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WELL-POSEDNESS AND STABILITY FOR A VISCOELASTIC WAVE EQUATION WITH DENSITY AND TIME-VARYING DELAY IN $\mathbb{R}^n$

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ABSTRACT. This paper concerns the well-posedness and energy decay of a linear wave equation with density, infinite memory and time-varying delay in the whole space $\mathbb{R}^n$ ($n \geq 3$). We consider the weighted spaces $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $L^2_\rho(\mathbb{R}^n)$ introduced by Karachalios and Stavrakakis (1999) to overcome the difficulty that some operators on $\mathbb{R}^n$ are not compact. We prove the global well-posedness of the Cauchy problem by using Faedo-Galerkin approximation and establish the exponential decay of energy when the amplitude of the time delay term is small by using suitable Lyapunov functional.

1. Introduction. In this paper, we study the following wave equation with density, infinite memory and time-varying delay in the whole space $\mathbb{R}^n$ $(n \geq 3)$:

$$
\begin{cases}
u_{tt}(x,t) - \phi(x)\left(\alpha\Delta u(x,t) - \displaystyle\int_{-\infty}^{t} g(t-s)\Delta u(x,s)\,ds\right) \\
\qquad\qquad + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t)) = 0, & (1.1)\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, & (1.2)\\
u_t(x,t) = f_0(x,t), \quad x \in \mathbb{R}^n, \quad t \in [-\tau(0), 0), & (1.3)
\end{cases}
$$

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where $u_0(x)$, $u_1(x)$ and $f_0(x,t)$ are given initial data belonging to appropriate spaces. The coefficient $\phi(x) := (\rho(x))^{-1}$ represents the speed of sound at the point $x \in \mathbb{R}^n$ and the function $\rho(x)$ is the density. The constants μ_1 and μ_2 are two real numbers and $\alpha > 0$ is a constant. The function $\tau(t)$ represents the time-varying delay.

By now there are many works dedicated to the mathematical analysis of wave equations in a bounded domain. It has been stabilized by adding some different controls, such as internal damping, boundary controls, dynamic boundary conditions, distributed damping and thermal nature, and many results may be found in the literature. For Cauchy problem of wave equation, here we mention Kafini and Messaoudi **[5, 6]**. In **[5]**, they studied a viscoelastic wave equation of the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds = 0, \quad x \in \mathbb{R}^n,\ t > 0,$$

and proved the energy decays polynomially for compactly supported initial data u_0, u_1 and for an exponentially decaying relaxation function $g(t)$. In **[6]**, they investigated a weakly damped viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds + u_t = |u|^{p-1}u, \quad x \in \mathbb{R}^n,\ t > 0,$$

and proved a blow-up result of the problem, and this result extended that of **[21]**.

When the density $\rho(x)$ is not a constant, Karachalios and Stavrakakis **[8]** considered the following problem:

$$\begin{cases} u_{tt} - \phi(x)\Delta u + \delta u_t + \lambda f(u) = \eta(x), & x \in \mathbb{R}^n,\ t > 0,\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

and proved the well-posedness of the problem. In addition they proved the existence of a global attractor in energy space $\mathcal{D}^{1,2}(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n)$ in the case where $(\phi(x))^{-1} := g(x) \in L^{n/2}(\mathbb{R}^n)$ and $n \geq 3$. Papadopoulos and Stavrakakis considered a quasilinear wave equation of Kirchhoff type with a weak dissipative term δu_t and $f(u) = |u|^a u$. They proved global existence, energy decay and blow-up result of solutions in the case where $n \geq 3$, $\delta \geq 0$ and $g(x) \in L^{n/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$; see **[17]**. Kafini **[4]** considered a viscoelastic wave equation with density, and proved a general result of the energy. For more results in this respect, we also refer the reader to Zhou [**22**] and Zennir [**20**], and so on.

In recent years, the study of problems related to differential equations with time delay has stimulated the interest of many researchers due to the extensive practical applications. The delay effects are often a source of instability. Generally speaking, if additional control terms (damping terms) are added the system will be uniformly asymptotically stable when the arbitrary coefficient is very small. For wave equations with delay in bounded domain, we refer to Kafini, Messaoudi and Nicaise [**7**], Nicaise and Pignotti [**13, 12, 14**], Nicaise, Valein and Fridman [**15**], Nicaise and Valein [**16**] and Xu, Yung and Li [**19**], etc.

For viscoelastic wave equations with delay, Kirane and Said-Houari [**9**] studied the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0.$$

Here they assume the constants μ_1 and μ_2 are positive constants. The authors established the global well-posedness of the system under some restrictions on the parameters μ_1 and μ_2, and obtained a general decay result of energy if $0 < \mu_2 \leq \mu_1$. Subsequently, the results were extended and improved by Liu [**10**] and Dai and Yang [**3**]. When the memory term is infinite, Liu and Zhang [**11**] considered the equation

$$u_{tt} - \alpha\Delta u + \int_{-\infty}^t \mu(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) + f(u) = h.$$

They proved the well-posedness of the problem and only established the exponential decay of energy under the assumption $0 < |\mu_2| < \mu_1$. Alabau-Boussouira, Nicaise and Pignotti [**1**] considered a time-delayed wave equation with history

$$u_{tt} - \Delta u + \int_0^\infty \mu(s)\Delta u(t-s)\,ds + k u_t(t-\tau) = 0,$$

and established exponential decay of energy if the coefficient k is small enough. And the result holds for anti-damping, i.e., $k < 0$.

For nonconstant density $\rho(x)$, the time-delayed viscoelastic wave equation was not investigated previously. Our goal in the present work is to analyze the well-posedness and exponential decay rate of energy for the system (1.1)–(1.3) in the whole spaces $\mathbb{R}^n$ $(n \geq 3)$. Here we assume the density $\rho(x)$ is not a constant. We can treat the equation in the classical Sobolev spaces if the density $\rho(x)$ is a constant. Since the non-compactness of some operators in unbounded domain (especially the Poincaré inequality and some Sobolev embedding inequalities) is not valid in the whole space, we consider some spaces weighted by density introduced by Karachalios and Stavrakakis [**8**] to overcome those difficulties. We prove the global well-posedness of the system without any restrictions of μ_1, μ_2, and establish the energy decay in the case

$$0 < |\mu_2| < \sqrt{1-d}\,\mu_1,$$

where d is assumed in (2.5), and in the case $\tau(t) = \tau > 0$, $\mu_1 = 0$, and $|\mu_2| > 0$, where τ is a constant. Hence we extend the results in [**11**] and [**10**].

To deal with the infinite memory, following the same arguments as in Dafermos [**2**], we define a new variable $\eta = \eta^t(x,s)$ by

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \quad (t,s) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.4}$$

which gives us

$$\eta_t + \eta_s = u_t, \quad (x,t,s) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.5}$$

with

$$\eta^t(0) = 0 \quad \text{in } \mathbb{R}^n,\ t \geq 0,$$

and initial condition

$$\eta^0(s) = \eta_0(s) \quad \text{in } \mathbb{R}^n,\ s \in \mathbb{R}^+.$$

Therefore the original history term can be rewritten as

$$\begin{aligned} \int_{-\infty}^{t} g(t-s)\Delta u(s)\,ds &= \int_0^\infty g(s)\Delta u(t-s)\,ds \\ &= \left(\int_0^\infty g(s)\,ds\right)\Delta u(t) - \int_0^\infty g(s)\Delta\eta^t(s)\,ds. \end{aligned} \tag{1.6}$$

Then combining (1.1)–(1.3) with (1.6) and assuming for simplicity that $\alpha - \int_0^\infty g(s)\,ds = 1$, problem (1.1)–(1.3) is transformed into the following system:

$$
\begin{cases}
\rho(x)u_{tt}(x,t) - \Delta u(x,t) - \int_0^\infty g(s)\Delta\eta^t(s)\,ds \\
\qquad\qquad + \mu_1\rho(x)u_t(x,t) + \mu_2\rho(x)u_t(x,t-\tau(t)) = 0, & (1.7)\\
\eta_t^t + \eta_s^t = u_t, & (1.8)\\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \eta^t(x,0) = 0, \quad x\in\mathbb{R}^n, & (1.9)\\
u_t(x,t) = f_0(x,t), \quad x\in\mathbb{R}^n,\ t\in[-\tau(0),0), & (1.10)\\
\eta^0(x,s) = \eta_0(x,s), \quad (x,s)\in\mathbb{R}^n\times\mathbb{R}^+, & (1.11)
\end{cases}
$$

where

$$\eta_0(x,s) = u_0(x) - u(x,-s).$$

The remaining of this paper is as follows. In Section 2, we give some preparations for our consideration. In Section 3, we state our main results. In Section 4, we prove the well-posedness of the problem. In Section 5, we establish the exponential decay of energy by using energy perturbation method. The conclusion will be given in Section 6.

2. Space setting and assumptions. In this section, we give space setting and some assumptions. We use the standard notations $L^q(\mathbb{R}^n)$ $(1 \le q \le \infty)$ and $H^1(\mathbb{R}^n)$ of Lebesgue integral and Sobolev spaces. And $\|\cdot\|_B$ denotes the norm in the space B; we write $\|u\|$ instead of $\|u\|_2$ for $q = 2$.

Space setting. As in **[8]**, we introduce the weighted spaces $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $L^p_\rho(\mathbb{R}^n)$. Firstly we assume the density $\rho(x): \mathbb{R}^n \to \mathbb{R}$ $(n \ge 3)$ satisfies the following condition

(A) $\rho(x) > 0$, $\rho \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^{n/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Now we define the weighted spaces $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $L^p_\rho(\mathbb{R}^n)$ $(1 < p < \infty)$.

(1) The space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is defined to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions of the form

$$\mathcal{D}^{1,2}(\mathbb{R}^n) = \{u \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n)\},$$

equipped with the norm $\|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla u|^2\,dx$.

(2) We introduce the weighted space $L^2_\rho(\mathbb{R}^n)$ to be defined as the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(u,v)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho uv \, dx,$$

and we know that $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space and

$$\|u\|^2_{L^2_\rho(\mathbb{R}^n)} = (u,u)_{L^2_\rho(\mathbb{R}^n)}.$$

(3) If u is a measurable function on $\mathbb{R}^n$, we define

$$\|u\|^p_{L^p_\rho(\mathbb{R}^n)}(\mathbb{R}^n) = \left(\int_{\mathbb{R}^n} \rho |u|^p \, dx\right)^{1/p}, \quad \text{for } 1 < p < \infty,$$

and let $L^p_\rho(\mathbb{R}^n)$ consist of all u for which $\|u\|_{L^p_\rho(\mathbb{R}^n)} < \infty$.

From **[8]**, we can get the following lemmas.

Lemma 2.1. *Assume the function $\rho(x)$ satisfies* (A)*. Then $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is compactly embedded in $L^2_\rho(\mathbb{R}^n)$.*

Lemma 2.2. *Assume the function $\rho(x)$ satisfies* (A)*. Then for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$,*

$$\|u\|_{L^q_\rho} \le \|\rho\|_{L^s} \|\nabla u\| \quad \text{with } s = \frac{2n}{2n - qn + 2q} \text{ and } 2 \le q \le \frac{2n}{n-2}.$$

In particular, if $q = 2$ and $\rho \in L^{n/2}(\mathbb{R}^n)$, we have

$$\|u\|_{L^2_\rho} \le c_* \|\nabla u\|, \tag{2.1}$$

where c_* is a positive constant.

Let

$$V_0 = L^2_\rho(\mathbb{R}^n), \qquad V_1 = \mathcal{D}^{1,2}(\mathbb{R}^n).$$

In order to consider the new variable η, we define the weighted L^2-spaces

$$\mathcal{M}_i = L^2_g(\mathbb{R}^+, V_i) = \left\{ \eta : \mathbb{R}^+ \to V_i : \int_0^\infty g(s) \|\eta(s)\|^2_{V_i} \, ds < \infty \right\}, \quad i = 0, 1,$$

which are Hilbert spaces endowed with inner products and norms

$$(\eta, \zeta)_{\mathcal{M}_i} = \int_0^\infty g(s) (\eta(s), \zeta(s))_{V_i} \, ds$$

and

$$\|\eta\|^2_{\mathcal{M}_i} = \int_0^\infty g(s)\|\eta(s)\|^2_{V_i}\, ds, \quad i = 0, 1.$$

Finally, we introduce the phase space

$$\mathcal{H} = V_1 \times V_0 \times \mathcal{M}_1.$$

Assumptions. Concerning the relaxation function g, we need the following conditions:

(G1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$g(0) > 0, \qquad \int_0^\infty g(s)\, ds = l_0 > 0 \quad \text{and} \quad 1 - \int_0^\infty g(s)\, ds = l > 0.$$

(G2) There exists a positive constant k such that

$$g'(t) \le -kg(t) \quad \text{for } t \ge 0. \tag{2.2}$$

Next we give some assumptions on the delay $\tau(t)$.

(D1) We assume that there exist two positive constants τ_0 and τ_1 such that

$$0 < \tau_0 \le \tau(t) \le \tau_1 \quad \text{for all } t > 0. \tag{2.3}$$

(D2) We assume further that

$$\tau(t) \in W^{2,\infty}([0, T]) \quad \text{for all } T > 0, \tag{2.4}$$

and

$$\tau'(t) \le d < 1 \quad \text{for all } t > 0. \tag{2.5}$$

3. Main results. First we give the definition of a weak solution of problem (1.7)–(1.11).

Definition 3.1. Given $T > 0$, we say that a function $U(t) = (u, u_t, \eta^t)$ in $C([0, T], \mathcal{H})$ is a weak solution of problem (1.7)–(1.11) if

$$\begin{cases} (\rho u_{tt}(t), \omega) + (\nabla u(t), \nabla\omega) + (\eta^t, \omega)_{\mathcal{M}_1} + \mu_1(\rho u_t(t), \omega) \\ \qquad\qquad + \mu_2(\rho u_t(t - \tau(t)), \omega) = 0, & (3.1) \\ (\partial_t \eta^t, \chi)_{\mathcal{M}_1} = -(\partial_s \eta^t, \chi)_{\mathcal{M}_1} + (u_t(t), \chi)_{\mathcal{M}_1}, & (3.2) \end{cases}$$

a.e. in $[0,T]$, for all $\omega \in V_1$, $\chi \in \mathcal{M}_1$ and

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad \eta_0(x,s)=\eta_0(s), \quad u_t(x,t)=f_0(x,t).$$

We can get the global well-posedness of problem (1.7)–(1.11) given in the following theorem.

Theorem 3.2. *Assume that* (A) *and* (G1) *hold. Then for any initial data* $U(0)=(u_0,u_1,\eta_0)\in\mathcal{H}$ *and* $f_0(x,t)\in L^2_\rho(\mathbb{R}^n\times(-\tau(0),0))$, *problem* (1.7)–(1.11) *has a weak solution such that for any* $T>0$,

$$u\in C(0,T;\mathcal{D}^{1,2}(\mathbb{R}^n)), \quad u_t\in C(0,T;L^2_\rho(\mathbb{R}^n)) \quad \text{and} \quad \eta^t\in C(0,T;\mathcal{M}_1).$$

Given any two weak solutions U_1 *and* U_2 *of* (1.7)–(1.11) *corresponding to the initial data* $U_1(0),U_2(0)\in\mathcal{H}$ *and* $f_0(x,t),\tilde{f}_0(x,t)\in L^2(\Omega\times(-\tau(0),0))$, *then there exists a constant* $C>0$ *such that for any time* $T>0$ *and all* $t\in[0,T]$,

$$\|U_1(t)-U_2(t)\|_{\mathcal{H}} \le e^{CT}\big(\|U_1(0)-U_2(0)\|_{\mathcal{H}} + \|f_0(x,t)-\tilde{f}_0(x,t)\|_{L^2(\Omega\times(-\tau(0),0))}\big).$$

In particular, the solution of problem (1.7)–(1.11) *is unique.*

The energy functional to problem (1.7)–(1.11) is defined by

$$E(t)=\frac{1}{2}\|u_t(t)\|^2_{L^2_\rho}+\frac{1}{2}\|\nabla u(t)\|^2+\frac{1}{2}\|\eta^t\|^2_{\mathcal{M}_1} + \frac{\xi}{2}\int_{t-\tau(t)}^{t}\int_{\mathbb{R}^n}\rho(x)e^{\lambda(s-t)}u_t^2(x,s)\,dx\,ds, \tag{3.3}$$

where ξ is a positive constant to be determined later and the constant $\lambda>0$ (see **[12]**) satisfies

$$\lambda<\frac{1}{\tau_1}\left|\log\frac{|\mu_2|}{\sqrt{1-d}}\right|.$$

Then we can obtain the following theorem.

Theorem 3.3. *Let the assumptions* (A), (G1)–(G2) *and* (D1)–(D2) *hold. Let the initial data* $U(0)=(u_0,u_1,\eta_0)$ *be in* $\mathcal{H}$ *and* $f_0(x,t)$ *in* $L^2_\rho(\mathbb{R}^n\times[-\tau(0),0))$. *We can get:*

(1) *If* $\mu_1 \neq 0$*, assume that*

$$0 < |\mu_2| < \sqrt{1-d}\,\mu_1 \quad \text{and} \quad \frac{e^{\lambda\tau_1}|\mu_2|}{\sqrt{1-d}} < \xi < \mu_1,$$

then there exist two constants $\beta > 0$ *and* $\alpha > 0$ *such that the energy* $E(t)$ *defined by* (3.3) *satisfies*

$$E(t) \leq \beta e^{-\alpha t} \quad \text{for all } t \geq 0. \tag{3.4}$$

(2) *Consider the delay term* $\tau(t) = \tau$*, where* τ *is a positive constant. Assume*

$$0 < |\mu_2| < a,$$

where the constant $a > 0$ *is defined in* (5.31)*. Then the estimate* (3.4) *also holds for* $\mu_1 = 0$*.*

4. Well-posedness. In this section, we study the global well-posedness of problem (1.7)–(1.11) to prove Theorem 3.2. First of all we prove the following proposition.

Proposition 4.1. *Consider positive constants* $T > 0$ *and* $R > 0$*, and initial data* $u_0 \in \mathcal{D}^{1,2}(B_R)$*,* $u_1 \in L^2_\rho(B_R)$*,* $\eta_0 \in \mathcal{M}_1(B_R)$ *and* $f_0(x,t) \in L^2_\rho(B_R \times [-\tau(0), 0))$*, where* B_R *represents the ball with a radius of* R*. Then for the problem* (1.7)–(1.11) *restricted on* $B_R \times (0,T)$ *satisfying the boundary condition* $u = 0$ *in* $\partial B_R \times (0,T)$*, there exists a weak solution such that*

$$u \in C(0,T;\mathcal{D}^{1,2}(B_R)), \quad u_t \in C(0,T;L^2_\rho(B_R)),$$

and

$$\eta^t \in C(0,T;\mathcal{M}_1(B_R)).$$

Proof. We use the Faedo-Galerkin method to prove this proposition, which will be divided into the following four steps.

<u>Step 1: Approximate problem.</u> Let $\{\omega_j\}_{j=1}^{\infty}$ be an orthonormal basis in V_0 given by eigenfunctions of Δ in B_R. Then it is easy to see that $\{\omega_j\}_{j=1}^{\infty}$ is smooth and can be taken orthogonal in $\mathcal{D}^{1,2}(B_R)$. Next we consider a smooth orthonormal basis $\{\chi_j\}_{j=1}^{\infty}$ for $\mathcal{M}_1$. We will find an

approximate solution in the form

$$u^m(t) = \sum_{j=1}^{m} a_{mj}(t)\omega_j(x) \quad \text{and} \quad \eta^{t,m}(s) = \sum_{j=1}^{m} b_{mj}(t)\chi_j(x),$$

satisfying the approximate problem

$$\begin{cases} (u_{tt}^m(t), \omega_j)_{L^2_\rho} + (\nabla u^m(t), \nabla \omega_j) + (\eta^{t,m}, \omega_j)_{\mathcal{M}_1} \\ \qquad + \mu_1 (u_t^m(t), \omega_j)_{L^2_\rho} + \mu_2 (u_t^m(t-\tau(t)), \omega_j)_{L^2_\rho} = 0, & (4.1) \\ (\partial_t \eta^{t,m}, \chi_j)_{\mathcal{M}_1} = -(\partial_s \eta^{t,m}, \chi_j)_{\mathcal{M}_1} + (u_t^m(t), \chi_j)_{\mathcal{M}_1}, & (4.2) \end{cases}$$

with initial conditions

$$(u^m(0), u_t^m(0), \eta^{0,m}, u_t^m(t)) = (u_0^m, u_1^m, \eta_0^m, f_0^m).$$

By using standard existence theory in ODE, we know that the approximate problem has a local solution $(u^m(t), u_t^m(t), \eta^{t,m})$ on some small time interval $[0, T_m)$ with $0 < T_m \le T$ for every $m \in \mathbb{N}$. Next we extend the local solutions to the time interval $[0, T]$, for any given $T > 0$.

Step 2: A priori estimate. Multiplying the equation (4.1) by u_t^m, integrating the result over B_R and using the second equation (4.2), we can obtain

$$\frac{d}{dt} E^m(t) = -(\partial_s \eta^{t,m}, \eta^{t,m})_{\mathcal{M}_1} - \mu_1 \|u_t^m\|^2_{L^2_\rho} - \mu_2 \int_{B_R} \rho u_t^m(t) u_t^m(t-\tau(t))\, dx, \tag{4.3}$$

where

$$E^m(t) = \frac{1}{2}\|u_t^m(t)\|^2_{L^2_\rho} + \frac{1}{2}\|\nabla u^m(t)\|^2 + \frac{1}{2}\|\eta^{t,m}\|_{\mathcal{M}_1}. \tag{4.4}$$

Since $\eta^{t,m}(0) = 0$, then we have

$$(\partial_s \eta^{t,m}, \eta^{t,m})_{\mathcal{M}_1} = -\frac{1}{2}\int_0^\infty g'(s)\|\nabla \eta^{t,m}(s)\|^2\, ds,$$

which gives us

$$-(\partial_s \eta^{t,m}, \eta^{t,m})_{\mathcal{M}_1} = \frac{1}{2}\int_0^\infty g'(s)\|\nabla \eta^{t,m}(s)\|^2\, ds \le 0. \tag{4.5}$$

By using Young's inequality, we can get

$$-\mu_2 \int_{B_R} \rho u_t^m(t) u_t^m(t-\tau(t))\,dx \leq \frac{|\mu_2|}{2}\|u_t^m(t)\|_{L^2_\rho}^2 + \frac{|\mu_2|}{2}\int_{B_R} \rho(u_t^m(t-\tau(t)))^2\,dx. \tag{4.6}$$

Inserting (4.4)–(4.5) into (4.3), we arrive at

$$\frac{d}{dt}E^m(t) \leq \left(|\mu_1| + \frac{|\mu_2|}{2}\right)\|u_t^m\|_{L^2_\rho}^2 + \frac{|\mu_2|}{2}\int_{B_R} \rho(u_t^m(t-\tau(t)))^2\,dx,$$

which, integrating over $(0,t)$, gives us

$$E^m(t) \leq E^m(0) + \left(|\mu_1| + \frac{|\mu_2|}{2}\right)\int_0^t \|u_t^m\|_{L^2_\rho}^2\,ds + \frac{|\mu_2|}{2}\int_0^t\int_{B_R} \rho(u_t^m(s-\tau(t)))^2\,dx\,ds. \tag{4.7}$$

By using the history values of u_t^m, $t \in [\tau(0), 0)$, we shall see that

$$\begin{aligned}
\int_0^t\int_{B_R} \rho(u_t^m(s-\tau(t)))^2\,dx\,ds
&= \int_{B_R}\int_{-\tau(t)}^{t-\tau(t)} \rho(u_t^m(s))^2\,ds\,dx \\
&= \int_{B_R}\int_{-\tau(t)}^{0} \rho(u_t^m(s))^2\,ds\,dx + \int_{B_R}\int_{0}^{t-\tau(t)} \rho(u_t^m(s))^2\,ds\,dx \\
&= \int_{B_R}\int_{-\tau(t)}^{0} \rho(f_0^m(s))^2\,ds\,dx + \int_{B_R}\int_{0}^{t-\tau(t)} \rho(u_t^m(s))^2\,ds\,dx \\
&\leq \int_{B_R}\int_{-\tau(t)}^{0} \rho(f_0^m(s))^2\,ds\,dx + \int_{B_R}\int_{0}^{t} \rho(u_t^m(s))^2\,ds\,dx.
\end{aligned} \tag{4.8}$$

Thus it follows from (4.7)–(4.8) that

$$E^m(t) \leq E^m(0) + \left(\frac{1}{2} + |\mu_1| + |\mu_2|\right)\int_0^t \|u_t^m(s)\|_{L^2_\rho}^2\,ds + \frac{|\mu_2|}{2}\int_{B_R}\int_{-\tau(t)}^{0} \rho(f_0^m(s))^2\,ds\,dx. \tag{4.9}$$

Noting (4.4), then (4.9) gives us

$$E^m(t) \le E^m(0) + 2\Big(\frac{1}{2} + |\mu_1| + |\mu_2|\Big) \int_0^t E^m(s)\,ds + \frac{|\mu_2|}{2} \int_{B_R} \int_{-\tau(t)}^0 \rho(f_0^m(s))^2\,ds\,dx,$$

which, performing Grönwall's inequality, implies

$$E^m(t) \le \left(E^m(0) + \frac{|\mu_2|}{2} \int_{B_R} \int_{-\tau(t)}^0 \rho(f_0^m(s))^2\,ds\,dx\right) e^{2((1/2)+|\mu_1|+|\mu_2|)t}.$$

Then we infer that there exists a positive constant C independent on m such that for every $m \in \mathbb{N}$,

$$E^m(t) \le C,$$

which gives us

$$\|u_t^m(t)\|^2 + \|\nabla u^m(t)\|^2 + \|\eta^{t,m}\|^2 \le C. \tag{4.10}$$

Thus we can obtain $T_m = T$, for all $T > 0$.

Step 3: Passage to limit. It follows from (4.10) that

$$\begin{aligned} \{u^m\} &\text{ is bounded in } L^\infty(0,T;\mathcal{D}^{1,2}(B_R)),\\ \{u_t^m\} &\text{ is bounded in } L^\infty(0,T;L^2_\rho(B_R)),\\ \{\eta^{t,m}\} &\text{ is bounded in } L^\infty(0,T;\mathcal{M}_1(B_R)). \end{aligned}$$

Then there exists a subsequence of $\{u^m\}$, still denoted by $\{u^m\}$, such that

$$\begin{aligned} u^m &\rightharpoonup u \text{ weakly star in } L^\infty(0,T;\mathcal{D}^{1,2}(B_R)),\\ \{u_t^m\} &\rightharpoonup z(t) \text{ weakly star in } L^\infty(0,T;L^2_\rho(B_R)),\\ \{\eta^{t,m}\} &\rightharpoonup \eta^t \text{ weakly star in } L^\infty(0,T;\mathcal{M}_1(B_R)),\\ \{u_{tt}^m\} &\rightharpoonup \gamma(t) \text{ weakly in } L^2(0,T;\mathcal{D}^{-1,2}(B_R)). \end{aligned}$$

Noting that for all $v(x,t) \in C_0^\infty([0,T] \times B_R)$ and using integration by parts, we obtain

$$\int_0^T (u_t^m(s), v(s))_{L^2_\rho(B_R)}\,ds = -\int_0^T (u^m(s), v_t(s))_{L^2_\rho(B_R)}\,ds,$$

$$\int_0^T (u_{tt}^m(s), v(s))_{L^2_\rho(B_R)}\,ds = \int_0^T (u^m(s), v_{tt}(s))_{L^2_\rho(B_R)}\,ds.$$

Then we can get, as $m \to \infty$,

$$\int_0^T (z(s), v(s))_{L^2_\rho(B_R)}\, ds = -\int_0^T (u(s), v_t(s))_{L^2_\rho(B_R)}\, ds,$$
$$\int_0^T (\gamma(s), v(s))_{L^2_\rho(B_R)} = \int_0^T (u(s), v_{tt}(s))_{L^2_\rho(B_R)}\, ds,$$

which yields

$$u_t(t) = z(t) \quad \text{and} \quad u_{tt}(t) = \gamma(t).$$

By compactness of the embedding and Lemma 4(ii) of **[18]**, we conclude that

$$u^m \to u \text{ strongly in } L^2(0,T; L^2_\rho(B_R)).$$

Therefore, the above limits are enough to pass to the limit the approximate problem (4.1)–(4.2) and get the desired weak solution, i.e,

$$\begin{cases} (u_{tt}(t),\omega)_{L^2_\rho} + (\nabla u(t), \nabla\omega) + (\eta^t,\omega)_{\mathcal{M}_1} + \mu_1(u_t(t),\omega)_{L^2_\rho} \\ \qquad + \mu_2(u_t(t-\tau(t)),\omega)_{L^2_\rho} = 0, \quad \text{for } \omega \in \mathcal{D}^{1,2}(B_R), & (4.11) \\ (\partial_t \eta^t, \chi)_{\mathcal{M}_1} = -(\partial_s \eta^t, \chi)_{\mathcal{M}_1} + (u_t(t), \chi)_{\mathcal{M}_1}, \quad \text{for } \chi \in \mathcal{M}_1. & (4.12) \end{cases}$$

Step 4: Initial data. In this step we verify that the limit function u satisfies the initial conditions and history value. From Step 3, we know that

$$u^m \to u \text{ strongly in } C(0,T; L^2_\rho(B_R)),$$

and therefore $u(0) = u_0$.

In the sequel we define a test function $\theta(t)$ satisfying

$$\theta(t) \in H^1(0,T), \qquad \theta(0) = 1 \quad \text{and} \quad \theta(T) = 0.$$

Multiplying the approximate equation (4.1) by $\theta(t)$ and integrating the result over $[0,T]$, we get

$$\begin{aligned} &(u_1,\omega)_{L^2_\rho} - \int_0^T (u_t^m,\omega)_{L^2_\rho}\theta_t\, dt \\ &\quad + \int_0^T (\nabla u^m, \nabla\omega)\theta\, dt + \int_0^T (\eta^{t,m},\omega)_{\mathcal{M}_1}\theta\, dt + \int_0^T (\mu_1 u_t^m, \omega)_{L^2_\rho}\theta\, dt \\ &\quad + \int_0^T (\mu_2 u_t^m(t-\tau(t)),\omega)_{L^2_\rho}\theta\, dt = 0, \quad \text{for all } \omega \in \mathcal{D}^{1,2}(B_R). \end{aligned}$$

Taking the limit as $m \to \infty$, we have

$$(4.13) \quad \begin{aligned} &(u_1, \omega)_{L^2_\rho} - \int_0^T (u_t, \omega)_{L^2_\rho} \theta_t \, dt \\ &+ \int_0^T (\nabla u, \nabla \omega) \theta \, dt + \int_0^T (\eta^t, \omega)_{\mathcal{M}_1} \theta \, dt + \int_0^T (\mu_1 u_t, \omega)_{L^2_\rho} \theta \, dt \\ &+ \int_0^T (\mu_2 u_t(t - \tau(t)), \omega)_{L^2_\rho} \theta \, dt = 0, \quad \text{for all } \omega \in \mathcal{D}^{1,2}(B_R). \end{aligned}$$

On the other hand, we multiply (4.11) by $\theta(t)$ and integrate the result over $[0, T]$ to obtain

$$(4.14) \quad \begin{aligned} &(u_t(0), \omega)_{L^2_\rho} - \int_0^T (u_t, \omega)_{L^2_\rho} \theta_t \, dt \\ &+ \int_0^T (\nabla u, \nabla \omega) \theta \, dt + \int_0^T (\eta^t, \omega)_{\mathcal{M}_1} \theta \, dt + \int_0^T (\mu_1 u_t, \omega)_{L^2_\rho} \theta \, dt \\ &+ \int_0^T (\mu_2 u_t(t - \tau(t)), \omega)_{L^2_\rho} \theta \, dt = 0, \quad \text{for all } \omega \in \mathcal{D}^{1,2}(B_R). \end{aligned}$$

It follows from (4.13) and (4.14) that

$$u_t(0) = u_1.$$

By using the same argument as above, we can derive

$$\eta^0(s) = \eta_0.$$

In addition, for all j, as $m \to \infty$,

$$\int_{B_R} u_t^m(t - \tau(t)) \omega_j \, dx \rightharpoonup \int_{B_R} f_0(t) \omega_j \, dx \quad \text{weakly in } L^2(0, T; L^2_\rho(B_R)),$$

which gives us $u_t(x, t - \tau(t)) = f_0(x, t)$.

Combining the above four steps completes the proof. □

Proof of Theorem 3.2. Following the same arguments as in [**8**], we extend the solutions to the whole spaces $\mathbb{R}^n$. For this purpose, let $R_0 > 0$ such that $\mathrm{supp}(u_0) \subset B_{R_0}$, $\mathrm{supp}(u_1) \subset B_{R_0}$ and $\mathrm{supp}(f_0) \subset B_{R_0}$. Then for $R \geq R_0$, $\mathbb{R} \in N$, we investigate the following problem:

$$\begin{cases} \rho(x)u^R_{tt}(x,t) - \Delta u^R(x,t) + \displaystyle\int_{-\infty}^t g(t-s)\Delta u^R(s)\,ds + \mu_1\rho(x)u^R_t(x,t) \\ \qquad\qquad + \mu_2\rho(x)u^R_t(x,t-\tau(t)) = 0,\ (x,t)\in B_R\times(0,T), \\ u^R(x,t) = 0, \quad (x,t)\in\partial B_R\times(0,T), \\ u^R(x,0) = u_0(x)\in C_0^\infty(B_R), \quad u^R_t(x,0) = u_1(x)\in C_0^\infty(B_R), \\ u^R_t(x,t) = f_0(x,t)\in C_0^\infty(B_R\times(-\tau(0),0)). \end{cases}$$

By using Proposition 4.1, we know the above approximating problem has a weak solution u^R such that

$$u^R\in C(0,T;\mathcal{D}^{1,2}(B_R)), \qquad u^R_t\in C(0,T;L^2_\rho(B_R)).$$

We extend the solution of the problem as

$$\tilde{u}^R(x,t) := \begin{cases} u^R(x,t), & \text{if } |x|\le R, \\ 0, & \text{otherwise.} \end{cases}$$

Following the same arguments as in Step 2 (see (4.10)), we can conclude that there exists a constant $C>0$ independent of R such that

$$\|\tilde{u}^R\|_{L^\infty(0,T;\mathcal{D}^{1,2}(\mathbb{R}^n))} \le C, \qquad |\tilde{u}^R_t\|_{L^\infty(0,T;L^2_\rho(\mathbb{R}^n))} \le C, \tag{4.15}$$

and

$$\|\tilde{u}^R_{tt}\|_{L^\infty(0,T;\mathcal{D}^{-1,2}(\mathbb{R}^n))} \le C, \tag{4.16}$$

which gives us the family of functions $\tilde{u}^R$ satisfies

$$\{\tilde{u}^R\}_R \text{ is relatively compact in } C(0,T;L^2_\rho(\mathbb{R}^n)).$$

Since the embedding $C(0,T;L^2_\rho(\mathbb{R}^n))\hookrightarrow L^2(0,T;L^2_\rho(\mathbb{R}^n))$ is continuous, we may extract a subsequence of $\tilde{u}^R$, denoted by $\tilde{u}^{R_m}$, such that as $R_m\to\infty$, we can get

$$\begin{aligned} \tilde{u}^{R_m} &\rightharpoonup \tilde{u} \text{ weakly star in } L^\infty(0,T;\mathcal{D}^{1,2}(\mathbb{R}^n)), \\ \{\tilde{u}^{R_m}_t\} &\rightharpoonup z(t) \text{ weakly star in } L^\infty(0,T;L^2_\rho(\mathbb{R}^n)), \\ \{\tilde{u}^{R_m}_{tt}\} &\rightharpoonup \gamma(t) \text{ weakly in } L^2(0,T;\mathcal{D}^{-1,2}(\mathbb{R}^n)). \end{aligned}$$

For any fixed $R=R_m$, we let L_m denote the operator satisfying

$$L_m : [0,T]\times\mathbb{R}^n \to [0,T]\times B_R.$$

Clearly the subsequence $L_m\tilde{u}^{R_m}$ satisfies (4.15) and (4.16). Then there exists a subsequence $\tilde{u}^{R_{m_j}}\equiv\tilde{u}^j$ such that $L_m\tilde{u}^j$ converges weakly to a weak solution $\tilde{u}_m$. Therefore we can get, for every $v \in C_0^\infty([0,T]\times B_R)$,

$$\begin{aligned}(4.17)\quad &\int_0^T (L_m\tilde{u}^j_{tt}, v)_{L^2_\rho(B_R)}\,dt + \int_0^T\int_{B_R} \nabla L_m\tilde{u}^j \nabla v\,dx\,dt \\ &\quad - \int_0^T \left(\int_{-\infty}^t g(t-s)\nabla L_m\tilde{u}^j(s), \nabla v\right) dt \\ &\quad + \int_0^T (\mu_1 L_m\tilde{u}^j_t, v)_{L^2_\rho(B_R)} dt + \int_0^T (\mu_2 L_m\tilde{u}^j_t(t-\tau(t)), v)_{L^2_\rho(B_R)} dt \\ &= \int_0^T (\tilde{u}^j_{tt}, v)_{L^2_\rho(B_R)}\,dt + \int_0^T (\nabla\tilde{u}^j, \nabla v)\,dt \\ &\quad - \int_0^T \left(\int_{-\infty}^t g(t-s)\nabla\tilde{u}^j(s), \nabla v\right) dt + \int_0^T (\mu_1\tilde{u}^j_t, v)_{L^2_\rho(B_R)} dt \\ &\quad + \int_0^T (\mu_2\tilde{u}^j_t(t-\tau(t)), v)_{L^2_\rho(B_R)}\,dt.\end{aligned}$$

Taking the limit as $j\to\infty$ in (4.17), we have $L_m\tilde{u} = \tilde{u}_m$. Since the radius R is arbitrarily chosen, we know that the equality (4.17) holds for any $v \in C_0^\infty([0,T]\times\mathbb{R}^n)$. Then $\tilde{u}$ is the weak solution of problem (1.7)–(1.11).

To prove the continuous dependence of solutions of problem (1.7)–(1.11), we assume that $U_1(t) = (u, u_t, \eta)$ and $U_1(t) = (\tilde{u}, \tilde{u}_t, \tilde{\eta})$ are two weak solutions with initial data $U_1(0) = (u_0, u_1, \eta_0)$, $u_t(x,t) = f_0(x,t)$ and $U_1(0) = (\tilde{u}_0, \tilde{u}_1, \tilde{\eta}_0)$, $\tilde{u}_t(x,t) = \tilde{f}_0(x,t)$, respectively. Let $\omega = u - \tilde{u}$ and $\chi = \eta - \tilde{\eta}$. Then the function (ω, ω_t, χ) satisfies the problem

$$\begin{cases} \rho\omega_{tt}(t) - \Delta\omega(t) + \int_0^\infty g(s)\Delta\chi(s)\,ds \\ \qquad\qquad + \mu_1\rho\omega_t(t) + \mu_2\rho\omega(t-\tau(t)) = 0, & (4.18) \\ \chi_t + \chi_s = \omega_t, & (4.19) \end{cases}$$

with initial conditions

$$(4.20)\quad (\omega(0), \omega_t(0), \chi^0) = (u_0 - \tilde{u}_0, u_1 - \tilde{u}_1, \eta_0 - \tilde{\eta}_0) = U_1(0) - U_2(0),$$

and

$$(4.21)\qquad \omega_t(x,t) = g_0(x,t) = f_0(x,t) - \tilde{f}_0(x,t).$$

The energy functional corresponding to problem (4.18)–(4.21) is defined by

$$W(t) = \frac{1}{2}\|\omega_t\|^2_{L^2_\rho} + \frac{1}{2}\|\nabla\omega\| + \frac{1}{2}\|\chi\|^2_{\mathcal{M}_1}.$$

Multiplying (4.18) by ω_t in V_0 and (4.19) by χ in $\mathcal{M}_1$, and integrating by parts, we can get

$$\frac{d}{dt}W(t) = -(\partial_s\chi^t, \chi^t)_{\mathcal{M}_1} - \mu_1\|\omega_t\|^2_{L^2_\rho} - \mu_2\int_{\mathbb{R}^n}\rho\omega_t(t)\omega_t(t-\tau(t))\,dx.$$

By using the same estimate as (4.8)–(4.9) in Step 2, we can arrive at

$$\frac{d}{dt}W(t) \le \Big(\frac{1}{2} + |\mu_1| + |\mu_2|\Big)\int_{\mathbb{R}^n}\rho\omega_t^2(x,t)\,dx + \frac{|\mu_2|}{2}\int_{\mathbb{R}^n}\rho g_0^2\,dx$$

and

$$W(t) \le W(0) + 2\Big(\frac{1}{2} + |\mu_1| + |\mu_2|\Big)\int_0^t W(s)\,ds \\ + \frac{|\mu_2|}{2}\int_{\mathbb{R}^n}\int_{-\tau(t)}^0 \rho g_0^2\,ds\,dx,$$

which, applying Grönwall's inequality, implies

$$W(t) \le \bigg(W(0) + \frac{|\mu_2|}{2}\int_{\mathbb{R}^n}\int_{-\tau(t)}^0\rho g_0^2\,ds\,dx\bigg)e^{2(\frac{1}{2}+|\mu_1|+|\mu_2|)t}.$$

From the definition of $W(t)$, we infer that for all $t \in [0,T]$,

$$\|u_t - \tilde{u}_t\|^2_{L^2_\rho} + \|\nabla u - \nabla\tilde{u}\|^2 + \|\eta - \tilde{\eta}\|^2_{\mathcal{M}_1} \\ \le e^{CT}\big(\|u_1 - \tilde{u}_1\|^2_{L^2_\rho} + \|\nabla u_0 - \nabla\tilde{u}_0\|^2 \\ + \|\eta - \tilde{\eta}\|^2_{\mathcal{M}_1} + \|f_0 - \tilde{f}_0\|^2_{L^2(\mathbb{R}^n\times(-\tau(0),0))}\big),$$

which gives us that solutions of problem (1.7)–(1.11) depend continuously on the initial data. In particular, the solution of problem (1.7)–(1.11) is unique. Then we complete the proof of Theorem 3.2. □

5. Exponential decay. To prove the exponential decay of energy for problem (1.7)–(1.11), we need the following lemmas.

Lemma 5.1. *Under the assumptions of Theorem 3.3, the energy functional defined by* (3.3) *satisfies for any* $t \geq 0$

$$E'(t) \leq \left(\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} \right) \|u_t(t)\|^2_{L^2_\rho} + \frac{1}{2} \int_0^\infty g'(s) \|\nabla \eta^t\|^2 \, ds + \left(\frac{|\mu_2|}{2} \sqrt{1-d} - \frac{\xi}{2} e^{-\lambda \tau_1} (1-d) \right) \int_{\mathbb{R}^n} \rho(x) u_t^2(t-\tau(t)) \, dx - \frac{\xi\lambda}{2} \int_{t-\tau(t)}^t \int_{\mathbb{R}^n} \rho(x) e^{-\lambda(t-s)} u_t^2(x,s) \, dx \, ds. \tag{5.1}$$

Proof. Clearly,

$$E'(t) = \int_{\mathbb{R}^n} \rho(x) u_{tt} u_t \, dx + \int_{\mathbb{R}^n} \nabla u \cdot \nabla u_t \, dx + \int_{\mathbb{R}^n} \int_0^\infty g(s) \nabla \eta^t \cdot \nabla \eta^t_t \, ds \, dx + \frac{\xi}{2} \|u_t\|^2_{L^2_\rho} - \frac{\xi}{2} e^{-\lambda \tau(t)} \int_{\mathbb{R}^n} \rho(x) u_t^2(t-\tau(t))(1-\tau'(t)) \, dx - \frac{\lambda\xi}{2} \int_{t-\tau(t)}^t \int_{\mathbb{R}^n} \rho(x) e^{-\lambda(t-s)} u_t^2(x,s) \, dx \, ds. \tag{5.2}$$

By using (1.7)–(1.8), (2.3)–(2.5) and integration by parts, we can derive

$$\begin{aligned} E'(t) = & -\mu_1 \|u_t\|^2_{L^2_\rho} - \mu_2 \int_{\mathbb{R}^n} \rho u_t(t) \cdot u_t(t-\tau(t)) \, dx - (\eta^t_s, \eta)_{\mathcal{M}_1} \\ & + \frac{\xi}{2} \|u_t\|^2_{L^2_\rho} - \frac{\xi}{2} e^{-\lambda \tau(t)} \int_{\mathbb{R}^n} \rho(x) u_t^2(t-\tau(t))(1-\tau'(t)) \, dx \\ & - \frac{\lambda\xi}{2} \int_{t-\tau(t)}^t \int_{\mathbb{R}^n} \rho(x) e^{-\lambda(t-s)} u_t^2(x,s) \, dx \, ds \\ \leq & -\mu_1 \|u_t\|^2_{L^2_\rho} - \mu_2 \int_{\mathbb{R}^n} \rho u_t(t) \cdot u_t(t-\tau(t)) \, dx - (\eta^t_s, \eta)_{\mathcal{M}_1} \\ & + \frac{\xi}{2} \|u_t\|^2_{L^2_\rho} - \frac{\xi}{2} (1-d) e^{-\lambda \tau_1} \int_{\mathbb{R}^n} \rho(x) u_t^2(t-\tau(t)) \, dx \\ & - \frac{\lambda\xi}{2} \int_{t-\tau(t)}^t \int_{\mathbb{R}^n} \rho(x) e^{-\lambda(t-s)} u_t^2(x,s) \, dx \, ds. \end{aligned} \tag{5.3}$$

Noting $\eta^t(0)=0$, we can obtain

$$(\partial_s\eta^t,\eta^t)_{\mathcal{M}_1} = -\frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta^t(s)\|^2\,ds. \tag{5.4}$$

It follows from Young's inequality that

$$\begin{aligned} -\mu_2\int_{\mathbb{R}^n}\rho u_t(t)\cdot u_t(t-\tau(t))\,dx \\ \le \frac{|\mu_2|}{2\sqrt{1-d}}\|u_t\|^2_{L^2_\rho} + \frac{|\mu_2|}{2}\sqrt{1-d}\int_{\mathbb{R}^n}\rho u_t^2(t-\tau(t))\,dx. \end{aligned} \tag{5.5}$$

Replacing (5.4)–(5.5) in (5.3), we can get (5.1). This ends the proof of this lemma. □

Lemma 5.2. *Under the assumptions of Theorem 3.3, let* (u,u_t,η^t) *be the solution of problem* (1.7)–(1.11)*. The functional* $\Phi(t)$ *defined by*

$$\Phi(t) = \int_{\mathbb{R}^n}\rho(x)u(t)u_t(t)\,dx \tag{5.6}$$

satisfies that there exist three positive constants c_1, c_2 *and* c_3 *such that for any* $t>0$,

$$\begin{aligned} \Phi'(t) \le -\frac{l}{2}\|\nabla u\|^2 + c_1\|u_t\|^2_{L^2_\rho} \\ + c_2\int_{\mathbb{R}^n}\rho u_t^2(t-\tau(t))\,dx + c_3\|\eta^t\|^2_{\mathcal{M}_1}. \end{aligned} \tag{5.7}$$

Proof. We take the derivative of $\Phi(t)$ and use equation (1.7) to obtain

$$\begin{aligned} \Phi'(t) &= \int_{\mathbb{R}^n}\rho u_t^2\,dx + \int_{\mathbb{R}^n}\rho u_{tt}u\,dx \\ &= \|u_t\|^2_{L^2_\rho} + \int_{\mathbb{R}^n}\left(\Delta u + \int_0^\infty g(s)\Delta\eta^t(s)\,ds\right)u\,dx \\ &\quad + \int_{\mathbb{R}^n}\rho(-\mu_1u_t - \mu_2u_t(t-\tau(t)))u\,dx \\ &= \|u_t\|^2_{L^2_\rho} - \|\nabla u\|^2 - \int_{\mathbb{R}^n}\nabla u(t)\int_0^\infty g(s)\nabla\eta^t(s)\,ds\,dx \\ &\quad - \mu_1\int_{\mathbb{R}^n}\rho uu_t\,dx - \mu_2\int_{\mathbb{R}^n}\rho uu_t(t-\tau(t))\,dx. \end{aligned} \tag{5.8}$$

From Hölder's inequality, Young's inequality and (2.1), we infer that for any $\epsilon > 0$,

$$-\int_{\mathbb{R}^n} \nabla u(t) \int_0^\infty g(s)\nabla\eta^t(s)\,ds\,dx \le \epsilon\|\nabla u\|^2 + \frac{l_0}{4\epsilon}\|\eta^t\|^2_{\mathcal{M}_1}, \tag{5.9}$$

$$\begin{aligned} -\mu_1 \int_{\mathbb{R}^n} \rho u u_t\,dx &\le |\mu_1|\epsilon\|u\|^2_{L^2_\rho} + \frac{|\mu_1|}{4\epsilon}\|u_t\|^2_{L^2_\rho} \\ &\le |\mu_1|\epsilon c_*^2\|\nabla u\|^2 + \frac{|\mu_1|}{4\epsilon}\|u_t\|^2_{L^2_\rho}, \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} -\mu_2 \int_{\mathbb{R}^n} \rho u u_t(t-\tau(t))\,dx \le{}& |\mu_2|\epsilon c_*^2\|\nabla u\|^2 \\ &+ \frac{|\mu_2|}{4\epsilon}\int_{\mathbb{R}^n} \rho u_t^2(t-\tau(t))\,dx. \end{aligned} \tag{5.11}$$

Combining (5.9)–(5.11) into (5.8) and choosing $\epsilon > 0$ small enough, we can get (5.7) with

$$c_1 := 1 + \frac{|\mu_1|}{4\epsilon}, \quad c_2 := \frac{|\mu_2|}{4\epsilon}, \quad c_3 := \frac{l_0}{4\epsilon}.$$

The proof is complete. □

Lemma 5.3. *Under the assumptions of Theorem 3.3, let* (u, u_t, η^t) *be the solution of problem* (1.7)–(1.11). *The functional* $\Psi(t)$ *defined by*

$$\Psi(t) = -\int_{\mathbb{R}^n} \rho u_t(t)\cdot\left(\int_0^\infty g(s)\eta^t(s)\,ds\right)dx \tag{5.12}$$

satisfies that there exists a positive constant c_4 *such that for any* $\delta > 0$,

$$\begin{aligned} \Psi'(t) \le -\left(\frac{l_0}{2} - 2\delta\right)\|u_t\|^2_{L^2_\rho} + \delta\|\nabla u\|^2 + \delta\int_{\mathbb{R}^n} \rho u_t^2(t-\tau(t))\,dx \\ - c_4\int_0^\infty g'(s)\|\nabla\eta^t(s)\|^2\,ds. \end{aligned} \tag{5.13}$$

Proof. Taking the derivative of $\Psi(t)$ and using (1.8), we have

$$\begin{aligned}\Psi'(t) &= -\int_{\mathbb{R}^n} \rho u_{tt} \cdot \left(\int_0^\infty g(s)\eta^t(s)\,ds\right) dx \\ &\quad - \int_{\mathbb{R}^n} \rho u_t \cdot \left(\int_0^\infty g(s)\eta^t_t(s)\,ds\right) dx \\ &= \underbrace{-\int_{\mathbb{R}^n} \rho u_{tt} \cdot \left(\int_0^\infty g(s)\eta^t(s)\,ds\right) dx}_{:=I_1} \\ &\quad \underbrace{+\int_{\mathbb{R}^n} \rho u_t \cdot \left(\int_0^\infty g(s)\eta^t_s(s)\,ds\right) dx}_{:=I_2} - l_0\|u_t\|^2_{L^2_\rho}.\end{aligned} \tag{5.14}$$

It follows from (1.7) that

$$\begin{aligned}I_1 = \int_{\mathbb{R}^n} \Big(&-\Delta u(t) - \int_0^\infty g(s)\Delta\eta^t(s)\,ds + \mu_1\rho u_t(t) \\ &+ \mu_2\rho u_t(t-\tau(t))\Big) \cdot \left(\int_0^\infty g(s)\eta^t(s)\,ds\right) dx.\end{aligned} \tag{5.15}$$

By using integration by parts, Young's inequality and Hölder's inequality, we have for any $\delta > 0$,

$$-\int_{\mathbb{R}^n} \Delta u(t) \cdot \left(\int_0^\infty g(s)\eta^t(s)\,ds\right) dx \le \delta\|\nabla u\|^2 + \frac{l_0}{4\delta}\|\eta^t\|^2_{\mathcal{M}_1}, \tag{5.16}$$

$$\begin{aligned}&-\int_{\mathbb{R}^n} \left(\int_0^\infty g(s)\Delta\eta^t(s)\,ds\right) \cdot \left(\int_0^\infty g(s)\eta^t(s)\,ds\right) dx \\ &= \int_{\mathbb{R}^n} \sum_{j=1}^n \left(\int_0^\infty g(s)\frac{\partial\eta^t}{\partial x_j}\right)^2 dx \\ &\le \int_{\mathbb{R}^n} \sum_{j=1}^n l_0 \left(\int_0^\infty g(s)\left|\frac{\partial\eta^t}{\partial x_j}\right|^2\right) dx \\ &\le l_0\|\eta^t\|^2_{\mathcal{M}_1},\end{aligned} \tag{5.17}$$

$$
\begin{aligned}
(5.18)\qquad & \mu_1 \int_{\mathbb{R}^n} \rho u_t(t) \cdot \left(\int_0^\infty g(s)\eta^t(s)\,ds \right) dx \\
&\leq \delta \|u_t\|_{L^2_\rho}^2 + \frac{l_0}{4\delta} \int_0^\infty g(s) \|\eta^t(s)\|_{L^2_\rho}^2\,ds \\
&\leq \delta \|u_t\|_{L^2_\rho}^2 + \frac{l_0 c_*^2}{4\delta} \int_0^\infty g(s) \|\eta^t(s)\|_{V_1}^2\,ds, \\
& \mu_2 \int_{\mathbb{R}^n} \rho u_t(t-\tau(t)) \cdot \left(\int_0^\infty g(s)\eta^t(s)\,ds \right) dx \\
&\leq \delta \int_{\mathbb{R}^n} \rho u_t^2(t-\tau(t))\,dx + \frac{l_0 c_*^2}{4\delta} \|\eta^t\|_{\mathcal{M}_1}^2,
\end{aligned}
$$

which, together with (5.15)–(5.18), gives us

$$
\begin{aligned}
(5.19)\quad I_1 \leq 2\delta \|u_t\|_{L^2_\rho}^2 + \delta \|\nabla u\|^2 + \delta \int_{\mathbb{R}^n} \rho u_t^2(t-\tau(t))\,dx \\
+ \left(l_0 + \frac{l_0}{4\delta} + \frac{l_0 c_*^2}{2\delta} \right) \|\eta^t\|_{\mathcal{M}_1}^2.
\end{aligned}
$$

Similarly, we can get

$$
\begin{aligned}
(5.20)\qquad I_2 &= -\int_{\mathbb{R}^n} \rho u_t(t) \cdot \left(\int_0^\infty g'(s)\eta^t(s)\,ds \right) dx \\
&\leq \frac{l_0}{2} \|u_t\|_{L^2_\rho}^2 - \frac{l_0'}{2l_0} \int_0^\infty g'(s) \|\eta^t(s)\|_{L^2_\rho}^2\,ds \\
&\leq \frac{l_0}{2} \|u_t\|_{L^2_\rho}^2 - \frac{l_0' c_*^2}{2l_0} \int_0^\infty g'(s) \|\eta^t(s)\|_{V_1}^2\,ds,
\end{aligned}
$$

where

$$
l_0' = -\int_0^\infty g'(s)\,ds.
$$

Inserting (5.19)–(5.20) into (5.14) and using (2.2), we can obtain (5.13) with

$$
c_4 := \frac{l_0}{4\delta k} + \frac{l_0}{k} + \frac{l_0 c_*^2}{2\delta k} + \frac{l_0' c_*^2}{2l_0}.
$$

The proof is done. □

In the sequel we define the Lyapunov functional

$$
(5.21)\qquad \mathcal{L}(t) := E(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \Psi(t),
$$

where ε_1 and ε_2 are positive constants to be chosen later. Then we can get the following lemma.

Lemma 5.4. *For $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough, the Lyapunov functional $\mathcal{L}(t)$ defined in* (5.21) *satisfies, for any $t > 0$,*

$$\frac{1}{2}E(t) \le \mathcal{L}(t) \le 2E(t). \tag{5.22}$$

Proof. It follows from Hölder's inequality, Young's inequality and (2.1) that for any $\epsilon > 0$,

$$\begin{aligned}
&|\mathcal{L}(t) - E(t)| \\
&\quad \le \varepsilon_1 \int_{\mathbb{R}^n} |\rho u u_t|\, dx + \varepsilon_2 \int_{\mathbb{R}^n} \left| \rho u_t \int_0^\infty g(s)\eta^t(s)\, ds \right| dx \\
&\quad \le \varepsilon_1 \Big(\epsilon \|u_t\|_{L^2_\rho}^2 + \frac{1}{4\epsilon}\|u\|_{L^2_\rho}^2 \Big) \\
&\qquad + \varepsilon_2 \left(\epsilon \|u_t\|_{L^2_\rho}^2 + \frac{l_0}{4\epsilon} \int_0^\infty g(s) \|\eta^t(s)\|_{L^2_\rho}^2\, ds \right) \\
&\quad \le \varepsilon_1 \left(\epsilon \|u_t\|_{L^2_\rho}^2 + \frac{c_*^2}{4\epsilon} \|\nabla u\|^2 \right) + \varepsilon_2 \left(\epsilon \|u_t\|_{L^2_\rho}^2 + \frac{l_0 c_*^2}{4\epsilon} \|\eta^t\|_{\mathcal{M}_1} \right) \\
&\quad \le \epsilon(\varepsilon_1 + \varepsilon_2)\|u_t\|_{L^2_\rho}^2 + \frac{\varepsilon_1 c_*^2}{4\epsilon}\|\nabla u\|^2 + \frac{\varepsilon_2 l_0 c_*^2}{4\epsilon}\|\eta^t\|_{\mathcal{M}_1},
\end{aligned}$$

which yields that there exists a positive constant $\varepsilon > 0$ such that

$$|\mathcal{L}(t) - E(t)| \le \varepsilon E(t),$$

i.e.,

$$(1 - \varepsilon)E(t) \le \mathcal{L}(t) \le (1 + \varepsilon)E(t). \tag{5.23}$$

Note that $\varepsilon > 0$ is small enough when ε_1 and ε_2 are small enough. Then we can get (5.22) if we take $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough. The proof is therefore complete. □

Proof of Theorem 3.3. We distinguish the following two cases to prove the theorem.

<u>Case 1:</u> $\mu_1 \neq 0$, $0 < |\mu_2| < \sqrt{1-d}\mu_1$.

By using (5.1), (5.7) and (5.13), we can obtain that for any $t \geq 0$,

$$
\begin{aligned}
\mathcal{L}'(t) &= E'(t) + \varepsilon_1 \Phi'(t) + \varepsilon_2 \Psi'(t) \\
&\leq \left[\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} + c_1 \varepsilon_1 + \varepsilon_2 \Big(2\delta - \frac{l_0}{2} \Big) \right] \|u_t\|_{L^2_\rho}^2 \\
&\quad + \Big(\varepsilon_2 \delta - \frac{\varepsilon_1}{2} \Big) \|\nabla u\|^2 + \bigg(\frac{|\mu_2|}{2} \sqrt{1-d} - \frac{\xi}{2} e^{-\lambda \tau_1} (1-d) \\
&\qquad + c_2 \varepsilon_1 + \varepsilon_2 \delta \bigg) \int_{\mathbb{R}^n} \rho u_t^2 (t - \tau(t)) \, dx \\
&\quad + \Big(\frac{1}{2} - \frac{c_3 \varepsilon_1}{k} - c_4 \varepsilon_2 \Big) \int_0^\infty g'(s) \|\nabla \eta^t(s)\|^2 \, ds.
\end{aligned} \tag{5.24}
$$

Clearly, $e^{\lambda \tau_1}$ goes to 1 as $\lambda \to 0$. By using the continuity of the set of real numbers, then we choose $\lambda < \frac{1}{\tau_1} |\log(|\mu_2|/\sqrt{1-d})|$ small enough so that there exists a constant $\xi > 0$ such that

$$
\frac{e^{\lambda \tau_1} |\mu_2|}{\sqrt{1-d}} < \xi < \mu_1. \tag{5.25}
$$

It follows from (5.25) that

$$
\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} < 0 \tag{5.26}
$$

and

$$
\frac{|\mu_2|}{2} \sqrt{1-d} - \frac{\xi}{2 e^{\lambda \tau_1}} (1-d) < 0. \tag{5.27}
$$

At this point we first choose $\delta > 0$ small enough so that

$$
\delta < \min \left\{ \frac{l_0}{8}, \frac{l_0}{8 c_1} \right\},
$$

which implies

$$
\frac{l_0}{2} - 2\delta > \frac{l_0}{4}, \qquad \frac{l_0}{4 c_1} > 2\delta.
$$

For any fixed $\delta > 0$, we take $\varepsilon_2 > 0$ and $\varepsilon_1 > 0$ satisfying

$$
2\delta \varepsilon_2 < \varepsilon_1 < \frac{l_0}{4 c_1} \varepsilon_2,
$$

so small that (5.22) holds, and further,

$$
\frac{1}{2} - \frac{c_3 \varepsilon_1}{k} - c_4 \varepsilon_2 > 0
$$

and
$$\frac{|\mu_2|}{2}\sqrt{1-d}-\frac{\xi}{2}e^{-\lambda\tau_1}(1-d)+c_2\varepsilon_1+\varepsilon_2\delta<0.$$

From above and (2.2) and (5.22), we infer that there exists a positive constant γ such that for any $t>0$,
$$\mathcal{L}'(t)\le -\gamma E(t)\le -\frac{\gamma}{2}\mathcal{L}(t),$$
which gives us
$$\mathcal{L}(t)\le \mathcal{L}(0)e^{-\frac{\gamma}{2}t}.$$

By using (5.22) again, we can obtain
$$E(t)\le 4E(0)e^{-\frac{\gamma}{2}t}. \tag{5.28}$$

Therefore, (3.4) follows by renaming the constants in (5.28).

Case 2: $\tau(t)=\tau>0$, $\mu_1=0$ and $|\mu_2|>0$.

Since $\mu_1=0$ and $\tau(t)=\tau$, by (5.1), the energy $E(t)$ satisfies
$$\begin{aligned} E'(t)\le{}&\left(\frac{|\mu_2|}{2}+\frac{\xi}{2}\right)\|u_t(t)\|^2_{L^2_\rho}+\frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta^t\|^2\,ds\\ &+\left(\frac{|\mu_2|}{2}-\frac{\xi}{2}e^{-\lambda\tau}\right)\int_{\mathbb{R}^n}\rho(x)u_t^2(t-\tau(t))\,dx. \end{aligned}\tag{5.29}$$

By using the same estimate as (5.24), we can obtain, for any $t\ge 0$,
$$\begin{aligned} \mathcal{L}'(t)={}&E'(t)+\varepsilon_1\Phi'(t)+\varepsilon_2\Psi'(t)\\ \le{}&\left[\frac{|\mu_2|}{2}+\frac{\xi}{2}+c_5\varepsilon_1+\varepsilon_2\left(2\delta-\frac{l_0}{2}\right)\right]\|u_t\|^2_{L^2_\rho}\\ &+\left(\varepsilon_2\delta-\frac{\varepsilon_1}{2}\right)\|\nabla u\|^2\\ &+\left(\frac{|\mu_2|}{2}-\frac{\xi}{2}e^{-\lambda\tau}+c_5\varepsilon_1+\varepsilon_2\delta\right)\int_{\mathbb{R}^n}\rho u_t^2(t-\tau(t))\,dx\\ &+\left(\frac{1}{2}-\frac{c_6\varepsilon_1}{k}-c_7\varepsilon_2\right)\int_0^\infty g'(s)\|\nabla\eta^t(s)\|^2\,ds, \end{aligned}\tag{5.30}$$
where the c_i, $i=5,6,7$, are positive constants.

At this point we take $\delta>0$ small enough so that
$$\delta<\min\left\{\frac{l_0}{6},\frac{l_0}{8c_5+6}\right\}.$$

Then we can get

$$\frac{l_0}{2} - 3\delta > 0, \qquad \frac{l_0 - 6\delta}{4c_5} > 2\delta.$$

For any fixed $\delta > 0$, we take $\varepsilon_2 > 0$ small enough so that

$$\varepsilon_2 < \min\left\{\frac{1}{4c_7}, \frac{k}{8c_6\delta}\right\},$$

which implies

$$\frac{1}{2} - c_7\varepsilon_2 > \frac{1}{4}, \qquad \frac{k}{4c_6} > 2\varepsilon_2\delta.$$

And then we take $\varepsilon_1 > 0$ satisfying

$$2\varepsilon_2\delta < \varepsilon_1 < \min\left\{\frac{l_0 - 6\delta}{4c_5}\varepsilon_2, \frac{k}{4c_6}\right\}.$$

We can obtain

$$\frac{1}{4} - \frac{c_6\varepsilon_1}{k} > 0, \qquad \varepsilon_2\delta - \frac{\varepsilon_1}{2} < 0,$$

and

$$\varepsilon_2\left(\frac{l_0}{2} - 2\delta\right) - c_5\varepsilon_1 > c_5\varepsilon_1 + \varepsilon_2\delta.$$

If we denote

$$\eta_1 = \varepsilon_2\left(\frac{l_0}{2} - 2\delta\right) - c_5\varepsilon_1 \quad \text{and} \quad \eta_2 = c_5\varepsilon_1 + \varepsilon_2\delta,$$

we know that $\eta_1 > \eta_2$.

Note that $e^{\lambda\tau_1}$ goes to 1 as $\lambda \to 0$. Now we take λ small enough so that there exists a positive constant ξ which satisfies

$$2\eta_2 e^{\lambda\tau} < \xi < 2\eta_1.$$

Then we can get

$$2\eta_1 - \xi > 0 \quad \text{and} \quad \frac{\xi}{e^{\sigma\tau}} - 2\eta_2 > 0.$$

We choose the constant μ_2 to satisfy

$$|\mu_2| < \min\left\{2\eta_1 - \xi, \frac{\xi}{e^{\lambda\tau}} - 2\eta_2\right\} := a, \tag{5.31}$$

which implies

$$\frac{|\mu_2|}{2} + \frac{\xi}{2} < \eta_1 \quad \text{and} \quad \frac{\xi}{2e^{\lambda\tau}} - \frac{|\mu_2|}{2} > \eta_2.$$

From above it follows that

$$\frac{|\mu_2|}{2}+\frac{\xi}{2}+c_5\varepsilon_1+\left(2\delta-\frac{l_0}{2}\right)\varepsilon_2<0,\qquad \varepsilon_2\delta-\frac{\varepsilon_1}{2}<0,$$

and

$$\frac{1}{2}-\frac{c_6\varepsilon_1}{k}-c_7\varepsilon_2>0,\qquad \frac{|\mu_2|}{2}-\frac{\xi}{2}e^{-\lambda\tau}+c_5\varepsilon_1+\varepsilon_2\delta<0.$$

Then we derive that there exists a positive constant γ such that for any $t>0$,

$$\mathcal{L}'(t)\leq-\gamma E(t)\leq-\frac{\gamma}{2}\mathcal{L}(t),$$

which implies

$$\mathcal{L}(t)\leq\mathcal{L}(0)e^{-\frac{\gamma}{2}t}.$$

By using (5.22) again, we can get

$$E(t)\leq 4E(0)e^{-\frac{\gamma}{2}t}. \tag{5.32}$$

Therefore, (3.4) follows by renaming the constants in (5.32). Thus the proof of Theorem 3.3 is complete. □

6. Conclusion. In the present work, we study a wave equation with density, infinite memory and time-varying delay in the whole space $\mathbb{R}^n$, $n\geq 3$. In order to overcome the difficulties in the non-compactness of some operators in unbounded domain, we consider some weighted spaces introduced by Karachalios and Stavrakakis [**8**]. This study contains the results of global well-posedness and energy decay of the Cauchy problem. We use the classical Faedo-Galerkin approximation to prove the well-posedness. We firstly prove the well-posedness of the problem in a ball with a radius of R, and then extend the solutions to the whole spaces $\mathbb{R}^n$, $n\geq 3$. At last we establish an exponential decay of the energy in the case $0<|\mu_2|<\sqrt{1-d}\,\mu_1$ and in the case $\tau(t)=\tau$, $\mu_1=0$, $0<|\mu_2|<a$, where the constants $a>0$ are defined in (5.31). Our results extend some recent works. But for the stability result, it is only valid for $0<|\mu_2|<\sqrt{1-d}\,\mu_1$ and $\mu_1=0$, $|\mu_2|>0$. Whether the stability property holds for $0<\sqrt{1-d}\,\mu_1=|\mu_2|$ is still open.

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