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SUPERCONVERGENT PRODUCT INTEGRATION METHOD FOR HAMMERSTEIN INTEGRAL EQUATIONS

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ABSTRACT. In this paper, we define a superconvergent projection method for approximating the solution of *Hammerstein* integral equations of the second kind. The projection is chosen either to be the orthogonal or an interpolatory projection at *Gauss* points onto the space of discontinuous piecewise polynomials. For a smooth kernel or *one* less smooth along the diagonal, the order of convergence of the proposed method improves upon the classical product integration method. Several numerical examples are given to demonstrate the effectiveness of the current method.

1. Introduction. Many problems that arise in the mathematical physics, engineering, biology, economics, etc., lead to mathematical models described by nonlinear integral equations [**1, 12, 29**]. For instance, *Hammerstein* integral equations appear in nonlinear physical phenomena such as electromagnetic fluid dynamics and reformulation of boundary value problems with a nonlinear boundary condition, see [**8**]. This equation is:

$$x - \mathcal{K}x = f, \tag{1.1}$$

where $\mathcal{K}$ is the *Hammerstein* integral operator defined on $\mathcal{X} = \mathcal{L}^\infty[0,1]$ by

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)\psi(t,x(t))\,dt, \quad s \in [0,1],\ x \in \mathcal{X},$$

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f and ψ are continuous functions, with $\psi(t,u)$ nonlinear in u, and x is the function to be determined. The kernel κ is continuous or may have a discontinuity of the first kind along the line $s=t$. Then, $\mathcal{K}$ is a compact operator from $\mathcal{X}$ to $\mathcal{C}[0,1]$.

Several numerical methods for approximating the solution of (1.1) are known. A variation of Nyström's method was proposed by Lardy [**28**]. A new collocation method was presented by Kumar and Sloan [**27**], and its superconvergence properties were studied by Kumar [**26**]. Moreover, an extrapolation of a discrete version of a collocation-type method was presented by Han [**19**]. The connection between Kumar and Sloan's method and the iterated spline collocation method for Hammerstein equations was discussed by Brunner [**11**]. Two discrete collocation methods were proposed by Kumar [**25**] and Atkinson and Flores [**9**]. A degenerate kernel method for Hammerstein equations was introduced by Kaneko and Xu [**21**]. The superconvergence of the iterated Galerkin solutions for Hammerstein equations with smooth as well as weakly singular kernels was probed by Kaneko and Xu [**22**]. Moreover, the superconvergence of the iterated collocation method for Hammerstein equations with smooth as well as weakly singular kernels was studied by Kaneko, Noren and Padila [**20**]. Hammerstein equations with less smooth kernels along the diagonal were considered in [**10**]. A nice review paper by Atkinson [**7**] is recommended to those readers who require more information on the numerical treatments of Hammerstein equations. Some theoretical results regarding these kinds of equations may be found in a book by Zeidler [**32**]. Recently, Kulkarni's method for more general Urysohn equations was proposed in [**18**].

More recently, the authors in [**4**] used superconvergent Nyström and degenerate kernel methods, which were inspired by Kulkarni's method [**17, 23**], to solve equation (1.1). They consist of approximating the operator $\mathcal{K}$ by one of the two finite rank operators:

$$\mathcal{T}_n = \pi_n\mathcal{K} + \mathcal{K}_{n,\iota} - \pi_n\mathcal{K}_{n,\iota}, \quad \iota = 1,2,$$
$$\mathcal{K} - \mathcal{T}_n = (\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,\iota}),$$

where π_n is a sequence of interpolatory projections, $\mathcal{K}_{n,1}$ is the degenerate kernel operator obtained by interpolating the kernel with respect to the second variable and $\mathcal{K}_{n,2}$ is the Nyström operator based on π_n. These methods were already used for linear integral equations in [**3**] and for the corresponding eigenvalue problems in [**2, 5**]. In this paper, similar projection methods are defined by replacing $\mathcal{K}_{n,\iota}$ with the following linear operator of product integration type, as defined by Kumar [**26**] and Kumar and Sloan [**27**]:

$$(\mathcal{K}_n x)(s) = \int_0^1 \kappa(s,t)(\pi_n z)(t)\,dt, \quad s \in [0,1],\ x \in \mathcal{X}, \tag{1.2}$$

where

$$z(t) = \psi(t, x(t))$$

and π_n is a sequence of finite rank projections converging to the identity operator pointwise. Thus, $\mathcal{K}_n$ is a variation of the Nyström operator $\mathcal{K}_{n,2}$, and, when the projection is orthogonal, we can show that

$$\mathcal{K}_n \equiv \mathcal{K}_{n,1}.$$

It has also been shown that the extra factor of $(\mathcal{I} - \pi_n)$ in $\mathcal{K} - \mathcal{K}_n^M$ exhibits superconvergence. More precisely, it is established that, if the kernel is sufficiently smooth, then, if π_n is either the orthogonal projection or the interpolatory projection at Gauss points onto a space of piecewise polynomials of degree less than or equal to $r-1$, the orders of convergence of the proposed method and its iterated version are, respectively, $3r$ and $4r$. This is an improvement over the order of convergence $2r$ in the product integration method. In the case of the orthogonal projection, it can be shown that, for a kernel which is less smooth along the diagonal, the iterated version of the method always improves upon the classical methods, such as the Galerkin and iterated Galerkin methods. The size of the system of equations that needs to be solved is at most twice as that of the dimension of the range of π_n. In particular, the method presented here could be viewed as an extension to the nonlinear case of the method introduced in [**24**].

The paper is organized in the following way. In Section 2, the proposed method is defined along with relevant notation, and the systems of nonlinear equations which need to be solved to obtain the approximations to the solution are discussed. Section 3 contains a general framework for the convergence analysis of the approximate and the iterated solutions. The case of kernels less smooth along the diagonal is discussed in Section 4. Numerical validation is given in Section 5.

2. Description of the method.

2.1. Preliminaries. We consider a quasi-uniform partition of $[0,1]$

$$\Delta_n : 0 = s_0 < s_1 < s_2 \cdots < s_{n-1} < s_n = 1. \tag{2.1}$$

For simplicity, we drop the index n and write $\Delta_n = \Delta$. Put $\mathcal{E}_i = [s_{i-1}, s_i]$, $h_i = s_i - s_{i-1}$ and $h = \max_{0 \le i \le n} h_i$. For a fixed $r \ge 1$, we denote by $\mathcal{P}_r$ the space of all polynomials of degree $\le r-1$. Let

$$\mathcal{X}_n = \{v : [0,1] \longrightarrow \mathbf{R} : v|_{\mathcal{E}_i} \in \mathcal{P}_r,\ 1 \le i \le n\}$$

be the space of piecewise polynomials of degree $\le r-1$, with breakpoints at $s_1, s_2, \ldots, s_{n-1}$. We consider two types of projections from $\mathcal{X}$ to $\mathcal{X}_n$.

1. The map π_n is the restriction to $\mathcal{X}$ of the orthogonal projection from $\mathcal{L}^2[0,1]$ to $\mathcal{X}_n$. The operator π_n is defined by

$$\begin{aligned}(\pi_n x)(s) &= \sum_{i=1}^{\mathfrak{n}_r} \langle x, \phi_i \rangle \phi_i(s), \\ \langle \pi_n x, \phi_i \rangle &= \langle x, \phi_i \rangle, \quad 1 \le i \le \mathfrak{n}_r,\end{aligned} \tag{2.2}$$

where $\mathfrak{n}_r = nr$, $\{\phi_i : i = 1, 2, \ldots, \mathfrak{n}_r\}$ is an orthonormal basis of $\mathcal{X}_n$ and $\langle \cdot, \cdot \rangle$ is the inner product defined by

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx.$$

for all $f, g \in \mathcal{L}^\infty[0,1]$.

2. Let $\mathcal{B}_r = \{\tau_1, \ldots, \tau_r\}$ be the set of r Gauss points in $[-1,1]$. Define a linear transformation

$$f_i : [-1,1] \longrightarrow [s_{i-1}, s_i]$$

as follows:

$$f_i(t) = \frac{1-t}{2} s_{i-1} + \frac{1+t}{2} s_i, \quad t \in [-1,1].$$

Then,

$$\begin{aligned}\mathcal{A} = \bigcup_{i=1}^{n} f_i(\mathcal{B}_r) &= \{\tau_{ij} = f_i(\tau_j) : 1 \le i \le n,\ 1 \le j \le r\} \\ &= \{t_i : i = 1, 2, \ldots, \mathfrak{n}_r\}\end{aligned}$$

is the set of $\mathfrak{n}_r$ interpolation Gauss points on $[0,1]$. Let

$$\pi_n : \mathcal{C}[0,1] \longrightarrow \mathcal{X}_n$$

be the interpolatory operator, defined by

$$\begin{aligned} (\pi_n x)(s) &= \sum_{i=1}^{\mathfrak{n}_r} x(t_i)\varphi_i(s), \\ (\pi_n x)(t_i) &= x(t_i), \quad 1 \le i \le \mathfrak{n}_r, \end{aligned} \tag{2.3}$$

where $\{\varphi_i : i = 1,2,\dots,\mathfrak{n}_r\}$ is the Lagrange basis of $\mathcal{X}_n$. This map, if necessary, is extended to $\mathcal{L}^\infty[0,1]$ as in Atkinson, et al. **[6]**, and then π_n is a projection. In both cases, π_n converge to the pointwise identity operator and, for $x \in \mathcal{C}^r[0,1]$ (see **[13,** page 328, Corollary 7.6]):

$$\|(\mathcal{I} - \pi_n)x\|_\infty \le c_1 \|x^{(r)}\|_\infty h^r, \tag{2.4}$$

where c_1 is a constant independent of n.

Let $z(t) = \psi(t, x(t))$, and consider the following approximate operator defined in **[26]** by

$$(\mathcal{K}_n x)(s) = \int_0^1 \kappa(s,t)(\pi_n z)(t)\, dt, \quad s \in [0,1],\ x \in \mathcal{X}. \tag{2.5}$$

We propose approximating $\mathcal{K}$ by the following finite rank operator

$$\begin{aligned} \mathcal{K}_n^M &= \pi_n \mathcal{K} + \mathcal{K}_n - \pi_n \mathcal{K}_n, \\ (\mathcal{K} - \mathcal{K}_n^M) &= (\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_n). \end{aligned} \tag{2.6}$$

The corresponding approximation of (1.1) becomes

$$x_n - (\pi_n \mathcal{K} + \mathcal{K}_n - \pi_n \mathcal{K}_n)x_n = f, \tag{2.7}$$

while the iterated solution is defined by

$$\widetilde{x}_n = \mathcal{K} x_n + f. \tag{2.8}$$

The reduction of (2.7) to a system of nonlinear equations is completed in the next section.

2.2. Implementation. Let π_n be the orthogonal projection defined by (2.2). The corresponding operator $\mathcal{K}_n$, defined by (2.5), can be written as

$$(\mathcal{K}_n x)(s) = \sum_{j=1}^{\mathfrak{n}_r} \langle z, \phi_j \rangle \kappa_j(s), \quad s \in [0,1],$$

where $\kappa_j(s) = \langle \kappa(s,\cdot), \phi_j \rangle$. Now, using equation (2.7), we can easily show that the approximate solution has the following form

$$x_n = f + \sum_{i=1}^{\mathfrak{n}_r} \mathcal{X}_i \phi_i + \sum_{j=1}^{\mathfrak{n}_r} \mathcal{Y}_j \kappa_j.$$

The coefficients $\{\mathcal{X}_i, \mathcal{Y}_i,\ i = 1, \ldots, \mathfrak{n}_r\}$ are obtained by substituting x_n in equation (2.7). Then, we successively have:

$$\begin{aligned}(\pi_n \mathcal{K}) x_n &= \sum_{i=1}^{\mathfrak{n}_r} \langle \mathcal{K} x_n, \phi_i \rangle \phi_i \\ &= \sum_{i=1}^{\mathfrak{n}_r} \Bigg[\int_0^1 \int_0^1 \kappa(s,t) \psi\Bigg(t, f(t) + \sum_{k=1}^{\mathfrak{n}_r} \mathcal{X}_k \phi_k(t) \\ &\qquad\qquad + \sum_{\ell=1}^{\mathfrak{n}_r} \mathcal{Y}_\ell \kappa_\ell(t) \Bigg) \phi_i(s)\, dt\, ds \Bigg] \phi_i,\end{aligned}$$

$$\mathcal{K}_n x_n = \sum_{j=1}^{\mathfrak{n}_r} \Bigg[\int_0^1 \psi\Bigg(t, f(t) + \sum_{k=1}^{\mathfrak{n}_r} \mathcal{X}_k \phi_k(t) + \sum_{\ell=1}^{\mathfrak{n}_r} \mathcal{Y}_\ell \kappa_\ell(t) \Bigg) \phi_j(t)\, dt \Bigg] \kappa_j,$$

$$\begin{aligned}(\pi_n \mathcal{K}_n) x_n &= \sum_{i=1}^{\mathfrak{n}_r} \langle \mathcal{K}_n x_n, \phi_i \rangle \phi_i \\ &= \sum_{i=1}^{\mathfrak{n}_r} \Bigg\{ \sum_{j=1}^{\mathfrak{n}_r} \Bigg[\int_0^1 \psi\Bigg(t, f(t) + \sum_{k=1}^{\mathfrak{n}_r} \mathcal{X}_k \phi_k(t) \\ &\qquad\qquad + \sum_{\ell=1}^{\mathfrak{n}_r} \mathcal{Y}_\ell \kappa_\ell(t) \Bigg) \phi_j(t)\, dt \Bigg] \langle \kappa_j, \phi_i \rangle \Bigg\} \phi_i.\end{aligned}$$

Except for some very specific situations, the family of functions $\{\phi_i, \kappa_j\}$ is linearly independent; therefore, we can identify the coefficients of ϕ_i, and κ_j, respectively, and we obtain the following system of size $2\mathfrak{n}_r$:

$$\mathcal{X}_i = \int_0^1 \int_0^1 \kappa(s,t)\psi\Big(t, f(t) + \sum_{k=1}^{\mathfrak{n}_r} \mathcal{X}_k \phi_k(t) + \sum_{\ell=1}^{\mathfrak{n}_r} \mathcal{Y}_\ell \kappa_\ell(t)\Big)\phi_i(s)\,dt\,ds$$
$$- \sum_{j=1}^{\mathfrak{n}_r} \mathcal{Y}_j \langle \kappa_j, \phi_i\rangle, \tag{2.9}$$
$$\mathcal{Y}_j = \int_0^1 \psi\Big(t, f(t) + \sum_{k=1}^{\mathfrak{n}_r} \mathcal{X}_k \phi_k(t) + \sum_{\ell=1}^{\mathfrak{n}_r} \mathcal{Y}_\ell \kappa_\ell(t)\Big)\phi_j(t)\,dt$$
$$i, j = 1, \ldots, \mathfrak{n}_r.$$

Remark 2.1. Despite that the size of system (2.9) is twice that of the Galerkin/iterated Galerkin methods, for a kernel κ which fails to be sufficiently differentiable due to discontinuities along the diagonal, the iterated solution (2.8) can converge faster than the iterated Galerkin solution and even faster than the solutions obtained by the proposed method using the interpolation projection.

For the interpolatory projection given by (2.3), applying π_n and $(\mathcal{I} - \pi_n)$ to equation (2.7), we obtain

$$\pi_n x_n - \pi_n \mathcal{K} x_n = \pi_n f, \tag{2.10}$$

$$(\mathcal{I} - \pi_n)x_n - (\mathcal{I} - \pi_n)\mathcal{K}_n x_n = (\mathcal{I} - \pi_n)f. \tag{2.11}$$

Replacing x_n by its expression from equation (2.11), $\mathcal{K}x_n$ becomes

$$\mathcal{K}x_n = \mathcal{K}(\pi_n x_n + (\mathcal{I} - \pi_n)\mathcal{K}_n x_n + (\mathcal{I} - \pi_n)f). \tag{2.12}$$

On the other hand, since $\mathcal{K}_n x_n = \mathcal{K}_n \pi_n x_n$, we obtain

$$\mathcal{K}x_n = \mathcal{K}(\pi_n x_n + (\mathcal{I} - \pi_n)\mathcal{K}_n \pi_n x_n + (\mathcal{I} - \pi_n)f). \tag{2.13}$$

Now, by replacing $\mathcal{K}x_n$ in equation (2.10), we obtain

$$\pi_n x_n - \pi_n \mathcal{K}(\pi_n x_n + (\mathcal{I} - \pi_n)\mathcal{K}_n \pi_n x_n + (\mathcal{I} - \pi_n)f) = \pi_n f,$$

and then we obtain the following system of size $\mathfrak{n}_r$:

$$x_n(t_i) - \mathcal{K}(\pi_n x_n + (\mathcal{I} - \pi_n)\mathcal{K}_n \pi_n x_n + (\mathcal{I} - \pi_n)f)(t_i) = f(t_i), \tag{2.14}$$
$$i = 1, \ldots, \mathfrak{n}_r.$$

Now, from equation (2.11), the approximate solution is given by

$$x_n = \pi_n x_n + (\mathcal{I} - \pi_n)\mathcal{K}_n \pi_n x_n + (\mathcal{I} - \pi_n) f.$$

Remark 2.2. In the iterated collocation method proposed by Sloan **[31]**, the approximate solution is given by $\overline{x}_n = f + \mathcal{K}\pi_n \overline{x}_n$ and satisfies $\overline{x}_n - \mathcal{K}\pi_n \overline{x}_n = f$. Thus, a system of the same size as in the case of our method is required to be solved. The solutions x_n and $\overline{x}_n$ are probably of equal complexity when being evaluated. The computational complexity in the method proposed here may lie in the evaluation of $\mathcal{K}(\pi_n x_n + (\mathcal{I} - \pi_n)\mathcal{K}_n \pi_n x_n + (\mathcal{I} - \pi_n) f)$ instead of $\mathcal{K}\pi_n x_n$ in Sloan's method. This addition in the cost is compensated by the improvement in the rate of convergence. On the other hand, there are integrals to be evaluated in solving nonlinear systems (2.9) and (2.14) and in evaluating $\widetilde{x}_n$. These integrals were numerically evaluated to high accuracy, to imitate exact integration.

In the next section, we prove the local existence and uniqueness of the solution of equation (2.7), and we give an estimation of its rate of convergence.

3. Orders of convergence. Let x^* be the unique solution of (1.1), and let a and b be real numbers such that

$$\Big[\min_{s\in[0,1]} x^*(s), \max_{s\in[0,1]} x^*(s)\Big] \subset [a, b].$$

Define $\Omega = [0,1]\times[a, b]$. We assume throughout this paper, unless stated otherwise, the following conditions on κ and ψ:

(i) $\Lambda = \sup_{s\in[0,1]} \int_0^1 |\kappa(s,t)|\, dt < \infty$.

(ii) The function $\psi(t,u)$ is Lipschitz continuous in $u \in [a, b]$, i.e., there exists a constant $q_1 > 0$, for which $|\psi(t,u) - \psi(t,v)| \le q_1 |u - v|$, for all $u, v \in [a, b]$.

(iii) The partial derivative $\partial\psi/\partial u$ of ψ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a $q_2 > 0$ such that

$$\left|\frac{\partial\psi}{\partial u}(t,x) - \frac{\partial\psi}{\partial u}(t,y)\right| \le q_2 |x - y|, \quad \text{for all } x, y \in [a, b].$$

Condition (iii) implies that the operator $\mathcal{K}$ is Fréchet differentiable and $\mathcal{L} = \mathcal{K}'(x^*)$ is Λq_2-Lipschitz. The Fréchet derivative is given by

$$(\mathcal{K}'(x^*)h)(s) = \int_0^1 \kappa(s,t)\frac{\partial \psi}{\partial u}(t, x^*(t))h(t)\,dt,$$

and the operator $\mathcal{K}'(x^*)$ is compact. Throughout this paper, we use the following notation:

$$\mathcal{L} = \mathcal{K}'(x^*), \quad \mathcal{L}_n = \mathcal{K}_n'(x^*), \quad \mathcal{L}_n^M = \left(\mathcal{K}_n^M\right)'(x^*), \quad z^*(t) = \psi(t, x^*(t)).$$

Note also that, throughout this paper, c, c_1, c_2 denote generic constants which may take different values but will be independent of n.

3.1. Approximate solution. The following result can be proven in the same manner as in [**16,** Theorem 1].

Theorem 3.1. *Suppose that $x^* \in \mathcal{X}$ is the unique solution of* (1.1) *with $f = 0$ and that* 1 *is not an eigenvalue of $\mathcal{L}$. Then, there exists a real number $\delta_0 > 0$ such that the approximate equation* (2.7) *has a unique solution x_n in $\mathcal{B}(u, \delta_0)$ for a sufficiently large n. Moreover,*

$$c_1\alpha_n \leq \|x^* - x_n\|_\infty \leq c_2\alpha_n, \tag{3.1}$$

where $\alpha_n = \|(\mathcal{I} - \mathcal{L}_n^M)^{-1}(\mathcal{K}(x^*) - \mathcal{K}_n^M(x^*))\|_\infty \to 0$ as $n \to \infty$ and $0 < c_1 < c_2$.

Lemma 3.2. *Assume that* 1 *is not an eigenvalue of $\mathcal{L}$. Then, for n large enough, $(\mathcal{I} - \mathcal{L}_n^M)^{-1}$ exists, and it is a bounded linear operator, i.e.,*

$$\|(\mathcal{I} - \mathcal{L}_n^M)^{-1}\|_\infty \leq c. \tag{3.2}$$

Proof. Since the operators π_n converge pointwise to the identity operator and $\mathcal{L}, \mathcal{L}_n$ are compact, it follows that

$$\max\{\|(\mathcal{I} - \pi_n)\mathcal{L}\|, \|(\mathcal{I} - \pi_n)\mathcal{L}_n\|\} \longrightarrow 0 \quad \text{as } n \to \infty.$$

From (2.6), we get

$$\mathcal{L} - \mathcal{L}_n^M = (\mathcal{I} - \pi_n)(\mathcal{L} - \mathcal{L}_n).$$

Thus,

$$\|\mathcal{L} - \mathcal{L}_n^M\| \leq \|(\mathcal{I} - \pi_n)\mathcal{L}\| + \|(\mathcal{I} - \pi_n)\mathcal{L}_n\| \longrightarrow 0 \quad \text{as } n \to \infty.$$

Hence, for n large enough, $\mathcal{I} - \mathcal{L}_n^M$ is invertible, and it is uniformly bounded by the geometric series theorem, see [**8**]. □

Choose $r \geq 1$ and $0 \leq p \leq 2r$. If $\kappa \in \mathcal{C}^p[0,1]^2$, then $R(\mathcal{K}) \subset \mathcal{C}^p[0,1]$. Thus, if $f \in \mathcal{C}^p[0,1]$, then $x \in \mathcal{C}^p[0,1]$. We set

$$\begin{aligned}\mathcal{D}^{i,j}\kappa &= \frac{\partial^{i+j}\kappa}{\partial s^i \partial t^j}(s,t), \quad s,t \in [0,1],\\ \|\kappa\|_{p,\infty} &= \sum_{i=0}^{p}\sum_{j=0}^{p}\|\mathcal{D}^{i,j}\kappa\|_\infty,\\ \|x\|_{p,\infty} &= \sum_{i=0}^{p}\|x^{(i)}\|_\infty\end{aligned}$$

and

$$\Psi_p = \sum_{i=0}^{p}\max_{t\in[0,1]}\left|\frac{\partial^i\psi}{\partial t^i}(t,x(t))\right|.$$

Let π_n be the orthogonal projection defined by (2.2). The result below is used to obtain the order of convergence of x_n to x^*.

Proposition 3.3. *We assume that* $\kappa \in \mathcal{C}^r[0,1]^2$, $\psi \in \mathcal{C}^r(\Omega)$ *and* $f \in \mathcal{C}^r[0,1]$. *Let* x^* *be the unique solution of* (1.1). *Then*:

$$\|(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_n)x^*\|_\infty \leq ch^{3r}. \tag{3.3}$$

Proof. From the definition of $\mathcal{K}_n$, we have

$$[(\mathcal{K}-\mathcal{K}_n)x^*]^{(r)}(s) = \int_0^1 \ell_s(t)(\mathcal{I}-\pi_n)z^*(t)\,dt,$$

where $\ell_s(t) = (\partial^r\kappa)/(\partial s^r)(s,t)$ and $z^*(t) = \psi(t,x^*(t))$. Let $\overline{\ell}_s$ denote the complex conjugate of ℓ_s. Then

$$\begin{aligned}[(\mathcal{K}-\mathcal{K}_n)x^*]^{(r)}(s) &= \langle(\mathcal{I}-\pi_n)z^*, \overline{\ell}_s\rangle\\ &= \langle(\mathcal{I}-\pi_n)z^*, (\mathcal{I}-\pi_n)\overline{\ell}_s\rangle\end{aligned}$$

since π_n is an orthogonal projection. Thus, by (2.4), we have, for each $s \in [0,1]$,

$$\begin{aligned}|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(r)}(s)| &\leq \|(\mathcal{I}-\pi_n)z^*\|_\infty \|(\mathcal{I}-\pi_n)\overline{\ell}_s\|_\infty \\ &\leq (c_1)^2\|z^{*(r)}\|_\infty \|(\overline{\ell}_s)^{(r)}\|_\infty h^{2r}.\end{aligned}$$

Hence, taking the supremum over $s \in [0,1]$, we obtain

$$\|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(r)}\|_\infty \leq (c_1)^2\Psi_r\|k\|_{r,\infty}h^{2r}.$$

Now, by replacing x by $(\mathcal{K}-\mathcal{K}_n)x^*$ in (2.4), we obtain

$$\begin{aligned}\|(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_n)x^*\|_\infty &\leq c_1\|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(r)}\|_\infty h^r \\ &\leq (c_1)^3\Psi_r\|k\|_{r,\infty}h^{3r}\end{aligned}$$

which completes the proof. □

Let π_n be the interpolatory projection at the Gauss point defined by (2.3). For $f \in \mathcal{C}^r[0,1]$ and $g \in \mathcal{C}^{2r}[0,1]$, we have from **[15]**:

$$\left|\int_0^1 f(t)(\mathcal{I}-\pi_n)g(t)\,dt\right| \leq c_2\|f\|_{r,\infty}\|g\|_{2r,\infty}h^{2r}. \tag{3.4}$$

Proposition 3.4. *Let* x^* *be the unique solution of* (1.1). *For* $f \in \mathcal{C}^{2r}[0,1]$, $\kappa \in \mathcal{C}^r[0,1]^2$ *and* $\psi \in \mathcal{C}^{2r}(\Omega)$, *we have*

$$\|(\mathcal{K}-\mathcal{K}_n)x^*\|_{2r,\infty} \leq c\Psi_{2r}\|\kappa\|_{2r,\infty}h^{2r}. \tag{3.5}$$

In addition,

$$\|(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_n)x^*\|_\infty \leq ch^{3r}. \tag{3.6}$$

Proof. For a fixed j such that $0 \leq j \leq 2r$, we have

$$[(\mathcal{K}-\mathcal{K}_n)x^*]^{(j)}(s) = \int_0^1 \ell_s(t)(\mathcal{I}-\pi_n)z^*(t)\,dt,$$

where $\ell_s(t) = (\partial^j\kappa)/(\partial s^j)(s,t)$. Then, from (3.4), it follows that

$$|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(j)}(s)| \leq c_2\|\ell_s\|_{r,\infty}\|z^*\|_{2r,\infty}h^{2r}.$$

Hence, taking the supremum over $s \in [0,1]$, we obtain

$$\|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(j)}\|_\infty \leq c_2\Psi_{2r}\|\kappa\|_{2r,\infty}h^{2r},$$

and hence,

$$\begin{aligned}\|(\mathcal{K}-\mathcal{K}_n)x^*\|_{2r,\infty} &= \sum_{j=0}^{2r} \|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(j)}\|_\infty \\ &\le c_2(2r+1)\Psi_{2r}\|\kappa\|_{2r,\infty}h^{2r},\end{aligned}$$

which proves (3.5). Now, by replacing x^* by $(\mathcal{K}-\mathcal{K}_n)x^*$ in (2.4), we obtain

$$\begin{aligned}\|(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_n)x^*\|_\infty &\le c_1\|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(r)}\|_\infty h^r \\ &\le c_1c_2\Psi_{2r}\|\kappa\|_{2r,\infty}h^{3r},\end{aligned}$$

which completes the proof. □

Now we are ready to state the following, main theorem.

Theorem 3.5. *Let $x^* \in \mathcal{X}$ be the unique solution of* (1.1)*, and assume that* 1 *is not an eigenvalue of $\mathcal{L}$. In the case of the orthogonal projection, we assume that $\kappa \in \mathcal{C}^r[0,1]^2$, $\psi \in \mathcal{C}^r(\Omega)$ and $f \in \mathcal{C}^r[0,1]$, while, in the case of the interpolatory projection, we assume that $\kappa \in \mathcal{C}^{2r}[0,1]^2$, $\psi \in \mathcal{C}^{2r}(\Omega)$ and $f \in \mathcal{C}^{2r}[0,1]$. Then:*

$$\|x^* - x_n\|_\infty = \mathcal{O}(h^{3r}). \tag{3.7}$$

Proof. Theorem 3.1 is applicable for $f = 0$. Then, by Lemma 3.1, we have

$$\|x^* - x_n\|_\infty \le c\|(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_n)x^*\|_\infty. \tag{3.8}$$

Thus, (3.7) follows from (3.3) or (3.6). □

Remark 3.6. Let χ_n be the approximate solutions of equation (1.1) obtained by the product integration method. Then, $\chi_n - \mathcal{K}_n\chi_n = f$. By using (2.4) or (3.5), we can show that

$$\|x^* - \chi_n\|_\infty = \mathcal{O}(h^{2r}).$$

Hence, x_n converges to x^* faster than χ_n.

In what follows, we show that the iterated solution defined by (2.8) converges to x^* faster than x_n.

3.2. Iterated solution. Since $\mathcal{K}_n$ is Fréchet differentiable, we define

$$\begin{aligned}\mathfrak{r}_\mathfrak{n} &= \frac{\|\mathcal{K}(x^*) - \mathcal{K}(x_n) - \mathcal{L}(x^* - x_n)\|_\infty}{\|x^* - x_n\|_\infty} \\ \mathfrak{q}_\mathfrak{n} &= \frac{\|\mathcal{K}_n(x^*) - \mathcal{K}_n(x_n) - \mathcal{L}_n(x^* - x_n)\|_\infty}{\|x^* - x_n\|_\infty}.\end{aligned} \tag{3.9}$$

By Theorem 3.5 and the definitions of $\mathcal{L}$ and $\mathcal{L}_n$, we deduce that

$$\{\mathfrak{r}_\mathfrak{n},\ \mathfrak{q}_\mathfrak{n}\} \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.10}$$

In addition, it can be shown that

$$\max\{\mathfrak{r}_\mathfrak{n},\ \mathfrak{q}_\mathfrak{n}\} \le \frac{c}{2}\|x^* - x_n\|_\infty, \tag{3.11}$$

see, for example, **[30]**. We use the following notation:

$$\begin{aligned}\mathfrak{a} &= \|(\mathcal{I} - \mathcal{L})^{-1}\|_\infty, \\ \mathfrak{a}_n &= \max\{\|(\mathcal{I} - \pi_n)\mathcal{L}\|_\infty, \|(\mathcal{I} - \pi_n)\mathcal{L}_n\|_\infty, \|(\mathcal{I} - \pi_n)\mathcal{L}^*\|_\infty\}, \\ \mathfrak{b}_n &= \|\mathcal{L}(\mathcal{I} - \pi_n)\|_\infty.\end{aligned}$$

The sequence $\mathfrak{b}_n$ is uniformly bounded

$$\mathfrak{b}_n \le \mathfrak{b}, \quad \text{for all } n \ge 1. \tag{3.12}$$

The error for the iterated solution is given in the next theorem.

Theorem 3.7. *Let x^* be the unique solution of* (1.1) *and assume that* 1 *is not an eigenvalue of $\mathcal{L}$. For n large enough, we have*

$$\begin{aligned}\|x^* - \widetilde{x}_n\|_\infty &\le c\|x^* - x_n\|_\infty^2 + \mathfrak{a}\|\mathcal{L}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_n)x^*\|_\infty \\ &\quad + \mathfrak{a}\|\mathcal{L}(\mathcal{I} - \pi_n)(\mathcal{L} - \mathcal{L}_n)\|_\infty\|x^* - x_n\|_\infty.\end{aligned} \tag{3.13}$$

Proof. We have

$$\begin{aligned}(\mathcal{I} - \mathcal{L})(x^* - \widetilde{x}_n) &= \mathcal{K}x^* - \mathcal{K}x_n - \mathcal{L}(x^* - x_n) + \mathcal{L}(\widetilde{x}_n - x_n) \\ &= \mathcal{K}x^* - \mathcal{K}x_n - \mathcal{L}(x^* - x_n) + \mathcal{L}(\mathcal{K} - \mathcal{K}_n^M)x_n \\ &= \mathcal{K}x^* - \mathcal{K}x_n - \mathcal{L}(x^* - x_n) + \mathcal{L}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_n)x_n \\ &= [\mathcal{I} - \mathcal{L}(\mathcal{I} - \pi_n)][\mathcal{K}x^* - \mathcal{K}x_n - \mathcal{L}(x^* - x_n)] \\ &\quad + \mathcal{L}(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_n)x^* - \mathcal{L}(\mathcal{I}-\pi_n)(\mathcal{L}-\mathcal{L}_n)(x^*-x_n) \\ &\quad + \mathcal{L}(\mathcal{I} - \pi_n)[\mathcal{K}_n x^* - \mathcal{K}_n x_n - \mathcal{L}_n(x^* - x_n)].\end{aligned}$$

Multiplying by $(\mathcal{I}-\mathcal{L})^{-1}$, we find that

$$\begin{aligned} x^* - \widetilde{x}_n = {} & [\mathcal{I} + (\mathcal{I}-\mathcal{L})^{-1}\mathcal{L}\pi_n][\mathcal{K}x^* - \mathcal{K}x_n - \mathcal{L}(x^* - x_n)] \\ & + (\mathcal{I}-\mathcal{L})^{-1}\mathcal{L}(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_n)x^* \\ & + (\mathcal{I}-\mathcal{L})^{-1}\mathcal{L}(\mathcal{I}-\pi_n)[\mathcal{K}_n x^* - \mathcal{K}_n x_n - \mathcal{L}_n(x^* - x_n)] \\ & - (\mathcal{I}-\mathcal{L})^{-1}\mathcal{L}(\mathcal{I}-\pi_n)(\mathcal{L}-\mathcal{L}_n)(x^* - x_n). \end{aligned}$$

By using (3.9), we deduce that

$$\begin{aligned} \|x^* - \widetilde{x}_n\|_\infty \le {} & c_1(\mathfrak{r}_n + \mathfrak{q}_n)\|x^* - x_n\|_\infty + \mathfrak{a}\|\mathcal{L}(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_n)x^*\|_\infty \\ & + \mathfrak{a}\|\mathcal{L}(\mathcal{I}-\pi_n)(\mathcal{L}-\mathcal{L}_n)\|_\infty \|x^* - x_n\|_\infty \end{aligned}$$

and, by (3.11), the proof is complete. □

A preliminary result is proven first below.

Lemma 3.8. *Let π_n be the interpolatory projection at Gauss points. For $f \in \mathcal{C}^r[0,1]$, $g \in \mathcal{C}^{2r}[0,1]$, $\kappa \in \mathcal{C}^r[0,1]$ and $(\partial\psi)/(\partial u) \in \mathcal{C}^r(\Omega)$, we have*

$$\|\mathcal{L}(\mathcal{I}-\pi_n)g\|_\infty \le c\|g\|_{2r,\infty} h^{2r}. \tag{3.14}$$

Proof. By the definition of $\mathcal{L}$, we have

$$\begin{aligned} (\mathcal{L}(\mathcal{I}-\pi_n)g)(s) &= \int_0^1 \kappa(s,t)\frac{\partial\psi}{\partial u}(t, x^*(t))(\mathcal{I}-\pi_n)g(t)\,dt \\ &= \int_0^1 q(s,t)(\mathcal{I}-\pi_n)g(t)\,dt, \end{aligned}$$

where

$$q(s,t) = \kappa(s,t)\frac{\partial\psi}{\partial u}(t, x^*(t)).$$

Thus, by using (3.4), we obtain

$$\begin{aligned} & \|\mathcal{L}(\mathcal{I}-\pi_n)g\|_\infty \\ & \qquad \le c_2 \sum_{j=0}^{r} \max_{(s,t)\in[0,1]^2} \left| \frac{\partial^{j+1}\kappa(s,t)\psi(t,x^*(t))}{\partial t^j \partial u} \right| \|k\|_{r,\infty}\|g\|_{2r,\infty} h^{2r}, \end{aligned}$$

which completes the proof. □

Now, we are ready for the main theorem.

Theorem 3.9. *Assume that the hypothesis of Theorem* 3.5 *are satisfied. Further, we assume for interpolatory projection that* $(\partial\psi)/(\partial u) \in \mathcal{C}^r(\Omega)$. *Then, for* n *sufficiently large, the iterated solution* $\widetilde{x}_n$, *defined by* (2.8), *satisfies*

$$\|x^* - \widetilde{x}_n\|_\infty = \mathcal{O}(h^{4r}). \tag{3.15}$$

Proof. For the orthogonal projection, we use the following identity given in the proof of **[20,** Theorem 2.3]

$$\|x^* - \widetilde{x}_n\|_\infty \leq c[(1 + \mathfrak{b}_\mathfrak{n})\mathfrak{r}_\mathfrak{n} + \mathfrak{a}\mathfrak{b}_\mathfrak{n}]\|x^* - x_n\|_\infty. \tag{3.16}$$

By using (2.4), it can easily be verified that

$$\mathfrak{a}_n = \mathcal{O}(h^r). \tag{3.17}$$

Hence, by the orthogonality of π_n and the argument of Sloan **[31,** Theorem 1], we have

$$\mathfrak{b}_n = \|(\mathcal{I} - \pi_n)^* \mathcal{L}^*\|_\infty = \|(\mathcal{I} - \pi_n)\mathcal{L}^*\|_\infty = \mathcal{O}(h^r). \tag{3.18}$$

Then, (3.15) follows by combining the estimates (3.7), (3.11), (3.16), (3.17) and (3.18).

If π_n is the interpolatory projection at r Gauss points, then, from (3.13),

$$\begin{aligned} \|x^* - \widetilde{x}_n\|_\infty \leq{}& c\|x^* - x_n\|_\infty^2 \\ &+ \mathfrak{a}\|\mathcal{L}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_n)x^*\|_\infty + c\mathfrak{a}_n\|x^* - x_n\|. \end{aligned} \tag{3.19}$$

On the other hand, by combining (3.14) and (3.5), we obtain

$$\|\mathcal{L}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_n)x^*\|_\infty = \mathcal{O}(h^{4r})$$

and thus, the desired result follows from estimates (3.7), (3.11)–(3.17) and (3.19). □

One step of the Richardson extrapolation can be used to further improve the order of convergence of $\widetilde{x}_n$. Let $\widetilde{x}_{2n}$ be the iterated solution associated with a uniform partition of $[0, 1]$ with $2n$ intervals of length $h/2$ and obtained by using the interpolatory projection at Gauss points.

Define

$$x_n^R = \frac{2^{4r}\widetilde{x}_{2n} - \widetilde{x}_n}{2^{4r}-1}.$$

Then, the following result can be proven.

Theorem 3.10. *Let $x^* \in \mathcal{X}$ be the unique solution of* (1.1)*, and assume that* 1 *is not an eigenvalue of $\mathcal{L}$. We assume that $\kappa \in \mathcal{C}^{2r+2}[0,1]^2$, $\psi \in \mathcal{C}^{2r+2}(\Omega)$, $f \in \mathcal{C}^{2r+2}[0,1]$ and $(\partial\psi)/(\partial u) \in \mathcal{C}^r(\Omega)$. Then:*

$$\|x^* - x_n^R\|_\infty = \mathcal{O}(h^{4r+2}). \tag{3.20}$$

4. Case of kernels less smooth along the diagonal. Let α and γ be two integers such that $\alpha \geq \gamma$, $\alpha \geq 0$ and $\gamma \geq -1$. We assume that the kernel κ has the following form:

$$\kappa(s,t) = \begin{cases} \kappa_1(s,t) & 0 \leq s \leq t \leq 1, \\ \kappa_2(s,t) & 0 \leq t \leq s \leq 1, \end{cases}$$

with $\kappa_1 \in \mathcal{C}^\alpha(\{0 \leq s \leq t \leq 1\})$, $\kappa_2 \in \mathcal{C}^\alpha\ (\{0 \leq t \leq s \leq 1\})$. If $\gamma \geq 0$, then it is assumed that $\kappa \in \mathcal{C}^\gamma[0,1]^2$ and, if $\gamma = -1$, then the kernel κ may have a discontinuity of the first kind along the line $s = t$. Following Chatelin-Lebbar **[14]**, the class of kernels of the above form is denoted by $\mathcal{C}(\alpha,\gamma)$. The obvious examples of such kernels are Green's functions of ordinary differential equations and kernels of Voltera integral operators.

The operator

$$\mathcal{K} : \mathcal{C}[0,1] \longrightarrow \mathcal{C}[0,1]$$

is compact, and the range of $\mathcal{K}$, $R(\mathcal{K})$, is contained in $\mathcal{C}^{\min\{\alpha,\gamma+1\}}[0,1]$. For $\nu \geq 0$, set

$$\mathcal{C}_\Delta^\nu = \{y \in \mathcal{L}^\infty : y|_{\mathcal{E}_i} \in \mathcal{C}^\nu(\mathcal{E}_i),\ 1 \leq i \leq n\},$$

where Δ is the quasi-uniform partition defined in Section 2 and $\mathcal{E}_i = [s_{i-1}, s_i]$. According to **[10]**, $\mathcal{K}$ is a continuous map from $\mathcal{C}_\Delta^\alpha$ to $\mathcal{C}_\Delta^\alpha$.

Set

$$\beta = \min\{\alpha, r\},$$

$$\gamma_1 = \min\{\alpha, \gamma+1\},$$

$$\beta_1 = \min\{\gamma_1, r\} = \min\{\alpha, r, \gamma+1\}.$$

If π_n is either the orthogonal projection or the interpolatory projection at Gauss points, then, from **[14]**, we have for any $x \in \mathcal{C}^{\alpha}_{\Delta}$,

$$\|(\mathcal{I} - \pi_n)x\|_\infty \le c_1 \|x^{(\beta)}\|_\infty h^{\beta}, \tag{4.1}$$

and, if $x \in \mathcal{C}^{\eta}_{\Delta}$ with $0 \le \eta \le \alpha$,

$$\|(\mathcal{I} - \pi_n)x\|_\infty \le c_1 \|x^{(\eta_1)}\|_\infty h^{\eta_1}, \tag{4.2}$$

where $\eta_1 = \min\{\eta, r\}$.

To remove any ambiguity, in the remainder of the paper, $\mathcal{Q}_n$ and π_n will denote the orthogonal projection and the interpolatory projection at Gauss points, defined by (2.2) and (2.3), respectively.

4.1. Orthogonal projection. Let $\mathcal{S}$ denotes the linear integral operator defined by

$$(\mathcal{S}x)(s) = \int_0^1 \kappa(s,t)x(t)\,dt, \quad s \in [0,1]. \tag{4.3}$$

If the kernel $\kappa \in \mathcal{C}(\alpha, \gamma)$, then, for $x \in \mathcal{C}^{\alpha}_{\Delta}$,

$$\|\mathcal{S}(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \le c_2 \|x^{(\beta)}\|_\infty h^{\beta+\beta_2}, \tag{4.4}$$

and, for $x \in \mathcal{C}^{\beta_1}_{\Delta}$,

$$\|\mathcal{S}(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \le c_2 \|x^{(\beta_1)}\|_\infty h^{\beta_1+\beta_2}, \tag{4.5}$$

where

$$\beta_2 = \min\{\beta, \gamma + 2\} = \min\{\alpha, r, \gamma + 2\}.$$

Using these estimates, we prove the following, preliminary result.

Lemma 4.1. *If the kernel $\kappa \in \mathcal{C}(\alpha, \gamma)$, $x \in \mathcal{C}^{\alpha}_{\Delta}$ and $\psi \in \mathcal{C}^{\alpha}(\Omega)$,*

$$\|(\mathcal{K} - \mathcal{K}_n)x\|_\infty \le c_2 \Psi_\beta h^{\beta+\beta_2}. \tag{4.6}$$

In addition, if $g \in \mathcal{C}^{\beta_1}_{\Delta}$ and $(\partial\psi)/(\partial u) \in \mathcal{C}^{\alpha}(\Omega)$, then

$$\|\mathcal{L}(\mathcal{I} - \mathcal{Q}_n)g\|_\infty \le c_2 \|g^{(\beta_1)}\|_\infty h^{\beta_1+\beta_2}. \tag{4.7}$$

Proof. Writing $(\mathcal{K}-\mathcal{K}_n)x = \mathcal{S}(\mathcal{I}-\mathcal{Q}_n)z$, where $z(t)=\psi(t,x(t))$, estimate (4.6) is immediate from (4.4). Note that, for $g \in \mathcal{C}^\alpha_\Delta$,

$$\begin{aligned}(\mathcal{L}(\mathcal{I}-\mathcal{Q}_n)g)(s) &= \int_0^1 \kappa(s,t)\frac{\partial\psi}{\partial u}(t,x(t))(\mathcal{I}-\mathcal{Q}_n)g(t)\,dt \\ &= \int_0^1 q(s,t)(\mathcal{I}-\mathcal{Q}_n)g(t)\,dt,\end{aligned}$$

with

$$q(s,t) = \kappa(s,t)\frac{\partial\psi}{\partial u}(t,x(t)).$$

Since, by assumption, $(\partial\psi)/(\partial u) \in \mathcal{C}^\alpha(\Omega)$, the kernel $q(s,t) \in \mathcal{C}(\alpha,\gamma)$, and, since $\mathcal{L}$ is a linear operator, the result follows from (4.5). $\square$

Theorem 4.2. *We assume that* $\kappa \in \mathcal{C}(2\alpha,\gamma)$, $f \in \mathcal{C}^\alpha_\Delta$ *and* $\psi \in \mathcal{C}^\alpha(\Omega)$. *Let* x^* *be the unique solution of* (1.1). *For all large* n, *we have*

$$\|x^* - x_n\|_\infty = \mathcal{O}(h^{\beta+\min\{\beta+\beta_1,\gamma+2\}}), \tag{4.8}$$

In addition, if $(\partial\psi)/(\partial u) \in \mathcal{C}^\alpha(\Omega)$, *then*

$$\|x^* - \widetilde{x}_n\|_\infty = \mathcal{O}(h^{\beta+\beta_2+\min\{\beta+\beta_1,\gamma+2\}}). \tag{4.9}$$

Proof. Since $f \in \mathcal{C}^\alpha_\Delta$, it follows from **[14]** that $x^* \in \mathcal{C}^\alpha_\Delta$, and, since $\psi \in \mathcal{C}^\alpha(\Omega)$, we deduce that $z^*(t) = \psi(t,x^*(t)) \in \mathcal{C}^\alpha_\Delta$. Now, since $\mathcal{Q}_n z^* \in \mathcal{C}^\infty_\Delta$, it follows that $z^* - \mathcal{Q}_n z^* \in \mathcal{C}^\alpha_\Delta$. The linear operator $\mathcal{S}$ is a continuous map from $\mathcal{C}^\alpha_\Delta$ to $\mathcal{C}^{\gamma_1}_\Delta$. Then, $(\mathcal{K}-\mathcal{K}_n)x^* = \mathcal{S}(\mathcal{I}-\mathcal{Q}_n)z^* \in \mathcal{C}^{\gamma_1}_\Delta$. By (4.2), we obtain

$$\|(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}-\mathcal{K}_n)x^*\|_\infty \le c_1\|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(\beta_1)}\|_\infty h^{\beta_1}. \tag{4.10}$$

We have

$$[(\mathcal{K}-\mathcal{K}_n)x^*]^{(\beta_1)}(s) = \int_0^1 \frac{\partial^{\beta_1}\kappa}{\partial s^{\beta_1}}(s,t)(\mathcal{I}-\mathcal{Q}_n)z^*(t)\,dt.$$

Since the kernel $\ell(s,t) = (\partial^{\beta_1}\kappa)/(\partial s^{\beta_1})(s,t) \in \mathcal{C}(2\alpha-\beta_1,\gamma-\beta_1) \subset \mathcal{C}(\alpha,\gamma-\beta_1)$, by (4.6), we obtain

$$\begin{aligned}\|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(\beta_1)}\|_\infty &= \max_{s\in[0,1]} |[(\mathcal{K}-\mathcal{K}_n)x^*]^{(\beta_1)}(s)| \\ &\le c_2\Psi_\beta h^{\beta+\min\{\beta,\gamma-\beta_1+2\}}.\end{aligned} \tag{4.11}$$

By combining (3.8) and (4.10), the estimate (4.8) follows.

For the iterated solution, by using estimate (3.13), we write

$$\begin{aligned}\|x^* - \widetilde{x}_n\|_\infty &\\ \leq c(\mathfrak{r}_\mathfrak{n} + \mathfrak{q}_\mathfrak{n})\|x^* - x_n\|_\infty + \mathfrak{a}\|\mathcal{L}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_n)x^*\|_\infty &\\ + \mathfrak{a}\|x^* - x_n\|_\infty \max\left\{\|\mathcal{L}(\mathcal{I} - \mathcal{Q}_n)\mathcal{L}\|_\infty, \|\mathcal{L}(\mathcal{I} - \mathcal{Q}_n)\mathcal{L}_n\|_\infty\right\}.\end{aligned} \tag{4.12}$$

From (3.11) and (4.8), the first term on right hand side of (4.12) is of the order h^{β^*}, where

$$\beta^* = 2\beta + 2\min\{\beta + \beta_1, \gamma + 2\} \geq \beta + \beta_2 + \min\{\beta + \beta_1, \gamma + 2\}. \tag{4.13}$$

In addition, from (4.7) and (4.11), we have

$$\begin{aligned}\|\mathcal{L}(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K} - \mathcal{K}_n)x^*\|_\infty &\leq \|[(\mathcal{K} - \mathcal{K}_n)x^*]^{(\beta_1)}\|_\infty h^{\beta_1+\beta_2},\\ &\leq (c_2)^2\Psi_\beta h^{\beta+\beta_2+\min\{\beta+\beta_1,\gamma+2\}}.\end{aligned} \tag{4.14}$$

On the other hand, since $\mathcal{L}g \in \mathcal{C}^\alpha_\Delta \subset \mathcal{C}^{\beta_1}_\Delta$, by (4.7), we get

$$\|\mathcal{L}(\mathcal{I} - \mathcal{Q}_n)\mathcal{L}g\|_\infty \leq c_2\|(\mathcal{L}g)^{(\beta_1)}\|_\infty h^{\beta_1+\beta_2}.$$

For $g \in \mathcal{C}_\Delta$, we have

$$(\mathcal{L}g)^{(\beta_1)}(s) = \int_0^1 \frac{\partial^{\beta_1}\kappa}{\partial s^{\beta_1}}(s,t)\frac{\partial\psi}{\partial u}(t, x^*(t))g(t)\,dt.$$

Thus,

$$\|(\mathcal{L}g)^{(\beta_1)}\|_\infty \leq \max_{t\in[0,1]}\left|\frac{\partial\psi}{\partial u}(t, x^*(t))\right| \max_{(s,t)\in[0,1]^2}\left|\frac{\partial^{\beta_1}\kappa}{\partial s^{\beta_1}}(s,t)\right|\|g\|_\infty.$$

Hence,

$$\|\mathcal{L}(\mathcal{I} - \mathcal{Q}_n)\mathcal{L}\|_\infty = \mathcal{O}(h^{\beta_1+\beta_2}). \tag{4.15}$$

In a similar manner, we show that

$$\|\mathcal{L}(\mathcal{I} - \mathcal{Q}_n)\mathcal{L}_n\|_\infty = \mathcal{O}(h^{\beta_1+\beta_2}). \tag{4.16}$$

Finally, combining estimates (4.12)–(4.16) and (4.19), the proof is complete. □

Remark 4.3. The iterated Galerkin solution satisfies the following equation:

$$x_n^G - \mathcal{K}\mathcal{Q}_n x_n^G = f.$$

If the kernel $\kappa \in \mathcal{C}(\alpha,\gamma)$, then, from **[14]**:

$$\|x^* - x_n^G\|_\infty = \mathcal{O}(h^{\beta+\beta_2}),$$

hence, for $\alpha \geq 0$, $\widetilde{x}_n$ converges to x^* faster than x_n^G.

4.2. Interpolatory projection. We quote the following estimates from **[14]**. If the kernel $\kappa \in \mathcal{C}(\alpha,\gamma)$ with $\alpha \geq r$, and $x \in \mathcal{C}_\Delta^{\beta_3}$,

$$\|\mathcal{S}(\mathcal{I}-\pi_n)x\|_\infty \leq c\|x\|_{\beta_3,\infty} h^{\beta_3}, \tag{4.17}$$

where $\beta_3 = \min\{\alpha, 2r, r+\gamma+2\}$. Since $(\mathcal{K}-\mathcal{K}_n)x = \mathcal{S}(\mathcal{I}-\pi_n)z$, then, for $x \in \mathcal{C}_\Delta^{\beta_3}$ and $\psi \in \mathcal{C}^{\beta_3}(\Omega)$,

$$\|(\mathcal{K}-\mathcal{K}_n)x\|_\infty \leq c\Psi_{\beta_3} h^{\beta_3}. \tag{4.18}$$

Theorem 4.4. *We assume that $\kappa \in \mathcal{C}(2\alpha,\gamma)$, $f \in \mathcal{C}_\Delta^\alpha$ and $\psi \in \mathcal{C}^\alpha(\Omega)$ with $\alpha \geq r$. Let x^* be the unique solution of* (1.1)*. For all large n, we have*

$$\|x^* - x_n\|_\infty = \mathcal{O}(h^{\beta_1+\min\{\alpha,2r,r+\gamma-\beta_1+2\}}). \tag{4.19}$$

Proof. Applying (4.2), we obtain

$$\|(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_n)x^*\|_\infty \leq c\|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(\beta_1)}\|_\infty h^{\beta_1}. \tag{4.20}$$

Since

$$[(\mathcal{K}-\mathcal{K}_n)x^*]^{(\beta_1)}(s) = \int_0^1 \frac{\partial^{\beta_1}\kappa}{\partial s^{\beta_1}}(s,t)(\mathcal{I}-\pi_n)z^*(t)\,dt,$$

and the kernel $\ell(s,t) = (\partial^{\beta_1}\kappa)/(\partial s^{\beta_1})(s,t) \in \mathcal{C}(\alpha,\gamma-\beta_1)$, by (4.18), we then obtain

$$\|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(\beta_1)}\|_\infty \leq c\Psi_\alpha h^{\min\{\alpha,2r,r+\gamma-\beta_1+2\}}. \tag{4.21}$$

Combining (4.20) and (4.21), we obtain the desired result. □

Remark 4.5. Note that, for $\alpha \geq 2r$ since $\beta = r$, we have

$$\begin{aligned}\beta_1 + \min\{\alpha, 2r, r+\gamma-\beta_1+2\} &= \beta_1 + \min\{2r, r+\gamma-\beta_1+2\}\\ &= r + \min\{r+\beta_1, \gamma+2\}\\ &= \beta + \min\{\beta+\beta_1, \gamma+2\}.\end{aligned}$$

Thus, the method has the same order of convergence as in the case of the orthogonal projection given by estimate (4.8).

4.3. Multi projection methods. In this section, we use two different projectors to define the approximate operator $\mathcal{K}_n^M$. Let $\mathcal{Q}_n$ and π_n be the orthogonal and interpolatory projections at Gauss points defined by (2.2) and (2.3), respectively. Define

$$\begin{aligned} \mathcal{K}_n^M &= \mathcal{Q}_n\mathcal{K} + \mathcal{K}_n - \mathcal{Q}_n\mathcal{K}_n, \\ \mathcal{K} - \mathcal{K}_n^M &= (\mathcal{I} - \mathcal{Q}_n)(\mathcal{K} - \mathcal{K}_n), \end{aligned} \tag{4.22}$$

where $\mathcal{K}_n$ is the approximate operator defined by (2.5), and based on π_n. We call this method the multi-projection 1 and, when the roles of $\mathcal{Q}_n$ and π_n are permuted, the multi-projection 2. As in subsection 2.2, the approximate solution is obtained by solving a nonlinear system of size $2\mathfrak{n}_r$.

For the multi-projection 1, we have the following result.

Theorem 4.6. *We assume that $\kappa \in \mathcal{C}(2\alpha, \gamma)$, $f \in \mathcal{C}^\alpha_\Delta$, $\psi \in \mathcal{C}^\alpha(\Omega)$ and $\alpha \geq r$. Let x^* be the unique solution of* (1.1). *For all large n, we have*

$$\|x^* - x_n\|_\infty = \mathcal{O}(h^{\beta_1 + \min\{\alpha, 2r, r+\gamma-\beta_1+2\}}), \tag{4.23}$$

In addition, if $(\partial\psi)/(\partial u) \in \mathcal{C}^\alpha(\Omega)$, then

$$\|x^* - \widetilde{x}_n\|_\infty = \mathcal{O}(h^{\beta_1+\beta_2 + \min\{\alpha, 2r, r+\gamma-\beta_1+2\}}). \tag{4.24}$$

Proof. From (4.2) and (4.18), the estimate (4.23) follows exactly in the same manner as (4.19). Recall from (3.13) that

$$\begin{aligned} &\|x^* - \widetilde{x}_n\|_\infty \\ &\quad \leq c\|x^* - x_n\|_\infty^2 + \mathfrak{a}\|\mathcal{L}(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K} - \mathcal{K}_n)x^*\|_\infty \\ &\quad\quad + \mathfrak{a}\|x^* - x_n\|_\infty \max\left\{\|\mathcal{L}(\mathcal{I} - \mathcal{Q}_n)\mathcal{L}\|_\infty, \|\mathcal{L}(\mathcal{I} - \mathcal{Q}_n)\mathcal{L}_n\|_\infty\right\}. \end{aligned} \tag{4.25}$$

From (4.23), the first term on the right hand side of (4.25) is of the order $h^{\overline{\beta}}$, where

$$\begin{aligned} \bar{\beta} &= 2\beta_1 + 2\min\{\alpha, 2r, r+\gamma-\beta_1+2\} \\ &\geq \beta_1 + \beta_2 + \min\{\alpha, 2r, r+\gamma-\beta_1+2\}. \end{aligned} \tag{4.26}$$

As before, we have from (4.7) and (4.21),

$$\begin{aligned}\|\mathcal{L}(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}-\mathcal{K}_n)x^*\|_\infty &\leq \|[(\mathcal{K}-\mathcal{K}_n)x^*]^{(\beta_1)}\|_\infty \\ &\leq c\Psi_\alpha h^{\beta_1+\beta_2+\min\{\alpha,2r,r+\gamma-\beta_1+2\}}.\end{aligned} \tag{4.27}$$

Now, by (4.15) and (4.16), the third term on the right hand side of (4.14) is of the order:

$$\mathcal{O}(h^{\beta_1+\beta_2+\min\{\alpha,2r,r+\gamma-\beta_1+2\}}).$$

By combining (4.23), (4.25)–(4.27) and the above estimate, the proof is complete. □

Note that, for $\alpha \geq 2r$, we obtain the same order of convergence obtained by using the orthogonal projection given by estimate (4.9).

For the multi-projection 2, we can show the following result.

Theorem 4.7. *We assume that $\kappa \in \mathcal{C}(2\alpha,\gamma)$, $f \in \mathcal{C}^\alpha_\Delta$ and $\psi \in \mathcal{C}^\alpha(\Omega)$. Let x^* be the unique solution of* (1.1)*. For all large n, we have*

$$\|x^* - x_n\|_\infty = \mathcal{O}(h^{\beta+\min\{\beta+\beta_1,\gamma+2\}}). \tag{4.28}$$

Remark 4.8.

(a) For $\alpha \geq 2r$, the multi projection 1 has the same convergence orders as the method using only orthogonal projection. The use of both operators $\mathcal{Q}_n$ and π_n can reduce computational costs since the expression of π_n does not contain integrals.

(b) When the kernel is sufficiently smooth, the order of convergence of these methods is also $3r$, and that of the iterated version is $4r$.

5. Numerical results. In this section, three examples are given to illustrate the theory established in the previous sections. Let $\mathcal{X}_n$ be the space of piecewise constant functions ($r = 1$) with respect to the uniform partition of $[0,1]$ on n subintervals with mesh length $h = 1/n$

$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1.$$

The projection π_n is chosen either to be the interpolatory projection or the orthogonal projection, with the range equal to $\mathcal{X}_n$. In the case of the interpolatory projection, the collocation points are the $\mathfrak{n}_\mathfrak{r} = nr = n$

midpoints

$$t_k = \frac{2k-1}{2n}, \quad k = 1, \dots, n.$$

In implementing the methods described in the previous sections, the associated nonlinear systems were solved using a Newton-Raphson algorithm. We denote

$$\|x^* - x_n\|_\infty = \mathcal{O}(h^{\delta_1}), \quad \|x^* - \widetilde{x}_n\|_\infty = \mathcal{O}(h^{\delta_2}).$$

Example 5.1. We consider the following Hammerstein integral equation with smooth kernel:

$$x(s) - \int_0^1 e^{st} log(-x^2(t) + t^2 + 2)\, dt = f(s), \quad s \in [0,1],$$

where the exact solution is $x^*(s) = \sqrt{s}$, and f is chosen accordingly. From Theorems 3.2 and 3.3, the expected orders of convergence are $\delta_1 = 3$ and $\delta_2 = 4$. The results are given in Tables 1 and 2. It can be seen that the computed orders of convergence match well with the theoretically predicted values.

TABLE 1. Orthogonal projection.

$\mathfrak{n}$	$\|x^* - x_n\|_\infty$	δ_1	$\|x^* - \widetilde{x}_n\|_\infty$	δ_2
2	1.15×10^{-3}		5.30×10^{-5}	
4	2.00×10^{-4}	2.53	3.48×10^{-6}	3.93
8	2.81×10^{-5}	2.83	2.20×10^{-7}	3.98
16	3.69×10^{-6}	2.93	1.38×10^{-8}	3.99
32	4.71×10^{-7}	2.97	8.65×10^{-10}	4.00

TABLE 2. Interpolatory projection.

$\mathfrak{n}$	$\|x^* - x_n\|_\infty$	δ_1	$\|x^* - \widetilde{x}_n\|_\infty$	δ_2
2	3.48×10^{-3}		3.23×10^{-4}	
4	5.01×10^{-4}	2.80	2.14×10^{-5}	3.92
8	6.67×10^{-5}	2.91	1.36×10^{-6}	3.98
16	8.59×10^{-6}	2.96	8.53×10^{-8}	3.99
32	1.06×10^{-6}	2.98	5.34×10^{-9}	4.00

Example 5.2. The second example is an equation quoted from **[10]**:

$$x(s) = \int_0^1 \kappa(s,t)[\psi(t,x(t)) + y(t)], \quad s \in [0,1],$$

with Green's kernel

$$\kappa(s,t) = \begin{cases} -(1-t)s & s \le t, \\ -(1-s)t & t \le s, \end{cases}$$

and $y(t)$ chosen so that $x^*(s) = (s(1-s))/(s+1)$. In fact, this equation is the reformulation of the boundary problem:

$$\begin{aligned} x''(t) &= \psi(t,x(t)) + y(t), \quad 0 < t < 1, \\ x(0) &= x(1) = 0. \end{aligned}$$

We consider the particular example

$$\psi(t,u) = \frac{1}{1+t+u}.$$

For this equation, we have $\gamma = 0$, $\alpha = \infty$, $r = 1$ and $\beta = \beta_1 = \beta_2 = 1$. From Theorems 4.1 and 4.3, the expected orders of convergence are $\delta_1 = 3$ and $\delta_2 = 4$. The results are given in Tables 3 and 4.

TABLE 3. Orthogonal projection.

$\mathfrak{n}$	$\|x^* - x_n\|_\infty$	δ_1	$\|x^* - \widetilde{x}_n\|_\infty$	δ_2
2	2.02×10^{-3}		1.74×10^{-5}	
4	4.71×10^{-4}	2.10	1.89×10^{-6}	3.20
8	8.12×10^{-5}	2.54	1.51×10^{-7}	3.65
16	1.25×10^{-5}	2.70	9.15×10^{-9}	4.04
32	1.73×10^{-6}	2.85	6.84×10^{-10}	3.74

TABLE 4. Multi projection 1.

$\mathfrak{n}$	$\|x^* - x_n\|_\infty$	δ_1	$\|x^* - \widetilde{x}_n\|_\infty$	δ_2
2	1.09×10^{-3}		1.27×10^{-5}	
4	2.62×10^{-4}	2.05	1.40×10^{-6}	3.18
8	4.52×10^{-5}	2.54	1.12×10^{-7}	3.65
16	6.87×10^{-6}	2.72	7.36×10^{-9}	3.92
32	7.46×10^{-7}	3.20	4.54×10^{-10}	4.02

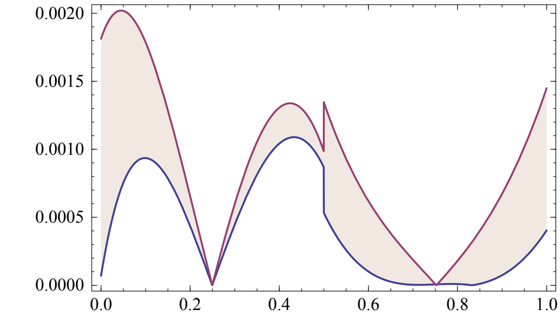

FIGURE 1. Errors of the two methods.

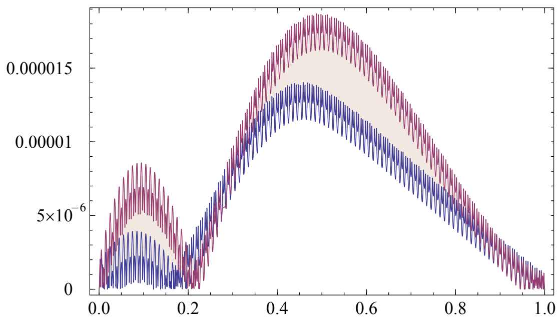

FIGURE 2. The iterated versions.

For the sake of completeness, we illustrate in Figures 1 and 2 the approximation errors $|x^*(s) - x_n(s)|$ and $|x^*(s) - \widetilde{x}_n(s)|$ obtained by the two methods (multi projection 1 in blue) with $n = 2$.

Example 5.3. In this example, we choose the next equation with discontinuous kernel along the diagonal, that is,

$$x(s) - \int_0^1 \kappa(s,t)x^2(t) = s - \tfrac{1}{4}(s - 2s^5), \quad s \in [0,1],$$

where

$$\kappa(s,t) = \begin{cases} st & s \leq t, \\ -st & t \leq s, \end{cases}$$

and the exact solution is $x^*(s) = s$. For this example, we have $\gamma = -1$,

TABLE 5. Multi projection 1.

$\mathfrak{n}$	$\|x^* - x_n\|_\infty$	δ_1	$\|x^* - \widetilde{x}_n\|_\infty$	δ_2
2	7.14×10^{-2}		8.04×10^{-3}	
4	2.16×10^{-2}	1.72	1.38×10^{-3}	2.54
8	5.63×10^{-3}	1.94	2.60×10^{-4}	2.41
16	1.28×10^{-3}	2.14	2.21×10^{-5}	3.56

TABLE 6. Multi projection 2.

$\mathfrak{n}$	$\|x^* - x_n\|_\infty$	δ_1	$\|x^* - \widetilde{x}_n\|_\infty$	δ_2
2	6.01×10^{-2}		1.01×10^{-2}	
4	2.03×10^{-2}	1.57	3.20×10^{-3}	1.66
8	5.76×10^{-3}	1.81	7.96×10^{-4}	2.01
16	1.77×10^{-3}	1.71	2.41×10^{-4}	1.72

$\alpha = \infty$, $r = 1$, $\beta_1 = 0$, and $\beta = \beta_2 = 1$. From Theorems 4.1 and 4.3, the expected orders of convergence are $\delta_1 = 2$ and $\delta_2 = 3$ for the multi projection 1 and $\delta_1 = 2$ for the multi projection 2. The results are given in Tables 5 and 6.

Note that the computed values of orders of convergence in all of the cases are as expected. The integrals appearing in the nonlinear systems have been evaluated by using the composite Gauss 2 point rule with respect to a uniform partition.

6. Conclusions. The method presented in this paper naturally extends to iterated schemes for less smooth kernels to further improve the order of convergence as well as multivariable integral equations. They may be extended to Hammerstein integral equations with weakly singular kernels. That is a consideration for future papers.

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