. They obtained a main result (see Theorem 3.4) under the stronger ρ-condition $\limsup_{t\to\infty} \frac{\rho(t+\tau)}{\rho(t)}$ and $\limsup_{T\to\infty} \frac{\mu(T+|\tau|,\rho)}{\mu(T,\rho)}$ are finite for any $\tau \in \mathbb{R}$, which is a particular case of Theorem 4.11. New conditions of ρ and (k,ρ) type, to ensure the existence and uniqueness of cl(ρ)-pseudo almost automorphic mild solution of the equation (4.17) are available in Theorem 4.11.

Example 4.13. Consider the abstract differential equation

$$u'(t) = Au(t) + f_1(t, u(t), u(h_1(t))), \quad t \in \mathbb{R}, \tag{4.18}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ and the functions $h_1 : \mathbb{R} \to \mathbb{R}$, $f_1 \in PAA(\mathbb{X}\times\mathbb{X}, \mathbb{X}, \rho)$ satisfy (H1), (H2) respectively.

Let $M > 0$, $\alpha > 0$ the constants such that $\|T(t)\| \leq Me^{-\alpha t}$ for each $t \geq 0$. Then the kernel $R_1(t,s) = T(t-s)$ satisfies $\|R_1(t,s)\| \leq |k_1(t-s)|$ for all $t \geq s$, where $k_1(t) = Me^{-\alpha t}$. Also we can see that R_1 is a bi-almost automorphic operator. Hence (H3) is satisfied. The mild solution of the equation (4.18) is expressed by the convolution integral equation

$$u(t) = \int_{-\infty}^{t} T(t-s) f_1(s, u(s), u(h_1(s)))\, ds, \quad t \in \mathbb{R},$$

which is a particular case of (4.1) setting $f_0 = f_2 = 0$ and R_2 by the null operator. As a consequence of Theorem 4.7 we have:

Theorem 4.14. *Assume that A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ and the functions $h_1 : \mathbb{R} \to \mathbb{R}$, $f_1 \in PAA(\mathbb{X}\times\mathbb{X}, \mathbb{X}, \rho)$ satisfy (H1), (H2) respectively. If the weight ρ satisfies at least one of the following three conditions:*

$$\rho \textit{ is such that } (3.6) \textit{ holds or}$$

$$\sup_{|s|\le r, r\in\mathbb{R}} \frac{e^{\alpha s}}{\rho(s)} \int_s^r e^{-\alpha t}\rho(t)\,dt < \infty,$$

$$\lim_{r\to\infty} \frac{e^{-\alpha r}}{\mu(r,\rho)} \int_{-r}^r e^{-\alpha t}\rho(t)\,dt = 0, \text{ or} \tag{4.19}$$

the condition (3.14) *is satisfied.*

Then the abstract differential equation (4.18) *has a unique cl*(ρ)*-pseudo almost automorphic mild solution whenever* $\||\mathcal{K}|_1 L_{f_1,1}\|_\infty + \||\mathcal{K}|_1 L_{f_1,2}\|_\infty < 1$. *If* $L_{f_1,1}$ *and* $L_{f_1,2}$ *are bounded the last inequality can be replaced by the stronger condition* $\|L_{f_1,1}\|_\infty + \|L_{f_1,2}\|_\infty < \alpha/M$.

Remark 4.15. The problem of existence of oscillatory type solutions of semilinear abstract differential equations have been considered by several authors, see for instance **[40, 3, 13]**. The existence and uniqueness of cl(ρ)-pseudo almost automorphic mild solution of the solution of (4.18) in the non deviated case was studied by Blot et al. **[3]**, their main result (see Theorem 3.2) was obtained under the hypothesis that A is the infinitesimal generator of a C_0-semigroup on a Banach space $\mathbb{X}$ such that $\|T(t)\| \le Ne^{-\omega t}$, $t \ge 0$ among others conditions of type as (H2). These conditions are compatible with the particular case of Theorem 4.14 making $L_{f_1,2} \equiv 0$ and $L_{f_1,1} \equiv L_f$ or equivalently $NL_f < \omega$. All these bounded conditions are weaker than these obtained from the convolution.

Our counterexample described in (3.19), (3.20) anticipates the necessity to impose conditions on the weight ρ to ensure the existence of cl(ρ)-pseudo almost automorphic mild solution of equation (4.18). For this reason we have considered different conditions of (k,ρ) or ρ type in the assumptions of Theorem 4.14.

Since $k_1(t) = e^{-\alpha t}$, if in (4.18) we take the function f_1 and ρ shown in Example 4.5 then, thanks to the conditions of (k,ρ)-type of (4.14), we can obtain existence and uniqueness of cl(ρ)-pseudo almost automorphic mild solution of equation (4.18). Here the ρ-type condition of (4.14) is not satisfied.

Example 4.16. Consider the abstract fractional differential equation

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f_1(t, u(t), u(h_1(t))), \quad t \in \mathbb{R}, \tag{4.20}$$

where $1 < \alpha < 2$ and the fractional derivative is understood in the Riemann-Liouville sense; the function $h_1 : \mathbb{R} \to \mathbb{R}$ satisfies (H1);

$f_1 \in PAA(\mathbb{X}\times\mathbb{X},\mathbb{X},\rho)$ satisfying the (H2) condition and $A: D(A)\subseteq\mathbb{X}\to\mathbb{X}$, is a linear operator, densely defined, of sectorial type ω and angle θ, that is, there are constants $\omega\in\mathbb{R}$, $\theta\in(0,\frac{\pi}{2})$, $M>0$ such that its resolvent exists outside the sector $\omega+\Sigma_\theta := \{\lambda+\omega : \lambda\in\mathbb{C},\ |\arg(-\lambda)|<\theta\}$, and

$$\|(\lambda-A)^{-1}\| \le \frac{M}{|\lambda-\omega|}, \quad \lambda\notin\omega+\Sigma_\theta.$$

Sectorial operators are well studied in the literature, usually in case $\omega=0$. For reference including several examples and properties we refer to **[26]**. We observe that equation (4.20) can be viewed as the limiting equation of the equation

$$\begin{aligned} v'(t) &= \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Av(s)\,ds + f_1(t,v(t),v(h_1(t))), \quad t>0, \\ v(0) &= u_0\in\mathbb{X}, \end{aligned} \tag{4.21}$$

in the sense that the solutions u of (4.20) and v of (4.21) are asymptotic of each other as $t\to\infty$. In fact, if we assume that A is sectorial of type ω with $0<\theta<\pi(1-\frac{\alpha}{2})$, then problem (3.2) is well posed, see **[15]**, and the variation of parameters formula allows us to write the solution of (4.21) as

$$v(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s) f_1(s,v(s),v(h_1(s)))\,ds, \quad t\ge 0,$$

where the family of operators $\{S_\alpha(t)\}_{t\ge0}$ on $\mathbb{X}$ are defined by

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha-1)^{-1}d\lambda, \quad t>0,$$

with γ a suitable path lying outside the sector $\omega+\Sigma_\theta$. Note that in the border case $\alpha=1$ the family $\{S_\alpha(t)\}_{t\ge0}$ corresponds to a C_0-semigroup. If $\{S_\alpha(t)\}_{t\ge0}$ is integrable the mild solution of (4.20) is given by

$$u(t) = \int_{-\infty}^t S_\alpha(t-s) f_1(s,u(s),u(h_1(s)))\,ds. \tag{4.22}$$

Cuesta in **[15,** Theorem 1] has proved that if A is a sectorial operator of type $\omega<0$ for some $M>0$ and $0<\theta<\pi(1-\frac{\alpha}{2})$, then there exists $C>0$ such that

$$\|S_\alpha(t)\| \le \frac{CM}{1+|\omega|t^\alpha}.$$

Hence $\{S_\alpha(t)\}_{t\geq 0}$ is integrable and the mild solution of (4.20) is given by (4.22). The kernel $R_1(t,s)=S_\alpha(t-s)$ satisfies $\|R_1(t,s)\|\leq |k_1(t-s)|$ for all $t\geq s$, where $k_1(t)=\frac{CM}{1+|\omega|t^\alpha}$. Since R_1 is bi-almost automorphic we obtain that (H3) and (H4) are satisfied. The following result follows from Theorem 4.7.

Theorem 4.17. *Assume that the functions* $h_1:\mathbb{R}\to\mathbb{R}$, $f_1\in PAA(\mathbb{X}\times\mathbb{X},\mathbb{X},\rho)$ *satisfy* (H1), (H2) *respectively and that* $A:D(A)\subseteq\mathbb{X}\to\mathbb{X}$ *is a linear operator, densely defined, of sectorial type* $\omega<0$ *and angle* θ. *Suppose the weight* ρ *satisfies at least one of the following conditions:*

$$\rho \text{ is such that } (3.6) \text{ holds or}$$

$$\sup_{|s|\leq r, r\in\mathbb{R}}\frac{1}{\rho(s)}\int_s^r\frac{\rho(t)}{1+|\omega|(t-s)^\alpha}dt<\infty,$$

$$\lim_{r\to\infty}\frac{1}{\mu(r,\rho)}\int_{-\infty}^{-r}\left(\int_{-r}^{r}\frac{\rho(t)}{1+|\omega|(t-s)^\alpha}dt\right)ds=0.$$

Then (4.20) *has a unique cl(ρ)-pseudo almost automorphic mild solution whenever* $\||\mathcal{K}|_1L_{f_1,1}\|_\infty+\||\mathcal{K}|_1L_{f_1,2}\|_\infty<1$. *Particularly if* $L_{f_1,1}$ *and* $L_{f_1,2}$ *are bounded the last inequality can be replaced by*

$$2CM\left(\|L_{f_1,1}\|_\infty+\|L_{f_1,2}\|_\infty\right)<\frac{\alpha\sin(\phi/\alpha)}{|\omega|^{-1/\alpha}\pi}.$$

This type of general result can be adapted to other works involving weighted pseudo-almost periodic function as for example [**2**] or as a projection to consider stepanov-like weighted pseudo almost automorphic solutions see [**1, 50, 9**].

Example 4.18. Consider the deviated semilinear integro differential equation of fractional order

$$D^\alpha u(t)=Au(t)+\int_{-\infty}^t a(t-s)Au(s)\,ds+f_1(t,u(t),u(h_1(t))), \tag{4.23}$$

$t\in\mathbb{R}$, where $\alpha>0$ and the fractional derivative is understood in Weyl's sense; the kernel $a\in L^1(\mathbb{R}_+)$; $f_1:\mathbb{R}\times\mathbb{X}\times\mathbb{X}\to\mathbb{X}$ satisfies (H2); h_1 satisfies (H1) and A is a closed operator defined on $\mathbb{X}$ which generates a resolvent $\{S(t)\}_{t\geq 0}\subseteq\mathcal{B}(\mathbb{X})$ in the sense of Prüss [**46,** Definition 1.3], associated to the kernel $a \in L^1_{loc}(\mathbb{R}_+)$, such that

$$\|S(t)\| \leq k_1(t), \text{ for all } t \geq 0 \text{ for some } k_1 \in L^1(\mathbb{R}_+).$$

In the linear version of the equation (4.23), i.e.,

$$D^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f_1(t), \quad t \in \mathbb{R}, \tag{4.24}$$

Ponce [**45,** Section 3] shows that, if $f_1(t) \in D(A)$ for all $t \in \mathbb{R}$, then the strict bounded solution of (4.24) is given by

$$u(t) := \int_{-\infty}^{t} S(t-s)f_1(s)\,ds, \quad t \in \mathbb{R}.$$

Since in general, we only have $f_1(t) \in \mathbb{X}$ or that $D^\alpha u$ does not exists, we say that u, defined as above, is a mild solution of (4.24). Then naturally a mild solution of equation (4.23) is defined by the solutions of

$$u(t) = \int_{-\infty}^{t} S(t-s)f_1(s, u(s), u(h_1(s)))\,ds, \quad t \in \mathbb{R}.$$

Again, the above integral equation is a particular case of (4.1) setting $f_0 = f_2 = 0$; R_2 by the null operator and $R(t,s) = S(t-s)$ for all $t \geq s$. Thus we have the following result.

Theorem 4.19. *Assume $f_1 : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ satisfies* (H2)*, h_1 satisfies* (H1) *and A is a closed operator defined on $\mathbb{X}$ that generates a resolvent $\{S(t)\}_{t\geq 0} \subseteq \mathcal{B}(\mathbb{X})$, associated to the kernel $a \in L^1_{loc}(\mathbb{R}_+)$, such that*

$$\|S(t)\| \leq k_1(t), \text{ for all } t \geq 0 \text{ for some } k_1 \in L^1(\mathbb{R}_+).$$

Suppose the weight ρ satisfies at least one of the following conditions:

$$\rho \text{ is such that (3.6) holds or}$$

$$\sup_{|s|\leq r, r\in\mathbb{R}} \frac{1}{\rho(s)} \int_{s}^{r} |k_1(t-s)|\rho(t)\,dt < \infty,$$

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-\infty}^{-r} \left(\int_{-r}^{r} |k_1(t-s)|\rho(t)\,dt \right) ds = 0.$$

Then the deviated semilinear fractional differential equation (4.23) *has a unique cl(ρ)-pseudo almost automorphic mild solution whenever*

$\||\mathcal{K}|_1 L_{f_1,1}\|_\infty + \||\mathcal{K}|_1 L_{f_1,2}\|_\infty < 1$. *In the bounded case, the last inequality can be replaced by* $\|L_{f_1,1}\|k_1\|_{L^1(\mathbb{R}_+)}\|_\infty + \|L_{f_1,2}\|_\infty \|k_1\|_{L^1(\mathbb{R}_+)} < 1$.

Remark 4.20. Theorems 4.17 and 4.19 are surely news since at present a convolution invariant result valid for any $k_1 \in L_1(\mathbb{R}_+)$ it was not known.

To end this section we consider an example of evolution operators with exponential dichotomy

Example 4.21. Consider the partial differential equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + g(t)u(t,x) + f(t,u(t,x),u(h(t),x)), \quad u(t,0) = u(t,\pi) = 0, \tag{4.25}$$

$t \in \mathbb{R}$, $x \in [0,\pi]$, where $f : \mathbb{R}^3 \to \mathbb{R}$ and $g \in AA(\mathbb{C})$ is a ergodic function, that is, the limit

$$M_0 := \lim_{T\to\infty} \frac{1}{2T} \int_{-T+\xi}^{T+\xi} g(s)\, ds \tag{4.26}$$

exists uniformly with respect to $\xi \in \mathbb{R}$ and its value is independent of ξ.

In order to write the partial differential equation (4.25) as an abstract differential equation we introduce the convenient Banach space $\mathbb{X} = L^2[0,\pi]$. We note the presence of the operator A defined on $\mathbb{X}$ by $Au = -u''$, with domain

$$D(A) = \{u \in L^2[0,\pi] : u'' \in L^2[0,\pi],\ u(0) = u(\pi) = 0\}.$$

It is well known that A is the generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on $L^2[0,\pi]$, and expressed by

$$T(t)\phi = \sum_{n=1}^{\infty} e^{-n^2 t} < \phi, \phi_n > \phi_n,\ \phi \in \mathbb{X},$$

with $\phi_n(x) = \sqrt{2/\pi}\sin(nx)$, $x \in [0,\pi]$, $n \in \mathbb{N}$. The partial differential equation (4.25) is associated with the linear abstract differential equation

$$u'(t) = A(t)u(t) \quad \text{for } t \in \mathbb{R}, \tag{4.27}$$

where $A(t) = A + g(t)I$, with $D(A(t)) = D(A)$. The system (4.27) has a corresponding evolution family of operators $U(t,s)$ on $\mathbb{X}$ related to the semigroup $T(t)$ by $U(t,s) = T(t-s)e^{\int_s^t g(\tau)d\tau}$, $t,s \in \mathbb{R}$, or

$$U(t,s)\phi = \sum_{n=1}^{\infty} e^{\int_s^t g(\tau)d\tau - n^2(t-s)} < \phi, \phi_n > \phi_n, \quad t,s \in \mathbb{R}. \tag{4.28}$$

Suposse that $\Re(M_0) \in (n_0^2, (n_0+1)^2)$ for some $n_0 \in \mathbb{N}$. Let α_1, $\alpha_2 > 0$ such that $\Re(M_0) - (n_0+1)^2 < -\alpha_1 < 0$ and $0 < \alpha_2 < \Re(M_0) - n_0^2$ respectively. For each $n \in \mathbb{N}$ consider $\mu : \mathbb{R} \to BC(\mathbb{R}, \mathbb{C})$ defined by $\mu(t) = g(t) - n^2$. Since the condition described in (4.26) is satisfied. If $n \geq n_0 + 1$ we have $\Re(M(\mu)) \leq \Re(M_0) - (n_0+1)^2 \in (-\infty, -\alpha_1) \subseteq \mathbb{R}_-$. Thus we can apply item (i) of Lemma 2.5 in **[13]** in order to obtain that there exists T_1 (big enough) such that

$$\Re\left(\int_s^t g(\tau)d\tau\right) - n^2(t-s) \leq -\alpha_1(t-s), \quad t-s \geq T_1,\ n \geq n_0+1. \tag{4.29}$$

In a similar way if $n \leq n_0$ we have $\Re(M(\mu)) \leq \Re(M_0) - n_0^2 \in (\alpha_2, \infty) \subseteq \mathbb{R}_+$. Then by item (ii) of Lemma 2.5 there exists T_2 (big enough) such that

$$\Re\left(\int_s^t g(\tau)d\tau\right) - n^2(t-s) \leq \alpha_2(t-s),\ s-t \geq T_2,\ n \leq n_0. \tag{4.30}$$

The estimates (4.29) and (4.30) allows us to deduce that there exist $c_1, c_2 > 0$ such that for each $n \geq n_0 + 1$

$$\left|e^{\int_s^t g(\tau)d\tau - n^2(t-s)}\right| \leq c_1 e^{-\alpha_1(t-s)}, \quad t \geq s,$$

and for each $1 \leq n \leq n_0$.

$$\left|e^{\int_s^t g(\tau)d\tau - n^2(t-s)}\right| \leq c_2 e^{\alpha_2(t-s)},\ s \geq t.$$

In view of (4.28) and the preceding two inequalities we obtain that the evolution family $U(t,s)$ has an (α_1, α_2, c)-exponential dichotomy with $\text{Rank}(P) = \infty$, $\text{Rank}(I-P) = n_0$ and $c = \max\{c_1, c_2\}$.

The nonlinear partial differential equation (4.25) can be rewritten

$$u'(t) = A(t)u(t) + f(t, u(t), u(h(t))) \tag{4.31}$$

where $u(t) = u(t, \cdot)$, and $f(t, u(t), u(h(t))) = f(t, u(t, \cdot), u(h(t, \cdot))) \in L^2[0, \pi]$. In the non-homogeneous linear version of (4.31), $f(t, u(t), u(h(t))) := f(t)$. If $f(t) \in D(A)$ for all $t \in \mathbb{R}$, then the strict bounded solution of

(4.31) is expressed by

$$u(t) := \int_{-\infty}^{\infty} G(t,s) f(s)\, ds, \quad t \in \mathbb{R}, \tag{4.32}$$

where

$$G(t,s) = \begin{cases} U(t,s)P & \text{for } t \geq s, \\ -U(s,t)Q & \text{for } t < s. \end{cases}$$

Since we are considering $f(t) \in \mathbb{X}$, it may be that $u'(t)$ does not exist; therefore we say that u defined by (4.32) is a mild solution of (4.31). A mild solution of equation (4.25) is defined by the solutions of

$$\begin{aligned} u(t) &= \int_{-\infty}^{\infty} G(t,s) f(s, u(s), u(h(s)))\, ds \\ &= \int_{-\infty}^{t} U(t,s) P f(s, u(s), u(h(s)))\, ds - \int_{t}^{\infty} U(s,t) Q f(s, u(s), u(h(s)))\, ds, \end{aligned}$$

which is a particular case of (4.1) setting, $h_1 = h_2 = h$; $f_0 = 0$, $f_1 = f_2 = f$; $R_1 = R_2 = G$ and $k_1(t) = ce^{-\alpha_1 t}$, $k_2(t) = ce^{\alpha_2 t}$. Note that (H3) and (H4) are satisfied since $g \in AA(\mathbb{C})$. As a consequence of Theorem 4.7 we have the following result.

Theorem 4.22. *Assume that f satisfies (H2), h satisfies (H1), $g \in AA(\mathbb{C})$ satisfies the ergodic condition described in* (4.26) *with* $\Re(M_0) \in (n_0^2, (n_0+1)^2)$ *for some* $n_0 \in \mathbb{N}$. *Suppose that* α_1, $\alpha_2 > 0$ *are such that* $0 < \alpha_1 < (n_0+1)^2 - \Re(M_0)$ *and* $0 < \alpha_2 < \Re(M_0) - n_0^2$ *respectively. If the weight ρ satisfies at least one of the following two conditions:*

$$\rho \textit{ is such that } (3.22) \textit{ holds or}$$

$$\begin{cases} \sup\limits_{|s| \leq r, r \in \mathbb{R}} \dfrac{e^{\alpha_1 s}}{\rho(s)} \displaystyle\int_s^r e^{-\alpha_1 t} \rho(t)\, dt < \infty, \quad \lim\limits_{r \to \infty} \dfrac{e^{-\alpha_1 r}}{\mu(r,\rho)} \displaystyle\int_{-r}^{r} e^{-\alpha_1 t} \rho(t)\, dt = 0, \\ \sup\limits_{|s| \leq r, r \in \mathbb{R}} \dfrac{e^{-\alpha_2 s}}{\rho(s)} \displaystyle\int_s^r e^{\alpha_2 t} \rho(t)\, dt < \infty, \quad \lim\limits_{r \to \infty} \dfrac{e^{-\alpha_2 r}}{\mu(r,\rho)} \displaystyle\int_{-r}^{r} e^{\alpha_2 t} \rho(t)\, dt = 0. \end{cases}$$

Then, the partial differential equation (4.25) *has a unique cl(ρ)-pseudo almost automorphic mild solution whenever* $\| |\mathcal{K}|_1 L_{f,1} \|_\infty + \| |\mathcal{K}|_1 L_{f,2} \|_\infty + \| |\mathcal{K}|_2 L_{f,1} \|_\infty + \| |\mathcal{K}|_2 L_{f,2} \|_\infty < 1$. *In particular, if* $L_{f,1}$, $L_{f,2}$ *are bounded the last inequality can be replaced by* $\|L_{f,1}\|_\infty + \|L_{f,2}\|_\infty < \frac{\alpha_1 \alpha_2}{2(\alpha_1+\alpha_2)}$.

Remark 4.23. It is not difficult to see that the first contractive condition in Theorem 4.22, involving convolutions, is better than the second which takes uniform bounds. Indeed, if $f(t,x,y)=\varphi(t)(x+y)$ $t\in\mathbb{R}$, $\alpha_1=\alpha_2=\alpha$. It is enough to choose φ satisfying

$$\|\varphi\| > \frac{\alpha}{4}, \quad \int_{-\infty}^{\infty} e^{-\alpha|t-s|}|\varphi(s)|ds < \frac{1}{2},$$

where $L_{f,1}=L_{f,2}=\varphi$.

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