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ON A SEMI-LINEAR SYSTEM OF NONLOCAL TIME AND SPACE REACTION DIFFUSION EQUATIONS WITH EXPONENTIAL NONLINEARITIES

B. AHMAD, A. ALSAEDI, D. HNAIEN AND M. KIRANE

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ABSTRACT. In this article, we investigate the local existence of a unique mild solution to a reaction diffusion system with time-nonlocal nonlinearities of exponential growth. Moreover, blowing-up solutions are shown to exist, and their time blow-up profile is presented.

1. Introduction.

In the past few years, anomalous diffusion equations have been extensively investigated due to their importance of applications in many fields: physics [**6, 9, 10**], mechanics [**18**], biology [**14**], chemistry [**15, 16**], financial engineering [**21**], control theory [**20**] and signal and image processing [**5**].

In this paper, we study the semi-linear system of nonlocal in time and space reaction diffusion equations

(1.1)
$$\begin{cases} u_t + (-\Delta)^{\eta/2} u = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} e^{v(s)}\, ds, & x \in \mathbb{R}^N,\ t>0, \\ v_t + (-\Delta)^{\eta/2} v = \dfrac{1}{\Gamma(1-\delta)} \displaystyle\int_0^t (t-s)^{-\delta} e^{u(s)}\, ds, & x \in \mathbb{R}^N,\ t>0, \end{cases}$$

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supplemented with the initial data

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

with $u_0, v_0 \in C_0(\mathbb{R}^N)$, $N \geq 1$, $0 < \eta \leq 2$, $0 < \gamma$, $\delta < 1$ and Γ is the Euler gamma function. Here, u_t stands for the derivative in time of u and $(-\Delta)^{\eta/2}$ for the fractional Laplacian operator defined by

$$(-\Delta)^{\eta/2} u(x) = \mathcal{F}^{-1}(|\xi|^{\eta} \mathcal{F}(u)(\xi))(x),$$

where $\mathcal{F}^{-1}$ is the inverse of the Fourier transform $\mathcal{F}$, for $u \in D((-\Delta)^{\eta/2}) = H^{\eta}(\mathbb{R}^N)$, where $H^{\eta}(\mathbb{R}^N)$ is the homogeneous Sobolev space defined by:

$$H^{\eta}(\mathbb{R}^N) = \left\{ u \in \mathcal{S}'; \, (-\Delta)^{\eta/2} u \in L^2(\mathbb{R}^N) \right\}, \quad \text{if } \eta \notin \mathbb{N},$$
$$H^{\eta}(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N); \, (-\Delta)^{\eta/2} u \in L^2(\mathbb{R}^N) \right\}, \quad \text{if } \eta \in \mathbb{N},$$

$\mathcal{S}'$ is the Schwartz space and

$$C_0(\mathbb{R}^N) = \left\{ u \in C(\mathbb{R}^N) \text{ such that } u(x) \longrightarrow 0, \text{ as } |x| \to \infty \right\}.$$

From an application point of view, the nonlinear memory term of exponential growth can be considered as a model of the Arrhenius reaction effect associated with either chemical kinetics or combustion phenomena [**2**].

Recently, the equation

$$u_t + (-\Delta)^{\eta/2} u = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} e^{u(s)} ds, \quad x \in \mathbb{R}^N, \ t > 0,$$

was considered in [**1**] which addressed local existence and blow-up questions. Our work generalizes the previous results of Ahmad, Alsaedi and Kirane [**1**] to the case of a system of two equations.

Our study is based on the observation that system (1.1) can be written in the form

$$\begin{cases} u_t + (-\Delta)^{\eta/2} u = J^{\alpha}_{0|t}(e^v) & x \in \mathbb{R}^N, \ t > 0, \\ v_t + (-\Delta)^{\eta/2} v = J^{\beta}_{0|t}(e^u) & x \in \mathbb{R}^N, \ t > 0, \end{cases} \tag{1.3}$$

where $\alpha := 1 - \gamma \in (0,1)$, $\beta := 1 - \delta \in (0,1)$ and $J^{\theta}_{0|t}$ is the Riemann-Liouville fractional integral of order $\theta \in (0, 1)$ defined in (2.11) below.

This paper is comprised of five sections. In Section 2, we present some definitions and properties. In Section 3, the existence of a unique local solution of (1.1)–(1.2) is presented. In the next section, we prove the existence of blowing-up solutions. Finally, in the last section, the blow-up rate of solutions is obtained.

2. Preliminaries. We begin by recalling some basic definitions and properties which will be useful throughout this paper.

First, the fundamental solution of the homogeneous linear fractional diffusion equation

$$u_t + (-\Delta)^{\eta/2} u = 0, \quad \eta \in (0,2],\ x \in \mathbb{R}^N,\ t > 0, \tag{2.1}$$

is given by

$$S_\eta(t)(x) := S_\eta(x,t) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix\cdot\xi - t|\xi|^\eta} d\xi. \tag{2.2}$$

It is well known that $S_\eta(x,t)$ satisfies

$$S_\eta(1) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad S_\eta(x,t) \geq 0, \quad \int_{\mathbb{R}^N} S_\eta(x,t)\, dx = 1, \tag{2.3}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Using the Young inequality for convolution and the self-similar form of S_η, we have

$$\|S_\eta(t) * v\|_q \leq Ct^{-(N/\eta)(1/r-1/q)} \|v\|_r \tag{2.4}$$

for all $v \in L^r(\mathbb{R}^N)$ and all $1 \leq r \leq q \leq \infty$, $t > 0$;

$$\|S_\eta(t) * v\|_q \leq \|v\|_q \tag{2.5}$$

for all $v \in L^q(\mathbb{R}^N)$ and all $1 \leq q \leq \infty$, $t > 0$, where $*$ stands for the space convolution.

Moreover, by Plancherel's formula, we have

$$\int_{\mathbb{R}^N} u(x)(-\Delta)^{\eta/2} v(x)\, dx = \int_{\mathbb{R}^N} v(x)(-\Delta)^{\eta/2} u(x)\, dx, \tag{2.6}$$

for all u, $v \in D((-\Delta)^{\eta/2}) = H^\eta(\mathbb{R}^N)$.

Let φ be a nonnegative, smooth and bounded function. Then, the following inequality [**4, 11**]

$$l\varphi^{l-1}(-\Delta)^{\eta/2}\varphi \geq (-\Delta)^{\eta/2}\varphi^l, \tag{2.7}$$

holds for all $l \geq 1$.

Let $\mathcal{S}(t) = e^{-t(-\Delta)^{\eta/2}}$. Since $(-\Delta)^{\eta/2}$ is a positive definite self-adjoint operator in $L^2(\mathbb{R}^N)$, $\mathcal{S}(t)$ is a strongly continuous semigroup on $L^2(\mathbb{R}^N)$ generated by $-(-\Delta)^{\eta/2}$ (see Yosida [**22**]). It holds $\mathcal{S}(t)v = S_\eta(t) * v$ for all $v \in L^2(\mathbb{R}^N)$, $t > 0$, where S_η is given by (2.2).

We denote by $\Delta_D^{\eta/2}$ the fractional Laplacian in an open bounded domain Ω with homogeneous Dirichlet boundary conditions. We recall the following facts:

If $\lambda_k (k \in \mathbb{N}^*)$ are the eigenvalues of $-\Delta_D$ with homogeneous Dirichlet boundary conditions considered in $L^2(\Omega)$ and φ_k is its corresponding eigenfunction, then

$$\begin{aligned} \Delta_D^{\eta/2}\varphi_k &= \lambda_k^{\eta/2}\varphi_k, \quad &&\text{in } \Omega, \\ \varphi_k &= 0, &&\text{on } \partial\Omega, \end{aligned} \tag{2.8}$$

with

$$D(\Delta_D^{\eta/2}) = \Big\{ u \in L^2(\Omega) \text{ such that } u_{|\partial\Omega} = 0, \\ \|\Delta_D^{\eta/2} u\|_{L^2(\Omega)} := \sum_{k=1}^{\infty} |\lambda_k^{\eta/2}\langle u, \varphi_k\rangle|^2 < \infty \Big\}.$$

Then, for $u \in D(\Delta_D^{\eta/2})$, we have

$$\Delta_D^{\eta/2} u = \sum_{k=1}^{\infty} \lambda_k^{\eta/2}\langle u, \varphi_k\rangle \varphi_k. \tag{2.9}$$

The formula of integration by parts

$$\int_\Omega u(x)\Delta_D^{\eta/2} v(x)dx = \int_\Omega v(x)\Delta_D^{\eta/2} u(x)\, dx \tag{2.10}$$

holds true for all $u,\ v \in D(\Delta_D^{\eta/2})$.

Next, the left- and right-handed Riemann-Liouville fractional integrals of order $\theta \in (0,1)$ are defined as

$$J^\theta_{0|t} f(t) := \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s)\, ds, \tag{2.11}$$

$$J^\theta_{t|T} f(t) := \frac{1}{\Gamma(\theta)} \int_t^T (s-t)^{\theta-1} f(s)\, ds, \tag{2.12}$$

for all $f \in L^p([0,T])$, $T > 0$, $1 \leq p \leq \infty$ and Γ is the Euler gamma function.

Let $AC([0,T])$ be the space of functions that is absolutely continuous on $[0,T]$. The left- and right-handed Riemann-Liouville fractional derivatives of order $\theta \in (0,1)$ are defined as

$$D^\theta_{0|t} f(t) := \frac{1}{\Gamma(1-\theta)} \frac{d}{dt} \int_0^t (t-s)^{-\theta} f(s)\, ds, \tag{2.13}$$

$$D^\theta_{t|T} f(t) := -\frac{1}{\Gamma(1-\theta)} \frac{d}{dt} \int_t^T (s-t)^{-\theta} f(s)\, ds, \tag{2.14}$$

for all $f \in AC([0,T])$. Furthermore, for every f, $g \in C([0,T])$ such that $D^\theta_{0|t} f, D^\theta_{t|T} g$ exist and are continuous, for all $t \in [0,T]$ and $\theta \in (0,1)$, the formula of integration by parts can be given by **[19]**

$$\int_0^T (D^\theta_{0|t} f)(t) g(t)\, dt = \int_0^T f(t) (D^\theta_{t|T} g)(t)\, dt. \tag{2.15}$$

Note also that, for all $f \in L^p([0,T])$, $1 \leq p \leq \infty$, we have **[12]**

$$(D^\theta_{0|t} J^\theta_{0|t} f)(t) = f(t). \tag{2.16}$$

Moreover, for all $f \in AC^2([0,T])$, we have **[12]**

$$-\frac{d}{dt} D^\theta_{t|T} f(t) = D^{1+\theta}_{t|T} f(t), \tag{2.17}$$

where $AC^2([0,T]) := \{f : [0,T] \to \mathbb{R};\ f \in AC([0,T]) \text{ and } f' \in AC([0,T])\}$.

Let $w_1(t) = (1 - t/T)^\sigma_+$, $t \geq 0$, $T > 0$, $\sigma \gg 1$. Then

$$D^\theta_{t|T} w_1(t) = \frac{(1-\theta+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\theta+\sigma)} T^{-\theta} \left(1 - \frac{t}{T}\right)^{\sigma-\theta}_+, \tag{2.18}$$

(2.19)
$$D^{\theta+1}_{t|T} w_1(t) = \frac{(1-\theta+\sigma)(\sigma-\theta)\Gamma(\sigma+1)}{\Gamma(2-\theta+\sigma)} T^{-(\theta+1)} \left(1-\frac{t}{T}\right)_+^{\sigma-\theta-1},$$

for all $\theta \in (0,1)$; hence,

$$(D^{\theta}_{t|T} w_1)(T) = 0; \qquad (D^{\theta}_{t|T} w_1)(0) = CT^{-\theta}, \tag{2.20}$$

where

$$C = \frac{(1-\theta+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\theta+\sigma)}. \tag{2.21}$$

3. Local existence. In this section, we show the local existence and uniqueness of a mild solution of (1.1)–(1.2) by applying the Banach fixed point theorem. We define a mild solution of (1.1)–(1.2) as follows.

Definition 3.1 (Mild solution)**.** Let $u_0,\ v_0 \in L^\infty(\mathbb{R}^N)$, $0 < \eta \leq 2$ and $T > 0$. We say that $(u,v) \in C([0,T]; C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))$ is a mild solution of system (1.1)–(1.2) if (u,v) satisfies the following integral equations

$$u(t) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-s) J^{\alpha}_{0|s}(e^{v(\tau)})\, ds, \quad t \in [0,T], \tag{3.1}$$

$$v(t) = \mathcal{S}(t)v_0 + \int_0^t \mathcal{S}(t-s) J^{\beta}_{0|s}(e^{u(\tau)})\, ds, \quad t \in [0,T]. \tag{3.2}$$

Theorem 3.2 (Local existence)**.** *Let* $u_0,\ v_0 \in C_0(\mathbb{R}^N)$. *Then, there exist a maximal time* $T_{\max} > 0$ *and a unique mild solution* $(u,v) \in C([0,T_{\max}); C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))$ *of problem* (1.1)–(1.2). *Furthermore, we have the alternative*:

(i) *either* $T_{\max} = +\infty$;

or

(ii) $T_{\max} < +\infty$ *and* $\lim_{t\to T_{\max}}(\|u(t)\|_{L^\infty(\mathbb{R}^N)} + \|v(t)\|_{L^\infty(\mathbb{R}^N)}) = +\infty$. *In addition, if* $u_0,\ v_0 \geq 0$, $u_0 \not\equiv 0$, $v_0 \not\equiv 0$, *then* $u(t),\ v(t) > 0$ *for all* $0 < t < T_{\max}$. *Furthermore, if* $u_0,\ v_0 \in L^r(\mathbb{R}^N)$, *for* $1 \leq r < \infty$, *then* $(u,v) \in C([0,T_{\max}); L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N))$.

Proof. For $T > 0$, we define the Banach space

$$E_T = \Big\{(u,v) \in C([0,T]; C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N)); \quad \|(u,v)\| \le 2(\|u_0\|_\infty + \|v_0\|_\infty)\Big\}, \tag{3.3}$$

where $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R}^N)}$ and

$$\|(u,v)\| := \|u\|_1 + \|v\|_1 := \|u\|_{L^\infty([0,T];L^\infty(\mathbb{R}^N))} + \|v\|_{L^\infty([0,T];L^\infty(\mathbb{R}^N))}.$$

Next, for every $(u,v) \in E_T$, we introduce the map Ψ defined on E_T by $\Psi(u,v) = (\Psi_1(u,v), \Psi_2(u,v))$, where

$$\Psi_1(u,v) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-s) J^\alpha_{0|s}(e^{v(\tau)})\, ds, \tag{3.4}$$

$$\Psi_2(u,v) = \mathcal{S}(t)v_0 + \int_0^t \mathcal{S}(t-s) J^\beta_{0|s}(e^{u(\tau)})\, ds. \tag{3.5}$$

We shall prove the existence of a local solution as a fixed point of Ψ via the Banach fixed point theorem. For this purpose, we first show that it maps E_T onto E_T. Let $(u,v) \in E_T$. Using (2.5), we obtain

$$\begin{aligned}
\|\Psi(u,v)\| &\le \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)}\left\| \int_0^t\int_0^s (s-\tau)^{-\gamma}\|e^{v(\tau)}\|_\infty \, d\tau\, ds\right\|_{L^\infty([0,T])} \\
&\quad + \|v_0\|_\infty + \frac{1}{\Gamma(1-\delta)}\left\| \int_0^t\int_0^s (s-\tau)^{-\delta}\|e^{u(\tau)}\|_\infty \, d\tau\, ds\right\|_{L^\infty([0,T])} \\
&\le \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)}\left\| \int_0^t\int_\tau^t (s-\tau)^{-\gamma} e^{\|v(\tau)\|_\infty} \, ds\, d\tau\right\|_{L^\infty([0,T])} \\
&\quad + \|v_0\|_\infty + \frac{1}{\Gamma(1-\delta)}\left\| \int_0^t\int_\tau^t (s-\tau)^{-\delta} e^{\|u(\tau)\|_\infty} \, ds\, d\tau\right\|_{L^\infty([0,T])} \\
&\le \|u_0\|_\infty + \frac{e^{\|v\|_1}}{\Gamma(1-\gamma)} \frac{T^{-\gamma+2}}{(1-\gamma)(2-\gamma)} \\
&\quad + \|v_0\|_\infty + \frac{e^{\|u\|_1}}{\Gamma(1-\delta)} \frac{T^{-\delta+2}}{(1-\delta)(2-\delta)}
\end{aligned}$$

$$\leq \|u_0\|_\infty+\|v_0\|_\infty+\frac{1}{\Gamma(3-\gamma)}e^{\|v\|_1}T^{-\gamma+2}+\frac{1}{\Gamma(3-\delta)}e^{\|u\|_1}T^{-\delta+2}$$
$$\leq \|u_0\|_\infty+\|v_0\|_\infty+e^{2(\|u_0\|_\infty+\|v_0\|_\infty)}\max\left\{\frac{T^{-\gamma+2}}{\Gamma(3-\gamma)},\frac{T^{-\delta+2}}{\Gamma(3-\delta)}\right\}.$$

Now, choosing T small enough such that

(3.6)
$$e^{2(\|u_0\|_\infty+\|v_0\|_\infty)}\max\left\{\frac{1}{\Gamma(3-\gamma)}T^{-\gamma+2},\ \frac{1}{\Gamma(3-\delta)}T^{-\delta+2}\right\}\leq \|u_0\|_\infty+\|v_0\|_\infty,$$

we conclude that $\Psi(u,v)\in E_T$.

Next, we show that $\Psi(u,v)$ is a contraction map. Letting (u,v), $(\widetilde{u},\widetilde{v})\in E_T$, we have

$$\begin{aligned}
&\|\Psi(u,v)-\Psi(\widetilde{u},\widetilde{v})\|\\
&\quad\leq \left\|\int_0^t \mathcal{S}(t-s)J^{\alpha}_{0|s}(e^{v(\tau)}-e^{\widetilde{v}(\tau)})\,ds\right\|_1\\
&\qquad+\left\|\int_0^t \mathcal{S}(t-s)J^{\beta}_{0|s}(e^{u(\tau)}-e^{\widetilde{u}(\tau)})\,ds\right\|_1\\
&\quad\leq \frac{1}{\Gamma(1-\gamma)}\left\|\int_0^t\int_\tau^t (s-\tau)^{-\gamma}\|e^{v(\tau)}-e^{\widetilde{v}(\tau)}\|_\infty\,ds\,d\tau\right\|_{L^\infty([0,T])}\\
&\qquad+\frac{1}{\Gamma(1-\delta)}\left\|\int_0^t\int_\tau^t (s-\tau)^{-\delta}\|e^{u(\tau)}-e^{\widetilde{u}(\tau)}\|_\infty\,ds\,d\tau\right\|_{L^\infty([0,T])}\\
&\quad\leq \frac{1}{\Gamma(3-\gamma)}e^{2(\|u_0\|_\infty+\|v_0\|_\infty)}T^{-\gamma+2}\|v-\widetilde{v}\|_1\\
&\quad+\frac{1}{\Gamma(3-\delta)}e^{2(\|u_0\|_\infty+\|v_0\|_\infty)}T^{-\delta+2}\|u-\widetilde{u}\|_1\\
&\quad\leq e^{2(\|u_0\|_\infty+\|v_0\|_\infty)}\max\left\{\frac{T^{-\gamma+2}}{\Gamma(3-\gamma)},\ \frac{T^{-\delta+2}}{\Gamma(3-\delta)}\right\}\|(u,v)-(\widetilde{u},\widetilde{v})\|\\
&\quad\leq \frac{1}{2}\|(u,v)-(\widetilde{u},\widetilde{v})\|,
\end{aligned}$$

due to the equality

(3.7)
$$|e^{u(\tau)}-e^{\widetilde{u}(\tau)}|=e^{\lambda u(\tau)+\mu\widetilde{u}(\tau)}|u(\tau)-\widetilde{u}(\tau)|,\quad 0<\lambda,\,\mu<1,\,\lambda+\mu=1,$$

and where T is chosen small enough such that

$$e^{2(\|u_0\|_\infty+\|v_0\|_\infty)}\max\left\{\frac{1}{\Gamma(3-\gamma)}T^{-\gamma+2},\ \frac{1}{\Gamma(3-\delta)}T^{-\delta+2}\right\}\le\frac{1}{2}. \tag{3.8}$$

Consequently, by the Banach fixed point theorem, system (1.1)–(1.2) admits a mild solution $(u,v)\in E_T$.

• *Uniqueness of the solution.* Let (u,v), $(\widetilde{u},\widetilde{v})\in E_T$ be two mild solutions in E_T for $T>0$. Using (2.5) and (3.7), we have, for $t\in[0,T]$,

$$\begin{aligned}
&\|u(t)-\widetilde{u}(t)\|_\infty+\|v(t)-\widetilde{v}(t)\|_\infty\\
&\quad\le\frac{1}{\Gamma(1-\gamma)}\int_0^t\int_0^s(s-\tau)^{-\gamma}\|e^{v(\tau)}-e^{\widetilde{v}(\tau)}\|_\infty\,d\tau\,ds\\
&\qquad+\frac{1}{\Gamma(1-\delta)}\int_0^t\int_0^s(s-\tau)^{-\delta}\|e^{u(\tau)}-e^{\widetilde{u}(\tau)}\|_\infty\,d\tau\,ds\\
&\le\frac{1}{\Gamma(1-\gamma)}\int_0^t\int_\tau^t(s-\tau)^{-\gamma}\|e^{v(\tau)}-e^{\widetilde{v}(\tau)}\|_\infty\,ds\,d\tau\\
&\qquad+\frac{1}{\Gamma(1-\delta)}\int_0^t\int_\tau^t(s-\tau)^{-\delta}\|e^{u(\tau)}-e^{\widetilde{u}(\tau)}\|_\infty\,ds\,d\tau\\
&\le\frac{1}{\Gamma(2-\gamma)}\int_0^t(t-\tau)^{1-\gamma}\|e^{v(\tau)}-e^{\widetilde{v}(\tau)}\|_\infty\,d\tau\\
&\qquad+\frac{1}{\Gamma(2-\delta)}\int_0^t(t-\tau)^{1-\delta}\|e^{u(\tau)}-e^{\widetilde{u}(\tau)}\|_\infty\,d\tau\\
&\quad\le\frac{1}{\Gamma(2-\gamma)}e^{2(\|u_0\|_\infty+\|v_0\|_\infty)}\int_0^t(t-\tau)^{1-\gamma}\|v(\tau)-\widetilde{v}(\tau)\|_\infty\,d\tau\\
&\qquad+\frac{1}{\Gamma(2-\delta)}e^{2(\|u_0\|_\infty+\|v_0\|_\infty)}\int_0^t(t-\tau)^{1-\delta}\|u(\tau)-\widetilde{u}(\tau)\|_\infty\,d\tau.
\end{aligned}$$

Hence, for

$$K:=e^{2(\|u_0\|_\infty+\|v_0\|_\infty)}\max\left\{\frac{1}{\Gamma(2-\gamma)},\frac{1}{\Gamma(2-\delta)}\right\}$$

and

$$f(\gamma,\delta):=\begin{cases}\min\{\gamma,\delta\} & \text{if } (t-\tau)>1,\\ \max\{\gamma,\delta\} & \text{if } (t-\tau)<1,\end{cases}$$

we conclude that

$$\begin{aligned}&\|u(t)-\widetilde{u}(t)\|_\infty+\|v(t)-\widetilde{v}(t)\|_\infty\\&\qquad\le K\int_0^t (t-\tau)^{1-f(\gamma,\delta)}\big[\|u(\tau)-\widetilde{u}(\tau)\|_\infty+\|v(\tau)-\widetilde{v}(\tau)\|_\infty\big]\,d\tau,\end{aligned}$$

and, by Gronwall's inequality **[3]**, we obtain the uniqueness.

As is standard, the solution may be extended to a maximal interval $[0,T_{\max})$ with the alternative described in the theorem.

• *Positivity of solutions.* If $u_0,\ v_0\ge 0$ and $u_0\not\equiv 0$, $v_0\not\equiv 0$, we have from (3.1) and (3.2),

$$\begin{aligned}u(t)\ge \mathcal{S}(t)u_0>0,\quad t\in(0,T_{\max}),\\ v(t)\ge \mathcal{S}(t)v_0>0,\quad t\in(0,T_{\max}).\end{aligned}$$

• *Regularity of solutions.* If $u_0,\ v_0\in L^r(\mathbb{R}^N)$, for $1\le r<\infty$, then applying the fixed point argument in the space

$$\begin{aligned}E_{T,r}:=\{&(u,v)\in C([0,T];(C_0(\mathbb{R}^N)\cap L^r(\mathbb{R}^N))\times(C_0(\mathbb{R}^N)\cap L^r(\mathbb{R}^N)));\\&\|(u,v)\|\le 2(\|u_0\|_\infty+\|v_0\|_\infty),\ \|(u,v)\|_{\infty,r}\le 2(\|u_0\|_{L^r}+\|v_0\|_{L^r})\},\end{aligned}$$

instead of E_T, where

$$\|(u,v)\|_{\infty,r}=\|u\|_{L^\infty([0,T];L^r(\mathbb{R}^N))}+\|v\|_{L^\infty([0,T];L^r(\mathbb{R}^N))},$$

we obtain a unique mild solution (u,v) in $E_{T,r}$. □

4. Blowing-up solutions. In this section, we prove the blow-up result for system (1.1)–(1.2). Before stating our result, we define the weak solution of problem (1.1)–(1.2).

Definition 4.1 (Weak solution)**.** Let $u_0,\ v_0\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and $T>0$. We say that

$$(u,v)\in L^p((0,T);L^\infty_{\mathrm{loc}}(\mathbb{R}^N)\times L^\infty_{\mathrm{loc}}(\mathbb{R}^N))$$

is a weak solution of (1.1)–(1.2) if it satisfies the following equations

$$\begin{aligned}(4.1)\quad &\int_{\mathbb{R}^N}u_0(x)\varphi(x,0)\,dx+\int_0^T\int_{\mathbb{R}^N}J^\alpha_{0|t}(e^{v(x,\tau)})\varphi(x,t)\,dx\,dt\\&=\int_0^T\int_{\mathbb{R}^N}u(x,t)(-\Delta)^{\eta/2}\varphi(x,t)\,dx\,dt\\&\qquad-\int_0^T\int_{\mathbb{R}^N}u(x,t)\varphi_t(x,t)\,dx\,dt,\end{aligned}$$

and

$$\int_{\mathbb{R}^N} v_0(x)\psi(x,0)\,dx + \int_0^T\int_{\mathbb{R}^N} J^{\beta}_{0|t}(e^{u(x,\tau)})\psi(x,t)\,dx\,dt = \int_0^T\int_{\mathbb{R}^N} v(x,t)(-\Delta)^{\eta/2}\psi(x,t)\,dx\,dt - \int_0^T\int_{\mathbb{R}^N} v(x,t)\psi_t(x,t)\,dx\,dt, \tag{4.2}$$

for any $\varphi,\ \psi \in C^1([0,T];H^{\eta}(\mathbb{R}^N))$ such that $\varphi(x,T)=\psi(x,T)=0$.

Lemma 4.2. *Let $T>0$ and $(u,v)\in C([0,T],C_0(\mathbb{R}^N)\times C_0(\mathbb{R}^N))$. If (u,v) is a mild solution of* (1.1)–(1.2)*, then (u,v) is a weak solution of* (1.1)–(1.2)*.*

The interested reader is referred to **[7]** for the proof of this lemma.

Theorem 4.3. *Let $u_0,\ v_0 \in C_0(\mathbb{R}^N)$ with $u_0\ge 0$, $u_0\not\equiv 0$, $v_0\ge 0$, $v_0\not\equiv 0$, $0<\eta\le 2$ and $\gamma,\ \delta\in(0,1)$. Then the solution of* (1.1)–(1.2) *blows-up in a finite time.*

Proof. The proof is based on a contradiction argument and follows along the lines of **[17]**. Suppose that (u,v) is a global mild solution of (1.1)–(1.2). Then, (u,v) is a mild solution of (1.1)–(1.2) where $u,\ v\in C([0,T],C_0(\mathbb{R}^N))$ for all $T\gg 1$, such that $u(t),\ v(t)>0$ for all $t\in(0,T]$. Moreover, according to Lemma 4.2, we have

$$\int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\,dx + \int_0^T\int_{\mathbb{R}^N} J^{\alpha}_{0|t}(e^{v(x,\tau)})\varphi(x,t)\,dx\,dt = \int_0^T\int_{\mathbb{R}^N} u(x,t)(-\Delta)^{\eta/2}\varphi(x,t)\,dx\,dt - \int_0^T\int_{\mathbb{R}^N} u(x,t)\varphi_t(x,t)\,dx\,dt, \tag{4.3}$$

and

$$\int_{\mathbb{R}^N} v_0(x)\psi(x,0)\,dx + \int_0^T\int_{\mathbb{R}^N} J^{\beta}_{0|t}(e^{u(x,\tau)})\psi(x,t)\,dx\,dt = \int_0^T\int_{\mathbb{R}^N} v(x,t)(-\Delta)^{\eta/2}\psi(x,t)\,dx\,dt - \int_0^T\int_{\mathbb{R}^N} v(x,t)\psi_t(x,t)\,dx\,dt, \tag{4.4}$$

where $T \gg 1$, φ, $\psi \in C^1([0,T];H^\eta(\mathbb{R}^N))$ such that $\varphi(x,T) = \psi(x,T) = 0$. Let $\varphi \in C^2([0,T];H^\eta(\mathbb{R}^N))$ with

$$\varphi(x,t) = \varphi_1(t)\varphi_2^l(x), \quad l \gg 1,$$

where

$$\varphi_1(t) = \left(1 - \frac{t}{T}\right)_+^\sigma, \quad \sigma \gg 1,\ \sigma \text{ even},$$

$$\varphi_2(x) = \xi\left(\frac{|x|}{T^{1/\eta}}\right),$$

and ξ is a regular function such that

$$\xi(x) = \begin{cases} 1 & \text{if } x \leq 1, \\ \searrow & \text{if } 1 \leq x \leq 2, \\ 0 & \text{if } x \geq 2, \end{cases}$$

with $\xi \in C^2(\mathbb{R})$.

At this stage, we take $D^\alpha_{t|T}\varphi = \varphi_2^l(x)D^\alpha_{t|T}(1-(t/T))_+^\sigma$ instead of φ in (4.3) and $D^\beta_{t|T}\varphi = \varphi_2^l(x)D^\beta_{t|T}(1-(t/T))_+^\sigma$ instead of ψ in (4.4). This yields

$$\begin{aligned} &\int_\Omega u_0(x)D^\alpha_{t|T}\varphi(x,0) + \int_{\Omega_T} J^\alpha_{0|t}(e^{v(x,\tau)})D^\alpha_{t|T}\varphi(x,t) \\ &= \int_{\Omega_T} u(x,t)(-\Delta)^{\eta/2}D^\alpha_{t|T}\varphi(x,t) \\ &\quad - \int_{\Omega_T} u(x,t)\frac{d}{dt}D^\alpha_{t|T}\varphi(x,t), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\int_\Omega v_0(x)D^\beta_{t|T}\varphi(x,0) + \int_{\Omega_T} J^\beta_{0|t}(e^{u(x,\tau)})D^\beta_{t|T}\varphi(x,t) \\ &= \int_{\Omega_T} v(x,t)(-\Delta)^{\eta/2}D^\beta_{t|T}\varphi(x,t) \\ &\quad - \int_{\Omega_T} v(x,t)\frac{d}{dt}D^\beta_{t|T}\varphi(x,t), \end{aligned} \tag{4.6}$$

with $\Omega_T = [0,T] \times \Omega$, $\Omega = \{x \in \mathbb{R}^N;\ |x| \le 2T^{1/\eta}\}$,

$$\int_\Omega = \int_\Omega dx \quad \text{and} \quad \int_{\Omega_T} = \int_0^T \int_\Omega dx\, dt.$$

Using the integration-by-parts formula (2.15), (2.16) and (2.20) on the left-hand sides of (4.5) and (4.6), and (2.17) on the right-hand side, we obtain

$$\begin{aligned} C_1 T^{-\alpha} \int_\Omega u_0(x)\varphi_2^l(x) + \int_{\Omega_T} e^{v(x,t)}\varphi(x,t) &\\ = \int_{\Omega_T} u(x,t)(-\Delta)^{\eta/2}\varphi_2^l(x) D^\alpha_{t|T}\varphi_1(t) &\\ + \int_{\Omega_T} u(x,t) D^{1+\alpha}_{t|T}\varphi(x,t), \end{aligned}$$

and

$$\begin{aligned} C_2 T^{-\beta} \int_\Omega v_0(x)\varphi_2^l(x) + \int_{\Omega_T} e^{u(x,t)}\varphi(x,t) &\\ = \int_{\Omega_T} v(x,t)(-\Delta)^{\eta/2}\varphi_2^l(x) D^\beta_{t|T}\varphi_1(t) &\\ + \int_{\Omega_T} v(x,t) D^{1+\beta}_{t|T}\varphi(x,t). \end{aligned}$$

Moreover, in light of (2.7) and the properties of φ_2, we have

$$\begin{aligned} C_1 T^{-\alpha} \int_\Omega u_0(x)\varphi_2^l(x) + \int_{\Omega_T} e^{v(x,t)}\varphi(x,t) &\\ \le l \int_{\Omega_T} u(x,t)\varphi_2^{l-1}(x)(-\Delta)^{\eta/2}\varphi_2(x) D^\alpha_{t|T}\varphi_1(t) &\\ + \int_{\Omega_T} u(x,t)\varphi_2^l(x) D^{1+\alpha}_{t|T}\varphi_1(t), &\\ \le l \int_{\Omega_T} u(x,t)|(-\Delta)^{\eta/2}\varphi_2(x)| D^\alpha_{t|T}\varphi_1(t) &\\ + \int_{\Omega_T} u(x,t) D^{1+\alpha}_{t|T}\varphi_1(t) = l\mathcal{I}_1 + \mathcal{J}_1, \end{aligned} \tag{4.7}$$

and

$$
\begin{aligned}
(4.8)\quad C_2T^{-\beta}\int_\Omega v_0(x)\varphi_2^l(x)+\int_{\Omega_T} e^{u(x,t)}\varphi(x,t)
&\le l\int_{\Omega_T} v(x,t)\varphi_2^{l-1}(x)(-\Delta)^{\eta/2}\varphi_2(x)D^{\beta}_{t|T}\varphi_1(t)\\
&\quad+\int_{\Omega_T} v(x,t)\varphi_2^l(x)D^{1+\beta}_{t|T}\varphi_1(t),\\
&\le l\int_{\Omega_T} v(x,t)|(-\Delta)^{\eta/2}\varphi_2(x)|D^{\beta}_{t|T}\varphi_1(t)\\
&\quad+\int_{\Omega_T} v(x,t)D^{1+\beta}_{t|T}\varphi_1(t)=l\mathcal{I}_2+\mathcal{J}_2.
\end{aligned}
$$

Using Young's inequality [**8**] ($e=\exp(1)$)

$$AB\le \varepsilon e^A+B\ln\frac{B}{e\,\varepsilon}\quad \text{for } A,B>0,\ \varepsilon>0,$$

with $\varepsilon=1/(4l)\varphi(x,t)$, $A=u$ and $B=|(-\Delta)^{\eta/2}\varphi_2(x)|D^{\alpha}_{t|T}\varphi_1(t)$ in $\mathcal{I}_1$, we obtain

$$
\begin{aligned}
\mathcal{I}_1\le \int_{\Omega_T}|(-\Delta)^{\eta/2}\varphi_2(x)|D^{\alpha}_{t|T}\varphi_1(t)\ln\left(\frac{4l|(-\Delta)^{\eta/2}\varphi_2(x)|D^{\alpha}_{t|T}\varphi_1(t)}{e\varphi_2^l(x)\varphi_1(t)}\right)\\
+\frac{1}{4l}\int_{\Omega_T} e^{u(x,t)}\varphi(x,t).
\end{aligned}
$$

Similarly, for $\mathcal{J}_1$ with $\varepsilon=(1/4)\varphi(x,t)$, $A=u$ and $B=D^{1+\alpha}_{t|T}\varphi_1(t)$, we obtain

$$\mathcal{J}_1\le\frac{1}{4}\int_{\Omega_T} e^{u(x,t)}\varphi(x,t)+\int_{\Omega_T} D^{1+\alpha}_{t|T}\varphi_1(t)\ln\left(\frac{4D^{1+\alpha}_{t|T}\varphi_1(t)}{e\varphi_2^l(x)\varphi_1(t)}\right).$$

Then, from (2.18) and (2.19), it follows that

$$
\begin{aligned}
(4.9)\quad \mathcal{I}_1\le \int_{\Omega_T}&|(-\Delta)^{\eta/2}\varphi_2(x)|D^{\alpha}_{t|T}\varphi_1(t)\\
&\cdot\ln\left(\frac{4lC_1|(-\Delta)^{\eta/2}\varphi_2(x)|T^{-\alpha}(1-(t/T))_+^{-\alpha}}{e\varphi_2^l(x)}\right)\\
&\qquad+\frac{1}{4l}\int_{\Omega_T} e^{u(x,t)}\varphi(x,t),
\end{aligned}
$$

and

$$\mathcal{J}_1 \le \frac{1}{4}\int_{\Omega_T} e^{u(x,t)}\varphi(x,t) + \int_{\Omega_T} D_{t|T}^{1+\alpha}\varphi_1(t) \cdot \ln\left(\frac{4C_3 T^{-(\alpha+1)}(1-(t/T))_+^{-\alpha-1}}{e\varphi_2^l(x)}\right), \tag{4.10}$$

where

$$C_1 = \frac{(1-\alpha+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}$$

and

$$C_3 = \frac{(1-\alpha+\sigma)(\sigma-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)}.$$

A similar argument applied to $\mathcal{I}_2$ with $\varepsilon = 1/(4l)\varphi(x,t)$, $A = v$ and $B = |(-\Delta)^{\eta/2}\varphi_2(x)|D_{t|T}^{\beta}\varphi_1(t)$, and for $\mathcal{J}_2$ with $\varepsilon = (1/4)\varphi(x,t)$, $A = v$ and $B = D_{t|T}^{1+\beta}\varphi_1(t)$, gives

$$\mathcal{I}_2 \le \int_{\Omega_T} |(-\Delta)^{\eta/2}\varphi_2(x)|D_{t|T}^{\beta}\varphi_1(t) \cdot \ln\left(\frac{4lC_2|(-\Delta)^{\eta/2}\varphi_2(x)|T^{-\beta}(1-(t/T))_+^{-\beta}}{e\varphi_2^l(x)}\right) + \frac{1}{4l}\int_{\Omega_T} e^{v(x,t)}\varphi(x,t), \tag{4.11}$$

and

$$\mathcal{J}_2 \le \frac{1}{4}\int_{\Omega_T} e^{v(x,t)}\varphi(x,t) + \int_{\Omega_T} D_{t|T}^{1+\beta}\varphi_1(t) \cdot \ln\left(\frac{4C_4 T^{-(\beta+1)}(1-(t/T))_+^{-\beta-1}}{e\varphi_2^l(x)}\right), \tag{4.12}$$

where

$$C_2 = \frac{(1-\beta+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\beta+\sigma)}$$

and

$$C_4 = \frac{(1-\beta+\sigma)(\sigma-\beta)\Gamma(\sigma+1)}{\Gamma(2-\beta+\sigma)}.$$

Using (4.9) and (4.10), inequality (4.7) becomes

$$
\begin{aligned}
&C_1T^{-\alpha}\int_\Omega u_0(x)\varphi_2^l(x)+\int_{\Omega_T}e^{v(x,t)}\varphi(x,t)\leq\frac{1}{2}\int_{\Omega_T}e^{u(x,t)}\varphi(x,t)\\
&\qquad+l\int_{\Omega_T}|(-\Delta)^{\eta/2}\varphi_2(x)|D^{\alpha}_{t|T}\varphi_1(t)\\
&\qquad\cdot\ln\left(\frac{4lC_1|(-\Delta)^{\eta/2}\varphi_2(x)|T^{-\alpha}(1-(t/T))_+^{-\alpha}}{e\varphi_2^l(x)}\right)+\int_{\Omega_T}D^{1+\alpha}_{t|T}\varphi_1(t)\\
&\qquad\cdot\ln\left(\frac{4C_3T^{-(\alpha+1)}(1-(t/T))_+^{-\alpha-1}}{e\varphi_2^l(x)}\right).
\end{aligned}
\tag{4.13}
$$

Similarly, taking into account (4.11) and (4.12), inequality (4.8) becomes

$$
\begin{aligned}
&C_2T^{-\beta}\int_\Omega v_0(x)\varphi_2^l(x)+\int_{\Omega_T}e^{u(x,t)}\varphi(x,t)\leq\frac{1}{2}\int_{\Omega_T}e^{v(x,t)}\varphi(x,t)\\
&\qquad+l\int_{\Omega_T}|(-\Delta)^{\eta/2}\varphi_2(x)|D^{\beta}_{t|T}\varphi_1(t)\\
&\qquad\cdot\ln\left(\frac{4lC_2|(-\Delta)^{\eta/2}\varphi_2(x)|T^{-\beta}(1-(t/T))_+^{-\beta}}{e\varphi_2^l(x)}\right)\\
&\qquad+\int_{\Omega_T}D^{1+\beta}_{t|T}\varphi_1(t)\ln\left(\frac{4C_4T^{-(\beta+1)}(1-(t/T))_+^{-\beta-1}}{e\varphi_2^l(x)}\right).
\end{aligned}
\tag{4.14}
$$

Now, combining (4.13) and (4.14) and as u_0, $v_0\geq 0$, we get

$$
\begin{aligned}
&\int_{\Omega_T}e^{v(x,t)}\varphi(x,t)\\
&\quad\leq\frac{4}{3}l\int_{\Omega_T}|(-\Delta)^{\eta/2}\varphi_2(x)|D^{\alpha}_{t|T}\varphi_1(t)\\
&\qquad\cdot\ln\left(\frac{4lC_1|(-\Delta)^{\eta/2}\varphi_2(x)|T^{-\alpha}(1-(t/T))_+^{-\alpha}}{e\varphi_2^l(x)}\right)\\
&\qquad+\frac{4}{3}\int_{\Omega_T}D^{1+\alpha}_{t|T}\varphi_1(t)\ln\left(\frac{4C_3T^{-(\alpha+1)}(1-(t/T))_+^{-\alpha-1}}{e\varphi_2^l(x)}\right)\\
&\qquad+\frac{2}{3}l\int_{\Omega_T}|(-\Delta)^{\eta/2}\varphi_2(x)|D^{\beta}_{t|T}\varphi_1(t)
\end{aligned}
\tag{4.15}
$$

$$\cdot \ln\left(\frac{4lC_2|(-\Delta)^{\eta/2}\varphi_2(x)|T^{-\beta}(1-(t/T))_+^{-\beta}}{e\varphi_2^l(x)}\right)$$
$$+\frac{2}{3}\int_{\Omega_T} D_{t|T}^{1+\beta}\varphi_1(t)\ln\left(\frac{4C_4T^{-(\beta+1)}(1-(t/T))_+^{-\beta-1}}{e\varphi_2^l(x)}\right),$$

and

(4.16)

$$\int_{\Omega_T} e^{u(x,t)}\varphi(x,t) \le \frac{4}{3}l\int_{\Omega_T}|(-\Delta)^{\eta/2}\varphi_2(x)|D_{t|T}^{\beta}\varphi_1(t)$$
$$\cdot \ln\left(\frac{4lC_2|(-\Delta)^{\eta/2}\varphi_2(x)|T^{-\beta}(1-(t/T))_+^{-\beta}}{e\varphi_2^l(x)}\right)$$
$$+\frac{4}{3}\int_{\Omega_T} D_{t|T}^{1+\beta}\varphi_1(t)\ln\left(\frac{4C_4T^{-(\beta+1)}(1-(t/T))_+^{-\beta-1}}{e\varphi_2^l(x)}\right)$$
$$+\frac{2}{3}l\int_{\Omega_T}|(-\Delta)^{\eta/2}\varphi_2(x)|D_{t|T}^{\alpha}\varphi_1(t)$$
$$\cdot \ln\left(\frac{4lC_1|(-\Delta)^{\eta/2}\varphi_2(x)|T^{-\alpha}(1-(t/T))_+^{-\alpha}}{e\varphi_2^l(x)}\right)$$
$$+\frac{2}{3}\int_{\Omega_T} D_{t|T}^{1+\alpha}\varphi_1(t)\ln\left(\frac{4C_3T^{-(\alpha+1)}(1-(t/T))_+^{-\alpha-1}}{e\varphi_2^l(x)}\right).$$

We pass to the scaled variables $\tau = t/T$ and $y = x/T^{1/\eta}$, $T \gg 1$. It follows that

$$(-\Delta_x)^{\eta/2}\varphi_2 = T^{-1}(-\Delta_y)^{\eta/2}\varphi_2,$$
$$D_{t|T}^{\alpha}\varphi_1 = C_1T^{-\alpha}(1-\tau)_+^{\sigma-\alpha},$$
$$D_{t|T}^{\beta}\varphi_1 = C_2T^{-\beta}(1-\tau)_+^{\sigma-\beta},$$
$$D_{t|T}^{1+\alpha}\varphi_1 = C_3T^{-(\alpha+1)}(1-\tau)_+^{\sigma-(\alpha+1)},$$

and

$$D_{t|T}^{1+\beta}\varphi_1 = C_4T^{-(\beta+1)}(1-\tau)_+^{\sigma-(\beta+1)}.$$

Now, we set $\Omega_2 = [0,1]\times\{y\in\mathbb{R}^N,\ |y|\le 2\}$ and

$$\int_{\Omega_2} = \int_{\Omega_2} dy\, d\tau.$$

Using these definitions, (4.15) and (4.16) can be rewritten as

(4.17)

$$\begin{aligned}
\int_{\Omega_T} e^{v(x,t)}\varphi(x,t) \leq C_1\frac{4}{3}lT^{(N/\eta)-\alpha}\int_{\Omega_2}|(-\Delta_y)^{\eta/2}\varphi_2(T^{1/\eta}y)|\\
\cdot\Big\{\ln\Big(\frac{4lC_1}{e}|(-\Delta_y)^{\eta/2}\varphi_2(T^{1/\eta}y)|\Big) - l\ln(\varphi_2(T^{1/\eta}y))\\
-\alpha\ln(1-\tau)_+ - (1+\alpha)\ln T\Big\}\\
+C_3\frac{4}{3}T^{N/\eta-\alpha}\int_{\Omega_2}\Big\{\ln\big(\frac{4C_3}{e}\big) - l\ln(\varphi_2(T^{1/\eta}y))\\
-(\alpha+1)\ln(1-\tau)_+ - (\alpha+1)\ln T)\Big\}\\
+C_2\frac{2}{3}lT^{N/\eta-\beta}\int_{\Omega_2}|(-\Delta_y)^{\eta/2}\varphi_2(T^{1/\eta}y)|\\
\cdot\Big\{\ln\Big(\frac{4lC_2}{e}|(-\Delta_y)^{\eta/2}\varphi_2(T^{1/\eta}y)|\Big) - l\ln(\varphi_2(T^{\frac{1}{\eta}}y))\\
-\beta\ln(1-\tau)_+ - (1+\beta)\ln T\Big\}\\
+C_4\frac{2}{3}T^{(N/\eta)-\beta}\int_{\Omega_2}\Big\{\ln\Big(\frac{4C_4}{e}\Big) - l\ln(\varphi_2(T^{1/\eta}y))\\
-(\beta+1)\ln(1-\tau)_+ - (\beta+1)\ln T\Big\},
\end{aligned}$$

and

(4.18)

$$\begin{aligned}
\int_{\Omega_T} e^{u(x,t)}\varphi(x,t) \leq C_2\frac{4}{3}lT^{(N/\eta)-\beta}\int_{\Omega_2}|(-\Delta_y)^{\eta/2}\varphi_2(T^{1/\eta}y)|\\
\cdot\Big\{\ln\Big(\frac{4lC_2}{e}|(-\Delta_y)^{\eta/2}\varphi_2(T^{1/\eta}y)|\Big) - l\ln(\varphi_2(T^{1/\eta}y))\\
-\beta\ln(1-\tau)_+ - (1+\beta)\ln T\Big\}\\
+C_4\frac{4}{3}T^{(N/\eta)-\beta}\int_{\Omega_2}\Big\{\ln\Big(\frac{4C_4}{e}\Big) - l\ln(\varphi_2(T^{1/\eta}y))
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - (\beta+1)\ln(1-\tau)_+ - (\beta+1)\ln T\Big\} \\
&+ C_1 \frac{2}{3} l T^{(N/\eta)-\alpha} \int_{\Omega_2} |(-\Delta_y)^{\eta/2}\varphi_2(T^{1/\eta}y)| \\
&\cdot \left\{ \ln\left(\frac{4lC_1}{e}|(-\Delta_y)^{\eta/2}\varphi_2(T^{1/\eta}y)|\right) - l\ln(\varphi_2(T^{1/\eta}y)) \right. \\
&\qquad\qquad \left. - \alpha\ln(1-\tau)_+ - (1+\alpha)\ln T \right\} \\
&+ C_3 \frac{2}{3} T^{(N/\eta)-\alpha} \int_{\Omega_2} \left\{ \ln\left(\frac{4C_3}{e}\right) - l\ln(\varphi_2(T^{1/\eta}y)) \right. \\
&\qquad\qquad \left. - (\alpha+1)\ln(1-\tau)_+ - (\alpha+1)\ln T \right\}.
\end{aligned}
$$

Thus, we have two bounded functions: φ_2 and $(-\Delta_y)^{\eta/2}\varphi_2$ in Ω_2 and

$$\varphi_2 \longrightarrow 1 \quad \text{as } T \to +\infty.$$

Using Lebesgue's dominated convergence theorem, we deduce that the right hand sides of (4.17) and (4.18) diverge to $-\infty$ when $T \to +\infty$, while the left hand sides are positives. This leads to a contradiction. □

5. Blow-up rate. In this section, we study the profile of solutions near the blow-up time. For this, we will derive an upper and lower bound for the blow-up rate.

Theorem 5.1. *Let $u_0,\ v_0 \in C_0(\mathbb{R}^N)$, $u_0 \geq 0$, $u_0 \not\equiv 0$, $v_0 \geq 0$, $v_0 \not\equiv 0$, and let (u,v) be the blowing-up solution of (1.1)–(1.2) at $T_{\max} = T^*$. Then, there exist four positive constants $c_i < C_i'$, $i = 1,\ 2$, such that*

$$
\begin{aligned}
\ln\left(c_1(T^*-t)^{-(2-\delta)}\right) &\leq \sup_{x\in\mathbb{R}^N} u(x,t), \\
\ln\left(c_2(T^*-t)^{-(2-\gamma)}\right) &\leq \sup_{x\in\mathbb{R}^N} v(x,t),
\end{aligned}
$$

for $t \in (0, T^*)$, and

$$
\begin{aligned}
u(x,t) &\leq \ln\left(C_1'(t^*-t)^{-(2-\delta)}\right), \\
v(x,t) &\leq \ln\left(C_2'(t^*-t)^{-(2-\gamma)}\right),
\end{aligned}
$$

for $t \in (0, t^*)$ where t^* is the blow-up time of the non-diffusive system.

Proof. The proof consists of two steps.

I. *The lower bound.* If we repeat the same proof of the local existence in Theorem 3.2 by taking $\|u\|_1 \le M_1$ and $\|v\|_1 \le M_2$ instead of $\|(u,v)\| \le 2\,(\|u_0\|_\infty + \|v_0\|_\infty)$ in the space E_T for all positive constants $M_1,\ M_2 > 0$ and all $0 < t < T$, then the condition on T will be

$$\|u_0\|_\infty + cT^{2-\gamma} e^{M_2} \le M_1, \quad cT^{2-\gamma} e^{M_2} \le \frac{1}{2},$$

and

$$\|v_0\|_\infty + c' T^{2-\delta} e^{M_1} \le M_2, \quad c' T^{2-\delta} e^{M_1} \le \frac{1}{2},$$

where

$$c = \frac{1}{\Gamma(3-\gamma)} \quad \text{and} \quad c' = \frac{1}{\Gamma(3-\delta)}.$$

By the same reasoning, we deduce that $\|u(t)\|_\infty \le M_1$ and $\|v(t)\|_\infty \le M_2$ for all $t \in (0,T)$, whereupon, if

$$\|u_0\|_\infty + ct^{2-\gamma} e^{M_2} \le M_1, \quad ct^{2-\gamma} e^{M_2} \le \frac{1}{2},$$

and

$$\|v_0\|_\infty + c' t^{2-\delta} e^{M_1} \le M_2, \quad c' t^{2-\delta} e^{M_1} \le \frac{1}{2},$$

then $\|u(t)\|_\infty \le M_1$ and $\|v(t)\|_\infty \le M_2$. Applying this to any point in the trajectories, we see that, if $0 \le s < t$ and

$$(5.1) \qquad (t-s)^{2-\gamma} \le \frac{M_1 - \|u(s)\|_\infty}{ce^{M_2}}, \qquad (t-s)^{2-\gamma} \le \frac{1}{2ce^{M_2}},$$

and

$$(t-s)^{2-\delta} \le \frac{M_2 - \|v(s)\|_\infty}{c' e^{M_1}}, \qquad (t-s)^{2-\delta} \le \frac{1}{2c' e^{M_1}},$$

then we deduce that $\|u(t)\|_\infty \le M_1$ and $\|v(t)\|_\infty \le M_2$ for all $0 < t < T$. Moreover, if $0 \le s < T^*$, $\|u(s)\|_\infty < M_1$ and $\|v(s)\|_\infty < M_2$, then

$$(5.2) \qquad (T^*-s)^{2-\gamma} > \frac{M_1 - \|u(s)\|_\infty}{ce^{M_2}}, \quad (T^*-s)^{2-\gamma} > \frac{1}{2ce^{M_2}},$$

and

$$(T^* - s)^{2-\delta} > \frac{M_2 - \|v(s)\|_\infty}{c' e^{M_1}}, \quad (T^* - s)^{2-\delta} > \frac{1}{2c' e^{M_1}}.$$

In fact, arguing by contradiction and assuming that, for some $M_1 > \|u(s)\|_\infty$, $M_2 > \|v(s)\|_\infty$ and all $t \in (s, T^*)$, we have

$$(t - s)^{2-\gamma} \le \frac{M_1 - \|u(s)\|_\infty}{c e^{M_2}}, \qquad (t - s)^{2-\gamma} \le \frac{1}{2c e^{M_2}},$$

or

$$(t - s)^{2-\delta} \le \frac{M_2 - \|v(s)\|_\infty}{c' e^{M_1}}, \qquad (t - s)^{2-\delta} \le \frac{1}{2c' e^{M_1}}.$$

Then, using (5.1), we infer that $\|u(t)\|_\infty \le M_1$ or $\|v(t)\|_\infty \le M_2$ for all $t \in (s, T^*)$; this is a contradiction to the fact that

$$\|u(t)\|_\infty \longrightarrow \infty$$

and

$$\|v(t)\|_\infty \longrightarrow \infty \quad \text{as } t \to T^*.$$

Next, letting $M_1 = \|u(s)\|_\infty + 1$ and $M_2 = \|v(s)\|_\infty + 1$ in (5.2), we see that, for $0 < s < T^*$, we have

$$(T^* - s)^{2-\gamma} > c_2 e^{-\|v(s)\|_\infty} \text{ and } (T^* - s)^{2-\delta} > c_1 e^{-\|u(s)\|_\infty}.$$

Since u and v are continuous and positive, we obtain

$$\ln(c_1 (T^* - s)^{-(2-\delta)}) \le \sup_{x \in \mathbb{R}^N} u(x, s)$$

and

$$\ln(c_2 (T^* - s)^{-(2-\gamma)}) \le \sup_{x \in \mathbb{R}^N} v(x, s),$$

for all $s \in (0, T^*)$.

II. *The upper bound.* Let $(\overline{u}(t), \overline{v}(t))$ be the solution of the system

$$\begin{aligned} \overline{u}'(t) &= \frac{1}{\Gamma(1-\gamma)} \int_0^t (t - s)^{-\gamma} e^{\overline{v}(s)}\, ds, \quad t > 0, \\ \overline{v}'(t) &= \frac{1}{\Gamma(1-\delta)} \int_0^t (t - s)^{-\delta} e^{\overline{u}(s)}\, ds, \quad t > 0, \end{aligned} \tag{5.3}$$

with the initial conditions

$$\overline{u}(0) = \max_{x\in\mathbb{R}^N} u_0(x) \quad \text{and} \quad \overline{v}(0) = \max_{x\in\mathbb{R}^N} v_0(x). \tag{5.4}$$

Through comparison, we see that $(\overline{u}, \overline{v})$ is an upper solution for (u, v). Moreover, following the lines of **[13]**, we can show that the solution to (5.3)–(5.4) blows up in a finite time t^*, and the profile near the blow-up time is given by

$$\overline{u}(t) \sim (2-\delta)\ln\left(\frac{1}{t^*-t}\right)$$

and

$$\overline{v}(t) \sim (2-\gamma)\ln\left(\frac{1}{t^*-t}\right), \quad \text{as } t \to t^*.$$

Consequently, we have the upper bound

$$u(x,t) \le \ln\left(C_1'(t^*-t)^{-(2-\delta)}\right)$$

and

$$v(x,t) \le \ln\left(C_2'(t^*-t)^{-(2-\gamma)}\right). \qquad \square$$

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REFERENCES

1. B. Ahmad, A. Alsaedi and M. Kirane, *On a reaction diffusion equation with nonlinear time-nonlocal source term*, Math. Meth. Appl. Sci. **39** (2016), 236–244.

2. J. Bebernes and D. Eberly, *Mathematical problems from combustion theory*, Springer, Berlin, 1989.

3. T. Cazenave and A. Haraux, *Introduction aux problèmes d'évolution semi-linéaires*, Ellipses, Paris, 1990.

4. A. Cordoba and D. Cordoba, *A maximum principle applied to quasi-geostrophic equations*, Comm. Math. Phys. **249** (2004), 511–528.

5. E. Cuesta, M. Kirane and S. Malik, *Image structure preserving denoising using generalized fractional time integrals*, Signal Process. **92** (2012), 553–563.

6. D. Del-Castillo-Negrete, B.A. Carreras and V.E. Lynch, *Fractional diffusion in plasma turbulence*, Phys. Plasmas **11** (1994), 3854–3864.

7. A. Fino and M. Kirane, *Qualitative properties of solutions to a time-space fractional evolution equation*, Quart. Appl. Math. **70** (2012), 133–157.

8. G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1952.

9. R. Herrmann, *Fractional calculus*: *An introduction for physicists*, World Scientific Publishing, Singapore, 2011.

10. R. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publishing, Singapore, 2000.

11. N. Ju, *The maximum principle and the global attractor for the dissipative* 2-D *quasi-geostrophic equations*, Comm. Pure Appl. Anal. (2005), 161–181.

12. A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, New York, 2006.

13. C.M. Kirk, W.E. Olmstead and C.A. Roberts, *A system of nonlinear Volterra equations with blow-up solutions*, J. Integral Equat. Appl. **25** (2013), 377–393.

14. R.L. Magin, *Fractional calculus models of complex dynamics in biological tissues*, Comp. Math. Appl. **59** (2010), 1586–1593.

15. V. Mendez, S. Fedotov and W. Horsthemke, *Reaction-transport systems*: *Mesoscopic foundations, fronts, and spatial instabilities*, Springer Series in Synergetics, Springer, New York, 2010.

16. R. Metzler and J. Klafter, *The restaurent at the end of the random walk*: *Recent developments in the description of a transport by fractional dynamics*, J. Physics **37**, R161–R208.

17. S.N. Pokhozhaev, *Concerning an equation in the theory of combustion* Math. Notes **88** (2010), 48–56.

18. F. Riewe, *Mechanics with fractional derivatives*, Phys. Rev. **55** (1997), 3581–3592.

19. S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach Science Publishers, Yverdon, 1993.

20. P. Wahi and A. Chatterjee, *Averaging oscillations with small fractional damping and delayed terms* Nonlinear Dynam. **38** (2004), 3–22.

21. W. Wyss, *The fractional Black-Scholes equation*, Fract. Calc. Appl. Anal. **3** (2000), 51–61.

22. K. Yosida, *Functional analysis*, *Classics in mhematics*, Springer-Verlag, Berlin, 1995.

NAAM Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
Email address: **bashirahmad_qau@yahoo.com**

NAAM Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
Email address: **aalsaedi@kau.edu.sa**

Institut Supérieur des Sciences Appliquées et de Technologie, Département de Mathématiques, Rue Omar ibn el Khattab, 6072 Gabes, Tunisie
Email address: **hnaien.dorsaf@gmail.com**

LASIE, Pôle Sciences et Technologies, Université de La Rochelle, Avenue Michel Crépeau, 17031 La Rochelle, France

and NAAM Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
Email address: **mkirane@univ-lr.fr**