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VOLTERRA-CHOQUET INTEGRAL EQUATIONS

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ABSTRACT. We study the classical Volterra integral equation of the second kind on an interval, in which the Lebesgue type integral is replaced by the more general Choquet integral with respect to a monotone, submodular and continuous from below and from above set function, including the so-called distorted Lebesgue measures.

1. Introduction. Let $\Omega \neq \emptyset$, $\mathcal{C}$ be a σ-algebra of subsets in Ω and $\mu : \mathcal{C} \to [0, +\infty]$ a monotone set function, that is, satisfying $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{C}$, with $A \subset B$.

In the very recent paper **[6]**, in the classical Fredholm integral equation of the second kind we have replaced the Lebesgue type integral $\int$ by the more general Choquet integral $(C)\int$ with respect to a monotone set function $\mu : \mathcal{C} \to [0, +\infty]$ and have studied the corresponding Fredholm-Choquet integral equation, given formally by

$$\varphi(x) = f(x) + \lambda \cdot (C)\int_{\Omega} K(x,s)\varphi(s)\, d\mu(s), \quad x \in \Omega, \tag{1}$$

with the given data $\lambda \in \mathbb{R}$, $f : \Omega \to \mathbb{R}$, $K : \Omega \times \Omega \to \mathbb{R}$ and the unknown function $\varphi : \Omega \to \mathbb{R}$.

Now, in what follows we consider $\mathcal{C}$ to be a σ-algebra of subsets in $[a, b]$ and $\mu : \mathcal{C} \to [0, +\infty]$ a monotone set function, that is, satisfying $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{C}$, with $A \subset B$.

In continuation of the idea in **[6]**, it is a natural and interesting problem that in the classical Volterra integral equation of the second kind, to replace the Lebesgue type integral $\int ds$ by the more general

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Choquet integral $(C)\int d\mu(s)$ with respect to a monotone set function $\mu:\mathcal{C}\to[0,+\infty]$ and to study the corresponding Volterra-Choquet integral equation, written formally as

$$\varphi(x)=f(x)+\lambda\cdot(C)\int_a^x K(x,s,\varphi(s))\,d\mu(s),\quad x\in[a,b], \tag{2}$$

with the given data $\lambda\in\mathbb{R}$, $f:[a,b]\to\mathbb{R}$, $K:[a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ and the unknown function $\varphi:[a,b]\to\mathbb{R}$.

It is clear that due to the essential generality of the Choquet integral (since the Choquet integral with respect to a countably additive and bounded set function becomes the Lebesgue type integral), such an integral equation does not have solutions for any choice of the monotone set function μ. Also, while the classical Volterra integral equation is linear, due to the nonlinearity of the Choquet integral, the Volterra-Choquet integral equation becomes nonlinear.

It is worth noting that as in the classical cases, although their proofs have in common the use of the classical Banach fixed point theorem, the approach of the Volterra-Choquet integral equation requires some different techniques from those used for the Fredholm-Choquet integral equation.

Section 2 contains some preliminaries on the Choquet integral. In Section 3, the existence and uniqueness of the solutions are obtained. Thus, for μ belonging to some classes of monotone, submodular and continuous from below and from above set functions, we show that these integral equations have unique solutions under some appropriate conditions on the given data f and K.

An important concrete class of such set functions are the so-called distorted Lebesgue measures defined in Section 2.

2. Preliminaries.

Some known concepts and results concerning the Choquet integral can be summarized by the following.

Definition 2.1. Suppose $\Omega\neq\varnothing$ and let $\mathcal{C}$ be a σ-algebra of subsets in Ω.

(i) (see, e.g., [**10,** p. 63]) The set function $\mu:\mathcal{C}\to[0,+\infty]$ is called a monotone set function (or capacity) if $\mu(\varnothing)=0$ and $\mu(A)\le\mu(B)$ for all $A, B \in \mathcal{C}$, with $A \subset B$. Also, μ is called submodular if

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B) \quad \text{for all } A, B \in \mathcal{C},$$

bounded if $\mu(\Omega) < +\infty$, and normalized if $\mu(\Omega) = 1$.

(ii) (see, e.g., [**10,** p. 233], or [**3**]) If μ is a monotone set function on $\mathcal{C}$ and if $f : \Omega \to \mathbb{R}$ is $\mathcal{C}$-measurable (that is, for any Borel subset $B \subset \mathbb{R}$ we have $f^{-1}(B) \in \mathcal{C}$), then for any $A \in \mathcal{C}$, the concept of Choquet integral is defined by

$$(C) \int_A f \, d\mu = \int_0^{+\infty} \mu(F_\beta(f) \cap A) \, d\beta + \int_{-\infty}^0 \big[\mu(F_\beta(f) \cap A) - \mu(A)\big] \, d\beta,$$

where we used the notation $F_\beta(f) = \{\omega \in \Omega : f(\omega) \geq \beta\}$. Notice that if $f \geq 0$ on A, then in the above formula we get $\int_{-\infty}^0 = 0$.

The function f is called Choquet integrable on A if $(C) \int_A f \, d\mu \in \mathbb{R}$.

(iii) (see, e.g., [**10,** p. 40]) We say that the set function $\mu : \mathcal{C} \to [0, +\infty]$ is continuous from below, if for any sequence $A_k \in \mathcal{C}$, $A_k \subset A_{k+1}$, for all $k = 1, 2, \ldots$, we have $\lim_{k\to\infty} \mu(A_k) = \mu(A)$, where $A = \bigcup_{k=1}^\infty A_k$.

Also, we say that μ is continuous from above, if for any sequence $A_k \in \mathcal{C}$, $A_{k+1} \subset A_k$, for all $k = 1, 2, \ldots$, we have $\lim_{k\to\infty} \mu(A_k) = \mu(A)$, where $A = \bigcap_{k=1}^\infty A_k$.

We list some known properties of the Choquet integral below.

Remark 2.2. If $\mu : \mathcal{C} \to [0, +\infty]$ is a monotone set function, then the following properties hold:

(i) For all $a \geq 0$ we have $(C) \int_A af \, d\mu = a \cdot (C) \int_A f \, d\mu$ (if $f \geq 0$ then see, e.g., [**10,** Theorem 11.2(5), p. 228] and if f is of arbitrary sign, then see, e.g., [**4,** Proposition 5.1(ii), p. 64]).

(ii) In general (that is, if μ is only monotone), the Choquet integral is not linear, i.e., $(C) \int_A (f + g) \, d\mu \neq (C) \int_A f \, d\mu + (C) \int_A g \, d\mu$.

In particular, for all $c \in \mathbb{R}$ and f of arbitrary sign, we have (see, e.g., [**10,** pp. 232–233] or [**4,** p. 65]) $(C) \int_A (f + c) \, d\mu = (C) \int_A f \, d\mu + c \cdot \mu(A)$.

Suppose that μ is monotone and submodular. Then, for all f, g of arbitrary sign and lower bounded, the property of subadditivity holds (see, e.g., [**4,** Theorem 6.3, p. 75])

$$(C)\int_A (f+g)\,d\mu \le (C)\int_A f\,d\mu + (C)\int_A g\,d\mu$$

and (see, e.g., [**4,** Corollary 6, p. 82])

$$\left|(C)\int_A f\,d\mu - (C)\int_A g\,d\mu\right| \le (C)\int_A |f-g|\,d\mu.$$

(iii) If $f \le g$ on A then $(C)\int_A f\,d\mu \le (C)\int_A g\,d\mu$ (see, e.g., [**10,** Theorem 11.2(3), p. 228] if $f, g \ge 0$ and p. 232 if f, g are of arbitrary sign).

(iv) Let $f \ge 0$. If $A \subset B$ then $(C)\int_A f\,d\mu \le (C)\int_B f\,d\mu$. In addition, if μ is finitely subadditive (that is $\mu\left(\bigcup_{k=1}^n A_k\right) \le \sum_{k=1}^n \mu(A_k)$, for all $n \in \mathbb{N}$), then

$$(C)\int_{A\cup B} f\,d\mu \le (C)\int_A f\,d\mu + (C)\int_B f\,d\mu.$$

(v) It is immediate that $(C)\int_A 1 \cdot d\mu(t) = \mu(A)$.

(vi) The formula $\mu(A) = \gamma(m(A))$, where $\gamma : [0, m(\Omega)] \to \mathbb{R}$ is an increasing and concave function, with $\gamma(0) = 0$ and m is a bounded measure (or bounded but only finitely additive) on a σ-algebra on Ω (that is, $m(\emptyset) = 0$ and m is countably additive), gives simple examples of monotone and submodular set functions (see, e.g., [**4,** pp. 16-17]). Such set functions μ are also called distortions of countably additive measures (or distorted measures).

If $\Omega = [a, b]$, then for the Lebesgue (or any Borel) measure m on $[a, b]$, $\mu(A) = \gamma(m(A))$ give simple examples of bounded, monotone and submodular set functions on $[a, b]$.

If, in addition we suppose that γ is continuous at 0 and at $m([a, b])$, then by the continuity of γ on the whole interval $[0, m([a, b])]$ and from the continuity from below and from above of any Borel measure (including therefore the Lebesgue measure), it easily follows that the corresponding distorted measure also is continuous from below and continuous from above.

For simple examples, we can take $\gamma(t) = t^p$ with $0 < p < 1$, $\gamma(t) = \frac{2t}{1+t}$, $\gamma(t) = 1 - e^{-t}$, $\gamma(t) = \ln(1+t)$ for $t \ge 0$ and $\gamma(t) = \sin(t/2)$ for $t \in [0, \pi]$.

(vii) If μ is a countably additive bounded measure, then the Choquet integral $(C)\int_A f\, d\mu$ reduces to the usual Lebesgue type integral (see, e.g., [**4,** p. 62] or [**10,** p. 226]).

3. Main results. In all that follows, $\mathcal{C}$ is the class of all Borel subsets in $[a,b]$ and is denoted by $\mathcal{B}$.

We make the following notations:

$$C[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b]\},$$

and for a fixed $\tau > 0$, we consider the following norm (equivalent with the Chebyshev/uniform norm):

$$\|f\|_\tau = \max\{|f(x)| \cdot e^{-\tau(x-a)} \mid x \in [a,b]\}.$$

We recall here that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called equivalent if there exist two positive constants c and C, such that $c\|f\|_1 \le \|f\|_2 \le C\|f\|_1$ for all f.

The first main result is the following.

Theorem 3.1. *Let $\mu : \mathcal{B} \to [0,+\infty)$ be a monotone, submodular and continuous from below and from above set function. Let us suppose that*

$$K \in C([a,b]\times[a,b]\times\mathbb{R}), \quad K \text{ nonnegative,} \tag{3}$$

$$K(x,\cdot,\varphi(\cdot)) \text{ is } \mathcal{B}\text{-measurable for any } x \in [a,b] \text{ and } \varphi \in C[a,b], \tag{4}$$

there exists $L > 0$ such that

$$|K(x,s,u) - K(x,s,v)| \le L \cdot |u-v| \quad \text{for all } x,s \in [a,b], u,v \in \mathbb{R} \tag{5}$$

and there exists $c(\tau) > 0$ with $\lim_{\tau\to\infty} c(\tau) = 0$, such that

$$(C)\int_a^x e^{\tau(s-x)}\, d\mu(s) \le c(\tau) \quad \text{for all } x \in [a,b] \text{ and } \tau > 0. \tag{6}$$

Then, for any $f \in C[a,b]$, the Volterra-Choquet integral equation (2) *has a unique solution $\varphi \in C[a,b]$.*

Proof. For any $\varphi \in C[a,b]$, let us denote

$$T(\varphi)(x) = (C)\int_a^x K(x,s,\varphi(s))\, d\mu(s), \quad x \in [a,b].$$

It is well-defined for any fixed $x \in [a,b]$, because from hypothesis on K and φ, it follows that as function of s, $K(x,s,\varphi(s))$ is $\mathcal{B}$-measurable, which implies that there exists $(C)\int_a^x K(x,s,\varphi(s))\,d\mu(s)$, for all $x \in [a,b]$. Also, by $\varphi \in C[a,b]$, denoting $[A,B] = \varphi([a,b])$, since the hypothesis implies $K \in C([a,b]\times[a,b]\times[A,B])$, there exists $M > 0$ such that $|K(x,s,\varphi(s))| \le M$, for all $x, s \in [a,b]$, which by **[4,** Corollary 6.6, p. 82] implies

$$\begin{aligned}\left|(C)\int_a^x K(x,s,\varphi(s))\,d\mu(s)\right| &\le (C)\int_a^x |K(x,s,\varphi(s)|\,d\mu(s)\\ &\le (C)\int_a^b |K(x,s,\varphi(s)|\,d\mu(s)\\ &\le M\cdot(C)\int_a^b d\mu(s) = M\cdot\mu([a,b]) < +\infty,\end{aligned}$$

for all $x \in [a,b]$.

In what follows, we prove that $T(\varphi) \in C[a,b]$. For that purpose, let $x \in [a,b]$ and $x_n \in [a,b]$ be with $\lim_{n\to\infty} x_n = x$. We have

$$\begin{aligned}&|T(\varphi)(x_n) - T(\varphi)(x)|\\ &\quad = \left|(C)\int_a^{x_n} K(x_n,s,\varphi(s))\,d\mu(s) - (C)\int_a^x K(x,s,\varphi(s))\,d\mu(s)\right|\\ &\quad \le \left|(C)\int_a^{x_n} K(x_n,s,\varphi(s))\,d\mu(s) - (C)\int_a^{x_n} K(x,s,\varphi(s))\,d\mu(s)\right|\\ &\qquad + \left|(C)\int_a^{x_n} K(x,s,\varphi(s))\,d\mu(s) - (C)\int_a^x K(x,s,\varphi(s))\,d\mu(s)\right|\\ &\quad \le (C)\int_a^{x_n} |K(x_n,s,\varphi(s)) - K(x,s,\varphi(s))|\,d\mu(s)\\ &\qquad + \left|(C)\int_a^{x_n} K(x,s,\varphi(s))\,d\mu(s) - (C)\int_a^x K(x,s,\varphi(s))\,d\mu(s)\right|\\ &\quad := E_1(x_n,x) + E_2(x_n,x).\end{aligned}$$

Now, since $K \in C([a,b]\times[a,b]\times[A,B])$, it follows that K is uniformly continuous on $[a,b]\times[a,b]\times[A,B]$ and therefore, for $\varepsilon > 0$ sufficiently small, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have

$$E_1(x_n,x) \le \varepsilon\cdot\mu([a,x_n]) \le \varepsilon\cdot\mu([a,b]),$$

that is, $\lim_{n\to\infty} E_1(x_n,x) = 0$.

On the other hand, denoting $A_n(x,\alpha)=\{s\in[0,x_n]:K(x,s,\varphi(s))\geq\alpha\}$ and $A(x,\alpha)=\{s\in[0,x]:K(x,s,\varphi(s))\geq\alpha\}$, we get

$$\begin{aligned}E_2(x_n,x) &= \bigg|\int_0^M \mu(\{s\in[0,x_n]:K(x,s,\varphi(s))\geq\alpha\})\,d\alpha \\ &\qquad -\int_0^M \mu(\{s\in[0,x]:K(x,s,\varphi(s))\geq\alpha\})\,d\alpha\bigg| \\ &= \bigg|\int_0^M \mu(A_n(x,\alpha))\,d\alpha-\int_0^M \mu(A(x,\alpha))\,d\alpha\bigg|.\end{aligned}$$

According to, e.g., [**8,** Exercise 5.5.21, p. 211], it suffices to consider that $x_n\nearrow x$ and that $x_n\searrow x$.

If $x_n\nearrow x$, then since $A_n(x,\alpha)\subset A_{n+1}(x,\alpha)\subset A(x,\alpha)\subset[a,b]$, for all $n\in\mathbb{N}$ and $\alpha\in[0,M]$, it follows that $\mu(A_n(x,\alpha))\leq\mu([a,b])$ for all $n\in\mathbb{N}$, $\alpha\in[0,M]$, and by the continuity from below of μ we get

$$\lim_{n\to\infty}\mu(A_n(x,\alpha))=\mu(A(x,\alpha))$$

for all $\alpha\in[0,M]$. Passing then to the limit under the integrals (which can be considered of Lebesgue type), we immediately obtain $\lim_{n\to\infty}E_2(x_n,x)=0$.

If $x_n\searrow x$, then since $A(x,\alpha)\subset A_{n+1}(x,\alpha)\subset A_n(x,\alpha)\subset[a,b]$ for all $n\in\mathbb{N}$ and $\alpha\in[0,M]$, it follows that $\mu(A_n(x,\alpha))\leq\mu([a,b])$ for all $n\in\mathbb{N}$ and $\alpha\in[0,M]$, and by the continuity from above of μ we get

$$\lim_{n\to\infty}\mu(A_n(x,\alpha))=\mu(A(x,\alpha))$$

for all $\alpha\in[0,M]$. Again, passing to the limit under the integrals, we immediately obtain $\lim_{n\to\infty}E_2(x_n,x)=0$.

In conclusion, $T(\varphi)\in C[a,b]$, and this also implies that $A(\varphi)=f+\lambda\cdot T(\varphi)\in C[a,b]$.

Now, from Remark 2.2(ii) coupled with Remark 2.2(iv) and with the hypothesis on K, we immediately obtain

$$\begin{aligned}|T(\varphi)(x)-T(\psi)(x)| &\leq (C)\int_a^x|K(x,s,\varphi(s))-K(x,s,\psi(s))|\,d\mu(s) \\ &\leq L\cdot(C)\int_a^x|\varphi(s)-\psi(s)|d\mu(s),\end{aligned}$$

and

$$
\begin{aligned}
|A(\varphi)(x) - A(\psi)(x)| & \\
&= |\lambda| \cdot |T(\varphi)(x) - T(\psi)(x)| \\
&\leq |\lambda| \cdot L \cdot (C)\int_a^x |\varphi(s) - \psi(s)|\, d\mu(s) \\
&= |\lambda| \cdot L \cdot (C)\int_a^x |\varphi(s) - \psi(s)| e^{-\tau(s-a)} e^{\tau(s-a)}\, d\mu(s) \\
&\leq |\lambda| \cdot L \cdot e^{\tau(x-a)} \cdot (C)\int_a^x |\varphi(s) - \psi(s)| e^{-\tau(s-a)} e^{\tau(s-x)}\, d\mu(s) \\
&\leq |\lambda| \cdot L \cdot e^{\tau(x-a)} \cdot \|\varphi - \psi\|_\tau \cdot (C)\int_a^x e^{\tau(s-x)}\, d\mu(s) \\
&\leq |\lambda| \cdot L \cdot e^{\tau(x-a)} \cdot \|\varphi - \psi\|_\tau \cdot c(\tau),
\end{aligned}
$$

which immediately implies

$$\|A(\varphi) - A(\psi)\|_\tau \leq |\lambda| \cdot L \cdot c(\tau) \cdot \|\varphi - \psi\|_\tau.$$

From condition (6), choosing sufficiently large $\tau > 0$ to get $|\lambda| \cdot L \cdot c(\tau) < 1$ implies that A is a contraction on the Banach space $C[a, b]$ endowed with the norm $\|\cdot\|_\tau$, and applying the Banach fixed point theorem we arrive at the desired conclusion. □

Remark 3.2. An important particular case is when $K(x, s, v) := K(x, s) \cdot v$. In this case, condition (3) becomes $K \in C([a, b] \times [a, b])$ and it immediately implies the conditions (4) and (5), with

$$L = \max\{|K(x, s)| : x, s \in [a, b]\}.$$

Remark 3.3. According to Remark 2.2(vi), if m is the Lebesgue measure and γ is an increasing, concave and continuous function with $\gamma(0) = 0$, then $\mu(A) = \gamma(m(A))$ is a monotone, submodular and continuous from above and from below set function. It is of interest, between these distorted Lebesgue measures, which ones also satisfy the condition (6). In this sense, let us suppose that $\gamma(0) = 0$, γ is continuous, increasing and concave and there exists $0 < \gamma'(0) < +\infty$. By concavity we immediately obtain $(\gamma(x) - \gamma(0))/x \leq \gamma'(0)$, for all $x \geq 0$, which implies $\mu(A) \leq \gamma'(0) \cdot m(A)$. It is worth noting that, excepting the cases when $\gamma(t) = t^p$ with $0 < p < 1$, all the other concrete examples in Remark 2.2(vi) satisfy the above property. For the set function satisfying the above mentioned property, we immediately obtain

$$
\begin{aligned}
(C)\int_a^x e^{\tau(s-x)}\,d\mu(s) &= \int_0^\infty \mu(\{s\in[a,x]: e^{\tau(x-s)}\ge\alpha\})\,d\alpha\\
&\le \gamma'(0)\cdot\int_0^\infty m(\{s\in[a,x]: e^{\tau(x-s)}\ge\alpha\})\,d\alpha\\
&= \gamma'(0)\int_a^x e^{\tau(s-x)}\,ds \le \frac{\gamma'(0)}{\tau}.
\end{aligned}
$$

Therefore, evidently (6) in Theorem 3.1 holds with $c(\tau)=\gamma'(0)/\tau$.

Remark 3.4. However, let us note that for the validity of Theorem 3.1, the condition $0<\gamma'(0)<+\infty$ in Remark 3.3 is not necessary. Indeed, let us consider $\gamma(x)=\sqrt{x}$, which satisfies $\gamma(0)=0$, is continuous, increasing and concave but $\gamma'(0)=+\infty$. In this case we show by direct calculation that condition (6) still holds.

For all $x\in[a,b]$ we have

$$
\begin{aligned}
I(x,\tau) &:= (C)\int_a^x e^{\tau(s-x)}\,d\mu(s)\\
&= \int_0^\infty \mu(\{s\in[a,x]: e^{\tau(s-x)}\ge\alpha\})\,d\alpha\\
&= \int_0^1 \mu(\{s\in[a,x]: e^{\tau(s-x)}\ge\alpha\})\,d\alpha\\
&= \int_0^{e^{-\tau(x-a)}} \mu([a,x])\,d\alpha + \int_{e^{-\tau(x-a)}}^1 \mu\Big(\Big[x-\frac{1}{\tau}\ln\Big(\frac{1}{\alpha}\Big),x\Big]\Big)\,d\alpha\\
&= \sqrt{x-a}\cdot e^{-\tau(x-a)} + \frac{1}{\sqrt{\tau}}\cdot\int_{e^{-\tau(x-a)}}^1 \sqrt{\ln(\alpha)}\,d\alpha\\
&= \sqrt{x-a}\cdot e^{-\tau(x-a)} + \frac{1}{\sqrt{\tau}}\cdot\int_0^{\tau(x-a)} \sqrt{t}\cdot e^{-t}\,dt\\
&\le \sqrt{x-a}\cdot e^{-\tau(x-a)} + \frac{1}{\sqrt{\tau}}\cdot\int_0^{\infty} \sqrt{t}\cdot e^{-t}\,dt\\
&= \sqrt{x-a}\cdot e^{-\tau(x-a)} + \frac{1}{\sqrt{\tau}}\cdot\Gamma\Big(\frac{3}{2}\Big)\\
&= \sqrt{x-a}\cdot e^{-\tau(x-a)} + \frac{1}{\sqrt{\tau}}\cdot\frac{\sqrt{\pi}}{2}.
\end{aligned}
$$

On the other hand, letting $F(v)=ve^{\tau v^2}$, from $F'(v)=e^{-\tau v^2}(1-2\tau v^2)=0$ we immediately get that $v=1/\sqrt{2\tau}$ is a maximum point for $v\ge 0$ and

$F(1/\sqrt{2\tau}) = (e^{-1/2}/\sqrt{2}) \cdot (1/\sqrt{\tau})$. Concluding, we obtain

$$I(x,\tau) \le \frac{e^{-1/2}}{\sqrt{2}} \cdot \frac{1}{\sqrt{\tau}} + \frac{1}{\sqrt{\tau}} \cdot \frac{\sqrt{\pi}}{2} := c(s),$$

which means that condition (6) holds for $\mu(A) = \sqrt{m(A)}$ too.

Remark 3.5. From Remarks 3.3 and 3.4, we immediately conclude that if m denotes the Lebesgue measure, for any $\lambda \in \mathbb{R}$, any $f \in C[a,b]$, any $K \in C([a,b] \times [a,b])$ with K nonnegative and any $\mu(A) = \gamma(m(A))$ with $\gamma(x) = \sqrt{x}$ or with $\gamma : [0, b-a] \to \mathbb{R}$ continuous, increasing and concave on $[0, b-a]$ with $\gamma(0) = 0$, $0 < \gamma'(0) < +\infty$, the Volterra-Choquet integral equation

$$\varphi(x) = f(x) + \lambda \cdot (C)\int_a^x K(x,s)\varphi(s)\, d\mu(s), \quad x \in [a,b], \tag{7}$$

always has a unique solution $\varphi \in C[a,b]$.

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