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GENERAL AND OPTIMAL DECAY IN A MEMORY-TYPE TIMOSHENKO SYSTEM

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ABSTRACT. This paper is concerned with the following memory-type Timoshenko system

$$\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0$$

$$\rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \int_0^t g(t-s)\psi_{xx}(s)\,ds = 0,$$

$(x,t) \in (0,L) \times (0,\infty)$, with Dirichlet boundary conditions, where g is a positive non-increasing function satisfying, for some constant $1 \leq p < 3/2$,

$$g'(t) \leq -\xi(t)g^p(t), \quad \text{for all } t \geq 0.$$

We prove some decay results which generalize and improve many earlier results in the literature. In particular, our result gives the optimal decay for the case of polynomial stability.

1. Introduction. In 1921, Timoshenko [**26**] presented the following system of hyperbolic partial differential equations

$$\begin{aligned} \rho u_{tt} &= (K(u_x - \phi))_x && \text{in } (0,L) \times (0,+\infty), \\ I_\rho \phi_{tt} &= (EI\phi_x)_x + K(u_x - \phi) && \text{in } (0,L) \times (0,+\infty), \end{aligned} \tag{1.1}$$

as a mathematical model describing the dynamics of a beam by taking the transverse shear strain into consideration. Here, t represents the time and x is the space variable along the beam of length L, u is the transverse displacement of the beam from its equilibrium configuration and ϕ is the rotational angle of the filament of the beam. The constant

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coefficients $\rho,\ I_\rho,\ E,\ I$ and K are the mass density, the polar moment of inertia of a cross-section, the Young modulus of elasticity, the moment of inertia of a cross-section, and the shear modulus, respectively.

For almost a century, a great number of researchers have devoted a considerable amount of time studying this model, and many results concerning the well-posedness and long-time behavior of the system have been established. Various types of dissipation mechanisms (such as boundary and/or internal controls) were employed in order to achieve different stability results. We mention a few of these results from the literature. For more details, we refer the reader to the references in this paper and the references therein.

In the case of boundary feedback controls, Kim and Renardy **[14]** investigated the uniform stabilization of (1.1) with clamped end at $x = 0$, that is,

$$u(0,t) = 0, \qquad \phi(0,t) = 0, \quad \text{for all } t \geq 0$$

and mixed boundary conditions of the form

$$\begin{aligned} &K\phi(L,t) - Ku_x(L,t) = \alpha u_t(L,t), && \text{for all } t \geq 0 \\ &EI\phi(L,t) = -\beta\phi_t(L,t), && \text{for all } t \geq 0. \end{aligned}$$

They used the multiplier method to prove that the energy associated to system (1.1) decays exponentially. Feng, et al. **[7]** considered the problem in **[14]** but replaced the linear boundary controls with some nonlinear feedback controls and established the asymptotic and exponential stability of the system by using the LaSalle invariance principle and energy perturbation method. Messaoudi and Mustafa in **[19]** investigated the long-time behavior of a Timoshenko system with internal and/or boundary feedback controls. Without imposing any restrictive growth assumption on the damping terms near the origin, they established explicit and general decay results.

In the presence of two internal feedback controls, Raposo, et al. **[23]** studied the exponential decay of the solution of a linear Timoshenko-type beam equation with frictional dissipative terms. Precisely, they studied the following system

$$\text{(1.2)} \quad \begin{cases} \rho_1 u_{tt} - K(u_x - \psi)_x + u_t = 0 & 0 < x < L,\ t > 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(u_x - \psi) + \psi_t = 0 & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = \psi(0,t) = \psi(L,t) = 0 & \text{for all } t > 0 \end{cases}$$

and used the semigroup method developed by Liu and Zheng **[16]** to prove the exponential decay of the solution of the system (1.2).

However, when a control is present on the rotation angle or on the transverse displacement alone the decay rates turn out to depend on the constants ρ, I_ρ, E, I and K. For instance, Soufyane and Wehbe **[25]** proved that one can uniformly stabilize a linear Timoshenko system under influence of one locally distributed damping. They considered

$$\tag{1.3} \begin{cases} \rho_1 u_{tt} = (K(u_x - \psi))_x & 0 < x < L,\ t > 0, \\ \rho_2 \psi_{tt} = (b\psi_x)_x + K(u_x - \psi) - \sigma\psi_t & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = \psi(0,t) = \psi(L,t) = 0 & \text{for all } t > 0, \end{cases}$$

where σ is any continuous function on $[0, L]$ satisfying

$$\sigma(x) \geq \gamma_0 > 0, \quad \text{for all } x \in [c,d] \subset [0,L].$$

Indeed, they proved the exponential stability for system (1.3) if and only if the system has equal speeds of wave propagation, that is, if and only if

$$\tag{1.4} \frac{\rho_1}{K} = \frac{\rho_2}{b}$$

holds. Otherwise, only the asymptotic stability is established. Fernandez Sare and Rivera **[8]** studied a Timoshenko system with infinite history of the form

$$\tag{1.5} \begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) \\ \quad + \displaystyle\int_0^{+\infty} g(s)\psi_{xx}(t-s)ds = 0, \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \end{cases}$$

where $x \in (0, L)$, $t > 0$, and the relaxation function g satisfies

$$\tag{1.6} \widetilde{b} := b - \int_0^{+\infty} g(s)\,ds > 0$$

and

$$\tag{1.7} \begin{cases} g(t) > 0, \\ \text{there exists } k_0, k_1, k_2 > 0 : -k_0 g(t) \leq g'(t) \leq -k_1 g(t), \\ |g''(t)| \leq k_2 g(t). \end{cases}$$

They showed that system (1.5) is exponentially stable if and only if relation (1.4) holds; otherwise, it is polynomially stable. Messaoudi and Said-Houari [**21**] investigated the same system with the following conditions on g:

(1.8)
$$\begin{cases} \widetilde{b} := b - \displaystyle\int_0^{+\infty} g(s)\,ds > 0 \\ g(t) > 0, \quad \text{there exists } k_0 > 0 : g'(t) \le -k_0 g^p(t), \quad 1 \le p < 3/2 \end{cases}$$

and proved that, if (1.4) holds, then the energy associated to the system decays exponentially for $p = 1$ and polynomially for $p > 1$. However, if (1.4) is not satisfied, they established the decay rate of the type $1/t^{1/(2p-1)}$. This result generalizes and improves that of [**8**]. In [**13**], Guesmia, et al., established general decay estimates for the solution of (1.5). Their results hold for the relaxation function g having more general decay, and they obtained general decay results from which the exponential and polynomial decay results are only special cases. Additionally, they improved the results of [**8, 21**].

The stability of a linear viscoelastic-type Timoshenko system (finite history) has also attracted the considerable attention of researchers. For example, Ammar-Khodja, et al. [**4**] studied the following system:

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \displaystyle\int_0^t g(t-s)\psi_{xx}(s)\,ds = 0, \end{cases} \tag{1.9}$$

with Dirichlet-boundary conditions, where $x \in (0, L)$ and $t > 0$. They proved that this system decays uniformly if and only if the coefficients satisfy (1.4). Concerning the rate of decay, they showed that if g satisfies hypotheses (1.6) and (1.7), then the system is exponentially stable. If g is of polynomial type, that is, if it satisfies, for some positive constants $b_0,\ b_1,\ b_2,\ b_3,\ b_4$ and $p > 2$,

$$\begin{cases} 0 < g(t) \le b_0(1+t)^{-p}, \\ -b_1 g^{(p+1)/p}(t) \le g'(t) \le -b_2 g^{(p+1)/p}(t), \\ -b_3 |g'(t)|^{(p+2)/(p+1)} \le g''(t) \le -b_4 |g'(t)|^{(p+2)/(p+1)}, \end{cases}$$

then the energy associated to the system decays polynomially to zero. In the case where the coefficients of system (1.9) satisfy (1.4), Guesmia and Messaoudi [**10**] established the same stabilization results of [**4**] by assuming that g satisfies conditions (1.8) which are weaker than those in [**4**]. Also, Messaoudi and Mustafa [**20**] discussed system (1.9) and proved a general decay result, from which the exponential and polynomial stability are only special cases, under the conditions

$$g(t) > 0, \qquad g'(t) \leq -\xi(t)g(t), \qquad b - \int_0^{+\infty} g(s)\,ds := l > 0,$$

where ξ is a positive non-increasing differentiable function. In fact, the result of [**20**] generalizes those of [**4, 10**] and allows a wider class of relaxation functions. Recently, Almeida Júnior et al. [**3**] considered the situation when the control is only on the transverse displacement equation, which is more realistic from the physical point of view. Precisely, they studied the following system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \mu\varphi_t = 0 & \text{in } (0,L)\times(0,+\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0 & \text{in } (0,L)\times(0,+\infty) \end{cases}$$

and showed that the affect of linear frictional damping on the first equation stabilizes the system exponentially if (1.4) holds; otherwise, the stabilization is of polynomial type. This result was later improved and generalized by Guesmia and Messaoudi [**9**]. For more recent results on this and viscoelastic systems in general, see [**1, 2, 6, 15, 18**].

Concerning stabilization by heat effect, Rivera and Racke [**22**] showed that it is possible to stabilize a Timoshenko system in such a way. In fact, they considered the following coupled system

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0 & \text{in } (0,L)\times(0,+\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0 & \text{in } (0,L)\times(0,+\infty) \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{xt} = 0 & \text{in } (0,L)\times(0,+\infty) \end{cases} \tag{1.10}$$

and proved many exponential stability results of the linearized system if relation (1.4) is satisfied and a polynomial stability result otherwise. Guesmia et al. [**13**] established various general decay estimates for system (1.10) depending on the regularity of the initial data and the validity of relation (1.4) by adding an infinite memory term on the first or second equation, where the heat propagation is given by Fourier's, Cattaneo's and Green and Naghdi's laws. Apalara et al. [**5**] proved the asymptotic stability of a one-dimensional linear thermoelastic Timoshenko system, where the heat conduction is given by Cattaneo's theory and the coupling is through the displacement equation. They proved their exponential and polynomial stability results under a stability number which was first introduced in [**24**]. For more recent results on this, see [**6, 12**].

Our main purpose in this paper is to study the following memory-type Timoshenko system

$$(P)\qquad \begin{cases} \rho_1\varphi_{tt} - K(\varphi_x+\psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) + \displaystyle\int_0^t g(t-s)\psi_{xx}(s)\,ds = 0, \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \end{cases}$$

where $(x,t) \in (0,L)\times(0,\infty)$, b, K, ρ_1 and ρ_2 are positive constants, φ_0, φ_1, ψ_0 and ψ_1 are given data, and g is a relaxation function satisfying some conditions to be specified in the next section. We prove generalized energy decay results for the system. Our results generalize and improve that of Messaoudi and Mustafa [**20**] in the case of equal speeds of wave propagation and that of Guesmia and Messaoudi [**11**] in the opposite case.

This paper is organized as follows. In Section 2, we state some preliminary results. In Section 3, we state and prove some technical lemmas. The statement and proof of our main results are given in Sections 4 and 5.

2. Preliminaries. In this section, we introduce our assumptions, present some useful lemmas and state the existence theorem. We use c to denote a positive generic constant.

Assumptions. We assume that the relaxation function g satisfies the following hypotheses:

(H1) $g:\mathbb{R}_+\to\mathbb{R}_+$ is a non-increasing differentiable function such that

$$g(0)>0, \qquad b-\int_0^{+\infty} g(s)\,ds =: l > 0.$$

(H2) There exist a non-increasing differentiable function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $1 \le p < 3/2$ such that

$$g'(t) \le -\xi(t) g^p(t), \quad \text{for all } t \ge 0.$$

Lemma 2.1. *Assume that g satisfies hypotheses* (H1) *and* (H2). *Then,*

$$\int_0^{+\infty} \xi(t) g^{1-\sigma}(t)\, dt < +\infty, \quad \textit{for all } 0 < \sigma < 2-p.$$

Proof. Using (H2), we have

$$\begin{aligned} \int_0^{+\infty} \xi(t) g^{1-\sigma}(t)\, dt &= \int_0^{+\infty} \xi(t) g^p(t) g^{1-\sigma-p}(t)\, dt \\ &\le -\int_0^{+\infty} g'(t) g^{1-\sigma-p}(t)\, dt \\ &= -\left[\frac{1}{2-\sigma-p} g^{2-\sigma-p}(t)\right]_{t=0}^{t=+\infty} < +\infty, \end{aligned}$$

since $\sigma < 2 - p$. □

For completeness, we state, without proof, the global existence and regularity result which can be easily established by a standard Galerkin argument.

Theorem 2.2. *Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0, L) \times L^2(0, L)$ be given. Assume that g satisfies hypothesis* (H1). *Then, problem* (P) *has a unique global* (*weak*) *solution*

$$\varphi, \psi \in C(\mathbb{R}_+; H_0^1(0, L)) \cap C^1(\mathbb{R}_+; L^2(0, L)).$$

Moreover, if $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L)$, then the problem has a unique strong solution

$$\varphi, \psi \in C(\mathbb{R}_+; H^2(0, L) \cap H_0^1(0, L)) \cap C^1(\mathbb{R}_+; H_0^1(0, L)) \cap C^2(\mathbb{R}_+; L^2(0, L)).$$

Now, we introduce the energy functional

$$E(t) := \frac{1}{2}\int_0^L \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \left(b - \int_0^t g(s)\,ds\right)\psi_x^2 + K(\varphi_x+\psi)^2\right]dx + \frac{1}{2}(g\circ\psi_x)(t), \tag{2.1}$$

where, for any $v \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(0,L))$,

$$(g\circ v)(t) := \int_0^L\int_0^t g(t-s)(v(t)-v(s))^2 ds\,dx.$$

Lemma 2.3 ([**20**])**.** *Let (φ,ψ) be the solution of (P). Then,*

$$\begin{aligned} E'(t) &= -\frac{1}{2}g(t)\int_0^L \psi_x^2\,dx + \frac{1}{2}(g'\circ\psi_x)(t) \\ &\le \frac{1}{2}(g'\circ\psi_x)(t) \le 0, \quad \textit{for all } t\ge 0. \end{aligned} \tag{2.2}$$

Lemma 2.4 ([**20**])**.** *There exists a constant $c>0$ such that, for any $v \in L^2_{\text{loc}}(\mathbb{R}_+; H^1_0(0,L))$, we have*

$$\int_0^L\left(\int_0^t g(t-s)(v(t)-v(s))\,ds\right)^2 dx \le c(g\circ v_x)(t), \quad \textit{for all } t\ge 0.$$

Lemma 2.5 ([**17**])**.** *Assume that conditions* (H1) *and* (H2) *hold and (φ,ψ) is the solution of (P). Then, for any $0<\sigma<1$, we have*

$$g\circ\psi_x \le \left[\frac{8}{l}E(0)\int_0^t g^{1-\sigma}(s)\,ds\right]^{(p-1)/(p+\sigma-1)}(g^p\circ\psi_x)^{\sigma/(p+\sigma-1)}.$$

For $\sigma = 1/2$, we obtain the following inequality:

$$g\circ\psi_x \le c\left(\int_0^t g^{1/2}(s)\,ds\right)^{(2p-2)/(2p-1)}(g^p\circ\psi_x)^{1/(2p-1)}. \tag{2.3}$$

Corollary 2.6. *Assume that g satisfies* (H1), (H2) *and (φ,ψ) is the solution of (P). Then,*

$$\xi(t)(g\circ\psi_x)(t) \le c(-E'(t))^{1/(2p-1)}, \quad \textit{for all } t\ge 0.$$

Proof. Multiplying both sides of the inequality (2.3) by $\xi(t)$ and using Lemmas 2.1 and 2.3, we obtain

$$\begin{aligned}
&\xi(t)(g \circ \psi_x)(t)\\
&\quad \le c\xi^{(2p-2)/(2p-1)}(t)\left(\int_0^t g^{1/2}(s)\,ds\right)^{(2p-2)/(2p-1)}(\xi g^p \circ \psi_x)^{1/(2p-1)}(t)\\
&\quad \le c\left(\int_0^t \xi(s) g^{1/2}(s)\,ds\right)^{(2p-2)/(2p-1)}(-g' \circ \psi_x)^{1/(2p-1)}\\
&\quad \le c(-E'(t))^{1/(2p-1)}.
\end{aligned}$$

□

Lemma 2.7 (Jensen's inequality)**.** *Let $G : [a,b] \to \mathbb{R}$ be a concave function. Assume that the functions $f : \Omega \to [a,b]$ and $h : [0,L] \to \mathbb{R}$ are integrable such that $h(x) \ge 0$, for any $x \in \Omega$ and $\int_\Omega h(x)\,dx = k > 0$. Then,*

$$\frac{1}{k}\int_\Omega G(f(x))h(x)\,dx \le G\left(\frac{1}{k}\int_\Omega f(x)h(x)\,dx\right).$$

In particular, for $G(y) = y^{1/p}$, $y \ge 0$, $p > 1$, we have

$$\frac{1}{k}\int_\Omega f^{1/p}(x)h(x)\,dx \le \left(\frac{1}{k}\int_\Omega f(x)h(x)\,dx\right)^{1/p}.$$

3. Technical lemmas. In this section, we state and prove some lemmas needed to establish our main results.

Lemma 3.1. *Assume that conditions* (H1) *and* (H2) *hold. Then, the functional F defined by*

$$F(t) := -\rho_2 \int_0^L \psi_t \int_0^t g(t-s)(\psi(t) - \psi(s))\,ds\,dx$$

satisfies, along with the solution of (P), the estimates

$$\begin{aligned}
F'(t) \le {} & -\rho_2\left(\int_0^t g(s)\,ds - \delta\right)\int_0^L \psi_t^2 dx + \delta K \int_0^L (\varphi_x + \psi)^2 dx \\
& + c\delta \int_0^L \psi_x^2 dx + c\left(\delta + \frac{1}{\delta}\right)(g \circ \psi_x) - \frac{c}{\delta}(g' \circ \psi_x),
\end{aligned} \tag{3.1}$$

for all $\delta > 0$.

Proof. Differentiating F and using the equations in (P), we get

$$\begin{aligned}F'(t) = &-\rho_2 \int_0^L \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))\,ds\,dx \\ &- \rho_2 \left(\int_0^t g(s)\,ds\right) \int_0^L \psi_t^2 dx \\ &+ b \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\,dx \\ &+ K \int_0^L (\varphi_x+\psi) \int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\,dx \\ &- \int_0^L \left(\int_0^t g(t-s)\psi_x(s)\,ds\right) \\ &\times \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right) dx.\end{aligned}$$

Next, we estimate the terms on the right-hand side of the above equation.

Using Young's inequality and Lemma 2.3 for $(-g')$, we obtain, for any $\delta > 0$,

$$-\rho_2 \int_0^L \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))\,ds\,dx \le \delta\rho_2 \int_0^L \psi_t^2 dx - \frac{c}{\delta}(g' \circ \psi_x).$$

Similarly, we have

$$b \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\,dx \le \delta \int_0^L \psi_x^2 + \frac{c}{\delta}(g \circ \psi_x),$$

$$\begin{aligned}K \int_0^L (\varphi_x+\psi) \int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\,dx \\ \le \delta K \int_0^L (\varphi_x+\psi)^2 dx + \frac{c}{\delta}(g \circ \psi_x),\end{aligned}$$

and

$$\int_0^L \left(\int_0^t g(t-s)\psi_x(s)\,ds\right)$$

$$\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)dx$$
$$\leq c\delta\int_0^L \psi_x^2 dx + c\left(\delta+\frac{1}{\delta}\right)(g\circ\psi_x).$$

A combination of these estimates gives the desired result. □

Lemma 3.2. *Under conditions* (H1) *and* (H2), *the functional* I_1 *defined by*

$$I_1(t) := -\int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t)\,dx$$

satisfies, along with the solution of (P), *the estimate*

$$I_1'(t) \leq -\int_0^L (\rho_1\varphi_t^2+\rho_2\psi_t^2)\,dx + K\int_0^L (\varphi_x+\psi)^2 dx + c\int_0^L \psi_x^2 dx + c(g\circ\psi_x). \tag{3.2}$$

Proof. Using equations of (P) and repeating the above computations, we arrive at

$$\begin{aligned} I_1'(t) &= -\int_0^L (\rho_1\varphi_t^2+\rho_2\psi_t^2)\,dx + b\int_0^L \psi_x^2 dx + K\int_0^L (\varphi_x+\psi)^2 dx \\ &\quad - \int_0^L \psi_x \int_0^t g(t-s)\psi_x(s)\,ds\,dx \\ &\leq -\int_0^L (\rho_1\varphi_t^2+\rho_2\psi_t^2)\,dx + K\int_0^L (\varphi_x+\psi)^2 dx \\ &\quad + c\int_0^L \psi_x^2 + c(g\circ\psi_x). \end{aligned}$$ □

Lemma 3.3. *Assume that the hypotheses* (H1) *and* (H2) *hold. Then, for any* $0<\varepsilon<1$, *the functional* I_2 *defined by*

$$\begin{aligned} I_2(t) &:= \rho_2\int_0^L \psi_t(\varphi_x+\psi)\,dx \\ &\quad + \frac{b\rho_1}{K}\int_0^L \varphi_t\psi_x\,dx - \frac{\rho_1}{K}\int_0^L \varphi_t\int_0^t g(t-s)\psi_x(s)\,ds\,dx \end{aligned}$$

satisfies, along with the solution of (P), *the estimate*

$$\begin{aligned} I_2'(t) \le {} & \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)\varphi_x\right]_{x=0}^{x=L} \\ & - K\int_0^L (\varphi_x+\psi)^2 dx + c\varepsilon\rho_1 \int_0^L \varphi_t^2 dx \\ & + \rho_2 \int_0^L \psi_t^2 dx + \frac{c}{\varepsilon}\int_0^L \psi_x^2 dx - \frac{c}{\varepsilon}(g'\circ\psi_x) \\ & + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_t\psi_{xt} dx. \end{aligned} \tag{3.3}$$

Proof. Using equations of (P), integrating by parts and applying Young's inequality, we obtain

$$\begin{aligned} I_2'(t) = {} & \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)\varphi_x\right]_{x=0}^{x=L} - K\int_0^L (\varphi_x+\psi)^2 dx \\ & + \rho_2 \int_0^L \psi_t^2 dx + \frac{\rho_1}{K}\int_0^L \varphi_t \int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,ds\,dx \\ & - \frac{\rho_1}{K} g(t)\int_0^L \varphi_t\psi_x\,dx + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_x\psi_{xt}\,dx \\ \le {} & \left[\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)\varphi_x\right]_{x=0}^{x=L} \\ & - K\int_0^L (\varphi_x+\psi)^2 dx + c\varepsilon\rho_1 \int_0^L \varphi_t^2 dx \\ & + \rho_2 \int_0^L \psi_t^2 dx + \frac{c}{\varepsilon}\int_0^L \psi_x^2 dx - \frac{c}{\varepsilon}(g'\circ\psi_x) \\ & + \left(\frac{b\rho_1}{K} - \rho_2\right)\int_0^L \varphi_x\psi_{xt}\,dx. \end{aligned}$$

□

Lemma 3.4. *Assume that the hypotheses* (H1) *and* (H2) *hold. Let* $m \in C^1([0,L])$ *be a real-valued function satisfying* $m(0) = -m(L) = 2$.

Then, for any $0 < \varepsilon < 1$*, the functional* I_3 *defined by*

$$I_3(t) := \frac{\rho_2}{4\varepsilon}\int_0^L m(x)\psi_t\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)dx + \varepsilon\frac{\rho_1}{K}\int_0^L m(x)\varphi_t\varphi_x\,dx$$

satisfies, along with the solution of (P)*, the estimate*

$$\begin{aligned} I_3'(t) \le -\frac{1}{4}\Bigg[&\left(b\psi_x(L,t) - \int_0^t g(t-s)\psi_x(L,s)\,ds\right)^2 \\ &+ \left(b\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)\,ds\right)^2\Bigg] \\ &- \varepsilon(\varphi_x^2(L,t) + \varphi_x^2(0,t)) \\ &+ \left(\frac{1}{4} + c\varepsilon\right)K\int_0^L(\varphi_x+\psi)^2dx + c\varepsilon\rho_1\int_0^L \varphi_t^2dx \\ &+ \frac{c}{\varepsilon}\rho_2\int_0^L \psi_t^2dx + \frac{c}{\varepsilon^2}\left(\int_0^L \psi_x^2dx + g\circ\psi_x\right) \\ &- \frac{c}{\varepsilon}(g'\circ\psi_x). \end{aligned} \tag{3.4}$$

Proof. Using equations of (P), the Young and Poicaré inequalities and the fact that

$$\varphi_x^2 \le 2(\varphi_x+\psi)^2 + 2\psi^2,$$

we have

$$\begin{aligned} I_3'(t) = \frac{1}{4\varepsilon}\Bigg[&-\left(b\psi_x(L,t) - \int_0^t g(t-s)\psi_x(L,s)\,ds\right)^2 \\ &- \left(b\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)\,ds\right)^2 \\ &- \frac{1}{2}\int_0^L m'(x)\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)^2dx \\ &- K\int_0^L m(x)\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right) \\ &\cdot(\varphi_x+\psi)\,dx - \frac{b\rho_2}{2}\int_0^L m'(x)\psi_t^2dx \end{aligned}$$

$$
\begin{aligned}
&- \rho_2 g(t) \int_0^L m(x)\psi_x\psi_t \, dx \\
&+ \rho_2 \int_0^L m(x)\psi_t \int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\, ds\, dx \Big] \\
&+ \varepsilon \Big[-(\varphi_x^2(L,t)+\varphi_x^2(0,t)) + \int_0^L m(x)\varphi_x\psi_x \, dx \\
&\quad - \frac{1}{2}\int_0^L m'(x)\varphi_x^2 dx - \frac{\rho_1}{2K}\int_0^L m'(x)\varphi_t^2 dx \Big] \\
\leq & -\frac{1}{4\varepsilon}\left[\left(b\psi_x(L,t) - \int_0^t g(t-s)\psi_x(L,s)\, ds \right)^2 \right. \\
&\quad \left. + \left(b\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)\, ds \right)^2 \right] \\
&- \varepsilon(\varphi_x^2(L,t)+\varphi_x^2(0,t)) + \left(\frac{1}{4} + c\varepsilon \right) K \int_0^L (\varphi_x+\psi)^2 dx \\
&+ c\varepsilon\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon}\rho_2 \int_0^L \psi_t^2 dx \\
&+ \frac{c}{\varepsilon^2}\left(\int_0^L \psi_x^2 dx + g\circ\psi_x \right) - \frac{c}{\varepsilon}(g'\circ\psi_x). \qquad \square
\end{aligned}
$$

Lemma 3.5. *Assume that conditions* (H1) *and* (H2) *hold. After fixing* ε *small enough, the functional* I *defined by*

$$I(t) := 3c\varepsilon I_1(t) + I_2(t) + I_3(t)$$

satisfies, along with the solution of (P), *the estimate*

$$
\begin{aligned}
I'(t) \leq & -\frac{K}{2}\int_0^L (\varphi_x+\psi)^2 dx - \tau\rho_1 \int_0^L \varphi_t^2 dx \\
& + c\rho_2 \int_0^L \psi_t^2 dx + c\int_0^L \psi_x^2 dx + c(g\circ\psi_x - g'\circ\psi_x) \\
& + \left(\frac{b\rho_1}{K} - \rho_2 \right) \int_0^L \varphi_x\psi_{xt}\, dx,
\end{aligned}
\tag{3.5}
$$

where $\tau = c\varepsilon$.

Proof. Using Lemmas 3.2–3.4 and the fact that

$$\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)\varphi_x \le \frac{1}{4\varepsilon}\left(b\psi_x - \int_0^t g(t-s)\psi_x(s)\,ds\right)^2 + \varepsilon\varphi_x^2,$$

then choosing ε such that $4c\varepsilon - (3/4) \le -(1/2)$, we obtain the required result. □

As in **[4, 20]**, we use the multiplier

$$w(x,t) = \frac{1}{L}\left(\int_0^L \psi(y,t)\,dy\right)x - \int_0^x \psi(y,t)\,dy \tag{3.6}$$

which satisfies, for some $c > 0$,

$$\int_0^L w_x^2 dx \le \int_0^L \psi^2 dx$$

and

$$\int_0^L w_t^2 dx \le c\int_0^L \psi_t^2 dx.$$

Lemma 3.6. *Assume that* (H1) *and* (H2) *hold. Then, the functional* J *defined by*

$$J(t) := \int_0^L (\rho_1 w\varphi_t + \rho_2\psi\psi_t)\,dx$$

satisfies, along with the solution of (P), *the estimate*

$$\begin{aligned} J'(t) \le &-\frac{l}{2}\int_0^L \psi_x^2 dx + \varepsilon_0\rho_1\int_0^L \varphi_t^2 dx \\ &+ \frac{c}{\varepsilon_0}\rho_2\int_0^L \psi_t^2 dx + c(g\circ\psi_x), \end{aligned} \tag{3.7}$$

for any $0 < \varepsilon_0 < l$.

Proof. Using Young's inequality and equation (3.6), we obtain

$$J'(t) = \int_0^L \psi\left(b\psi_{xx} - K(\varphi_x+\psi) - \int_0^t g(t-s)\psi_{xx}(s)\,ds\right)dx$$

$$
\begin{aligned}
&+ \rho_2 \int_0^L \psi_t^2 dx + K \int_0^L w(\varphi_x + \psi)_x dx + \rho_1 \int_0^L w_t \varphi_t \, dx \\
&= \rho_2 \int_0^L \psi_t^2 dx - b \int_0^L \psi_x^2 dx \\
&\quad + K \int_0^L (w_x^2 - \psi^2) \, dx + \rho_1 \int_0^L w_t \varphi_t \, dx \\
&\quad + \left(\int_0^t g(s) \, ds \right) \int_0^L \psi_x^2 dx \\
&\quad + \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(s) - \psi_x(t)) \, ds \, dx \\
&\leq \left(\rho_2 + \frac{c}{\varepsilon_0} \right) \int_0^L \psi_t^2 dx + \left(\frac{\varepsilon_0}{2} - l \right) \int_0^L \psi_x^2 dx \\
&\quad + \varepsilon_0 \rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon_0} (g \circ \psi_x) \\
&\leq -\frac{l}{2} \int_0^L \psi_x^2 dx + \varepsilon_0 \rho_1 \int_0^L \varphi_t^2 dx \\
&\quad + \frac{c}{\varepsilon_0} \rho_2 \int_0^L \psi_t^2 dx + c(g \circ \psi_x),
\end{aligned}
$$

provided that $\varepsilon_0 < l$. □

4. General and optimal decay rates for equal speeds of wave propagation.

In this section, we state and prove a general decay result under the equal speed of waves propagation condition. The exponential and polynomial decay results are merely special cases.

Theorem 4.1. *Let*

$$(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0, L) \times L^2(0, L).$$

Assume that the hypotheses (H1) *and* (H2) *and identity* (1.4) *hold. Then, for any* $t_0 > 0$, *there exist two positive constants* C *and* λ, *for which the solution of* (P) *satisfies, for* $t \geq t_0$,

$$E(t) \leq C \exp\left(-\lambda \int_{t_0}^t \xi(s) \, ds \right) \quad \text{for } p = 1, \tag{4.1}$$

and

$$E(t) \leq C\left(\frac{1}{1+\int_{t_0}^{t} \xi^{2p-1}(s)\,ds}\right)^{1/(2p-2)} \quad \textit{for } 1 < p < \frac{3}{2}. \tag{4.2}$$

Moreover, if

$$\int_{t_0}^{+\infty} \left(\frac{1}{1+\int_{t_0}^{t} \xi^{2p-1}(s)\,ds}\right)^{1/(2p-2)} dt < +\infty \quad \textit{for } 1 < p < \frac{3}{2}, \tag{4.3}$$

then

$$E(t) \leq C\left(\frac{1}{1+\int_{t_0}^{t} \xi^{p}(s)\,ds}\right)^{1/(p-1)} \quad \textit{for } 1 < p < \frac{3}{2}. \tag{4.4}$$

Remark 4.2. Inequalities (4.2) and (4.3) together give

$$\int_0^{+\infty} E(t)\,dt < +\infty.$$

Proof of Theorem 4.1. For the case when $p = 1$, see **[20]**. Assume that $1 < p < 3/2$, and let $N_1,\ N_2,\ N_3 > 0$, define a functional

$$\mathcal{L}(t) := N_1 E(t) + N_2 F(t) + I(t) + N_3 J(t),$$

and set

$$g_0 = \int_0^{t_0} g(s)\,ds \quad \text{and} \quad \delta = \frac{1}{4N_2}.$$

Then, from (2.2), (3.1), (3.5) and (3.7), we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq &-\frac{K}{4}\int_0^L (\varphi_x+\psi)^2 dx - \left(\frac{lN_3}{2} - \frac{5}{4}c\right)\int_0^L \psi_x^2 dx \\
&- (\tau - \varepsilon_0 N_3)\rho_1 \int_0^L \varphi_t^2 dx \\
&- \left(N_2 g_0 - \frac{1}{4} - \frac{cN_3}{\varepsilon_0} - c\right)\rho_2 \int_0^L \psi_t^2 dx \\
&+ c\left(4N_2^2 + N_3 + \frac{5}{4}\right)(g \circ \psi_x) \\
&+ \left(\frac{N_1}{2} - 4cN_2^2 - c\right)(g' \circ \psi_x),
\end{aligned} \tag{4.5}$$

for all $t \geq t_0$ and $0 < \varepsilon_0 < l$. First, we choose N_3 large enough so that

$$c_1 := \left(\frac{lN_3}{2} - \frac{5}{4}c\right) > 0,$$

then ε_0 very small so that

$$c_2 := (\tau - \varepsilon_0 N_3) > 0.$$

Next, we pick N_2 large enough so that

$$c_3 := \left(N_2 g_0 - \frac{1}{4} - \frac{cN_3}{\varepsilon_0} - c\right) > 0.$$

Finally, we select N_1 large enough so that

$$\left(\frac{N_1}{2} - 4cN_2^2 - c\right) > 0.$$

Thus, (4.5) becomes

$$\begin{aligned} \mathcal{L}'(t) &\leq -c_1 \int_0^L \psi_x^2 dx - c_2 \rho_1 \int_0^L \varphi_t^2 dx \\ &\quad - c_3 \rho_2 \int_0^L \psi_t^2 dx - \frac{K}{4} \int_0^L (\varphi_x + \psi)^2 dx \\ &\quad + c(g \circ \psi_x) \leq -kE(t) + c(g \circ \psi_x)(t), \end{aligned} \tag{4.6}$$

for all $t \geq t_0$, and some $k > 0$. On the other hand, we can choose N_1 even larger (if necessary) so that

$$\mathcal{L}(t) \sim E(t).$$

Therefore, by using Corollary 2.6 and estimate (4.6), we arrive at

$$\begin{aligned} \xi(t)\mathcal{L}'(t) &\leq -k\xi(t)E(t) + c\xi(t)(g \circ \psi_x)(t) \\ &\leq -k\xi(t)E(t) + c(-E'(t))^{1/(2p-1)}. \end{aligned}$$

Multiplying both sides of the above inequality by $(\xi E)^\alpha(t)$, for $\alpha = 2p - 2$, we obtain

$$\xi^{\alpha+1}(t)E^\alpha(t)\mathcal{L}'(t) \leq -k(\xi E)^{\alpha+1}(t) + c(\xi E)^\alpha(t)(-E'(t))^{1/(2p-1)}.$$

Applying Young's inequality with $q = (\alpha + 1)/\alpha$ and $q' = \alpha + 1$, we obtain

$$\xi^{\alpha+1}(t)E^\alpha(t)\mathcal{L}'(t) \leq -(k - c\gamma)(\xi E)^{\alpha+1}(t) - \frac{c}{\gamma}E'(t), \quad \text{for all } \gamma > 0.$$

We choose γ such that $\lambda_1 := k - c\gamma > 0$ and use the non-increasing property of ξ and E, to have

$$(\xi^{\alpha+1}E^\alpha\mathcal{L})'(t) \le \xi^{\alpha+1}(t)E^\alpha(t)\mathcal{L}'(t) \le -\lambda_1(\xi E)^{\alpha+1}(t) - cE'(t),$$

which entails that

$$(\xi^{\alpha+1}E^\alpha\mathcal{L} + cE)'(t) \le -\lambda_1(\xi E)^{\alpha+1}(t).$$

Let $\mathcal{F} = \xi^{\alpha+1}E^\alpha\mathcal{L} + cE \sim E$. Then

$$\mathcal{F}'(t) \le -\lambda\xi^{\alpha+1}\mathcal{F}^{\alpha+1}(t),$$

for some $\lambda > 0$. Integration over (t_0, t) gives

$$E(t) \le C\left(\frac{1}{1 + \int_{t_0}^t \xi^{2p-1}(s)\,ds}\right)^{1/(2p-2)} \quad \text{for all } t \ge t_0.$$

This establishes (4.2).

Next, we prove (4.4). From (4.6), we have

$$\begin{aligned}\xi(t)\mathcal{L}'(t) &\le -k\xi(t)E(t) + c\xi(t)(g\circ\psi_x)(t)\\ &\le -k\xi(t)E(t)\\ &\quad + c\frac{\eta(t)}{\eta(t)}\int_0^t \left(\xi^p(s)g^p(s)\right)^{1/p}\|\psi_x(t) - \psi_x(t-s)\|_2^2 ds,\end{aligned} \tag{4.7}$$

for any $t \ge t_0$, where

$$\begin{aligned}\eta(t) &= \int_0^t \|\psi_x(t) - \psi_x(t-s)\|_2^2 ds\\ &\le 2\int_0^t (\|\psi_x(t)\|_2^2 + \|\psi_x(t-s)\|_2^2)\,ds\\ &\le \frac{4}{\ell}\int_0^t (E(t) + E(t-s))\,ds\\ &\le \frac{8}{\ell}\int_0^t E(t-s)\,ds = \frac{8}{\ell}\int_0^t E(s)\,ds\\ &\le \frac{8}{\ell}\int_0^{+\infty} E(s)\,ds < +\infty,\end{aligned}$$

by Remark 4.2. Applying Jensen's inequality to the second term on the right-hand side of (4.7), with $G(y) = y^{1/p}$, $y > 0$, $f(s) = \xi^p(s)g^p(s)$

and $h(s) = \|\psi_x(t) - \psi_x(t-s)\|_2^2$, we obtain

$$\xi(t)\mathcal{L}'(t) \leq -k\xi(t)E(t)+c\eta(t)\left(\frac{1}{\eta(t)}\int_0^t \xi^p(s)g^p(s)\|\psi_x(t)-\psi_x(t-s)\|_2^2 ds\right)^{1/p},$$

where we assume that $\eta(t) > 0$; otherwise we get, from (4.6),

$$E(t) \leq C\exp(-kt) \quad \text{for all } t \geq t_0.$$

Therefore,

$$\begin{aligned}\xi(t)\mathcal{L}'(t) &\leq -k\xi(t)E(t)\\ &\quad + c\eta^{(p-1)/p}(t)\left(\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\psi_x(t) - \psi_x(t-s)\|_2^2 ds\right)^{1/p}\\ &\leq -k\xi(t)E(t) + c(-g' \circ \psi_x)^{1/p}(t)\\ &\leq -k\xi(t)E(t) + c(-E'(t))^{1/p}.\end{aligned}$$

Multiplying both sides of the above inequality by $(\xi E)^\alpha(t)$, for $\alpha = p-1$, and repeating the above computations, we arrive at

$$E(t) \leq C\left(\frac{1}{1+\int_{t_0}^t \xi^p(s)\,ds}\right)^{1/(p-1)} \quad \text{for all } t \geq t_0,$$

which establishes (4.4). □

Example 4.3. Let $g(t) = a/(1+t)^q$ with $q > 2$, and let $a > 0$ be chosen such that (H1) is satisfied. Then,

$$g'(t) = -\frac{aq}{(1+t)^{q+1}}. \tag{4.8}$$

Assume that $\rho_1/K = \rho_2/b$. If we write the above identity as

$$g'(t) = -\frac{q}{1+t} \cdot \frac{a}{(1+t)^q} = -\xi(t)g(t),$$

with $\xi(t) = q/(1+t)$ and $p = 1$, then it follows from (4.1) that, for any $t_0 > 0$, there exist $\lambda > 0$ and $C > 0$ such that

$$E(t) \leq C\exp\left(-\lambda\int_{t_0}^t \xi(s)\,ds\right) = \frac{c}{(1+t)^{\lambda q}},$$

with the decay rate λq which is not necessarily the optimal rate.

Now, by writing (4.8) as

$$g'(t) = -a_0\left(\frac{a}{(1+t)^q}\right)^{(q+1)/q} = -\xi(t)g^p(t), \tag{4.9}$$

with $\xi(t) = a_0 = q/a^{1/q}$ and $p = (q+1)/q < 3/2$, we have, for any fixed $t_0 > 0$,

$$\int_{t_0}^{+\infty}\left(\frac{1}{1+\int_{t_0}^t \xi^{2p-1}(s)\,ds}\right)^{1/(2p-2)} dt = \int_{t_0}^{+\infty}\left(\frac{1}{1+c(t-t_0)}\right)^{1/(2p-2)} dt < +\infty.$$

Therefore, inequality (4.3) entails that a constant $C > 0$ exists such that

$$E(t) \le C\left(\frac{1}{1+\int_{t_0}^t \xi^p(s)\,ds}\right)^{1/(p-1)} = \frac{c}{(1+t)^q},$$

with the optimal decay rate q. For more examples, see **[18, 20]**.

5. General decay rate for different speeds of wave propagation. In this section, we state and prove a generalized decay result in the case of non-equal speeds of wave propagation. We begin by differentiating both sides of the differential equations in (P) with respect to t and use the fact that

$$\begin{aligned}\frac{\partial}{\partial t}\left[\int_0^t g(t-s)\psi_{xx}(s)\,ds\right] &= \frac{\partial}{\partial t}\left[\int_0^t g(s)\psi_{xx}(t-s)\,ds\right] \\ &= g(t)\psi_{xx}(0) + \int_0^t g(s)\psi_{xxt}(t-s)\,ds \\ &= \int_0^t g(t-s)\psi_{xxt}(s)\,ds + g(t)\psi_{0xx},\end{aligned}$$

to obtain the following system

(P_*)

$$\begin{cases}\rho_1\varphi_{ttt} - K(\varphi_{xt}+\psi_t)_x = 0, \\ \rho_2\psi_{ttt} - b\psi_{xxt} + K(\varphi_{xt}+\psi_t) + \int_0^t g(t-s)\psi_{xxt}(s)\,ds + g(t)\psi_{0xx} = 0.\end{cases}$$

The energy functional associated to (P_*) is given by

(5.1)
$$E_*(t) := \frac{1}{2}\int_0^L \left[\rho_1\varphi_{tt}^2 + \rho_2\psi_{tt}^2 + \left(b - \int_0^t g(s)\,ds\right)\psi_{xt}^2 + K(\varphi_{xt}+\psi_t)^2\right] dx + \frac{1}{2}(g\circ\psi_{xt})(t).$$

Lemma 5.1 ([**11**]). *Let (φ,ψ) be the strong solution of (P). Then, the energy of (P_*) satisfies, for all $t\geq 0$,*

$$E_*'(t) = -\frac{1}{2}\,g(t)\int_0^L \psi_{xt}^2\,dx + \frac{1}{2}(g'\circ\psi_{xt}) - g(t)\int_0^L \psi_{tt}\psi_{0xx}\,dx \tag{5.2}$$

and

$$E_*(t) \leq c\left(E_*(0) + \int_0^L \psi_{0xx}^2\,dx\right). \tag{5.3}$$

By repeating the steps of [**17**] in proving Lemma 2.5 and using (5.3), we easily obtain the following lemma.

Lemma 5.2. *Assume that hypotheses* (H1) *and* (H2) *hold and (φ,ψ) is the strong solution of (P). Then, for any $0<\sigma<1$, we have*

$$g\circ\psi_{xt} \leq \left[\frac{8}{l}c\left(E_*(0) + \int_0^L \psi_{0xx}^2 dx\right) \cdot \int_0^t g^{1-\sigma}(s)\,ds\right]^{(p-1)/(p+\sigma-1)} \big(g^p\circ\psi_{xt}\big)^{\sigma/(p+\sigma-1)}.$$

In particular, for $\sigma = 1/2$, we get the following inequality:

$$g\circ\psi_{xt} \leq c\left(\int_0^t g^{1/2}(s)\,ds\right)^{(2p-2)/(2p-1)} (g^p\circ\psi_{xt})^{1/(2p-1)}. \tag{5.4}$$

Corollary 5.3. *Assume that conditions* (H1) *and* (H2) *hold and (φ,ψ) is the strong solution of (P). Then,*

$$\xi(t)(g\circ\psi_{xt})(t) \leq c\big(-E_*'(t) + c_1 g(t)\big)^{1/(2p-1)} \quad \textit{for all } t\geq 0,$$

for some positive constant c_1.

Proof. From equation (5.2) and inequality (5.3), we have

$$\begin{aligned} 0 &\leq -g' \circ \psi_{xt} \\ &= -2E_*'(t) - g(t)\int_0^L \psi_{xt}^2 dx - 2g(t)\int_0^L \psi_{tt}\psi_{0xx}\, dx \\ &\leq 2\left(-E_*'(t) + c_1 g(t)\right), \end{aligned} \tag{5.5}$$

for some positive constant c_1. Multiplication on both sides of (5.4) by $\xi(t)$ and the use of Lemma 2.1 and inequality (5.5) give

$$\begin{aligned} &\xi(t)(g \circ \psi_{xt})(t) \\ &\quad\leq c\bigg(\xi(t)\int_0^t g^{1/2}(s)\, ds\bigg)^{(2p-2)/(2p-1)} \big(\xi g^p \circ \psi_{xt}\big)^{1/(2p-1)}(t) \\ &\quad\leq c\bigg(\int_0^t \xi(s) g^{1/2}(s)\, ds\bigg)^{(2p-2)/(2p-1)} \big(-g' \circ \psi_{xt}\big)^{1/(2p-1)}(t) \\ &\quad\leq c\big(-E_*'(t) + c_1 g(t)\big)^{1/(2p-1)}. \end{aligned}$$

□

Let $t_0 > 0$ and $N_1,\ N_2,\ N_3 > 1$. We set $g_0 = \int_0^{t_0} g(s)\, ds$ and $\delta = 1/(4N_2)$ in (3.1), and define a functional $\mathscr{L}$ by

$$\mathscr{L}(t) := N_1(E(t) + E_*(t)) + N_2 F(t) + I(t) + N_3 J(t).$$

Then, from (2.2), (5.2), (3.1), (3.5) and (3.7), we get

$$\begin{aligned} \mathscr{L}'(t) \leq &-\frac{K}{4}\int_0^L (\varphi_x + \psi)^2 dx - \left(\frac{lN_3}{2} - \frac{5}{4}c\right)\int_0^L \psi_x^2 dx \\ &- (\tau - \varepsilon_0 N_3)\int_0^L \rho_1 \varphi_t^2 dx + c\bigg(4N_2^2 + N_3 + \frac{5}{4}\bigg) g \circ \psi_x \\ &- \left(N_2 g_0 - \frac{1}{4} - \frac{cN_3}{\varepsilon_0} - c\right)\rho_2 \int_0^L \psi_t^2 dx \\ &+ \left(\frac{N_1}{2} - 4cN_2^2 - c\right) g' \circ \psi_x + \frac{N_1}{2} g' \circ \psi_{xt} \\ &- N_1 g(t)\int_0^L \psi_{tt}\psi_{0xx}\, dx \\ &+ \left(\frac{\rho_1 b}{K} - \rho_2\right)\int_0^L \varphi_t \psi_{xt}\, dx. \end{aligned} \tag{5.6}$$

Now, we estimate the last term on the right side of (5.6) as in **[11]**.

Lemma 5.4. *Let* (φ, ψ) *be the strong solution of* (P)*. Then, for any* $\varepsilon > 0$*, we have*

$$\left(\frac{\rho_1 b}{K} - \rho_2\right) \int_0^L \varphi_t \psi_{xt}\, dx \le \varepsilon \rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon}(g \circ \psi_{xt} - g' \circ \psi_x) + \frac{c}{\varepsilon} E(0) g(t) \quad \text{for all } t \ge t_0. \tag{5.7}$$

Proof.

$$\left(\frac{\rho_1 b}{K} - \rho_2\right) \int_0^L \varphi_t \psi_{xt}\, dx = \frac{((\rho_1 b/K) - \rho_2)}{\int_0^t g(s)\, ds} \times \int_0^L \varphi_t \int_0^t g(t-s)(\psi_{xt}(t) - \psi_{xt}(s))\, ds\, dx + \frac{((\rho_1 b/K) - \rho_2)}{\int_0^t g(s)\, ds} \times \int_0^L \varphi_t \int_0^t g(t-s)\psi_{xt}(s)\, ds\, dx. \tag{5.8}$$

By observing that

$$\frac{1}{\int_0^t g(s)\, ds} \le \frac{1}{g_0} \quad \text{for all } t \ge t_0$$

and exploiting Young's inequality and Lemma 2.4 (for ψ_{xt}), we get, for $\varepsilon > 0$ and $t \ge t_0$,

$$\frac{((\rho_1 b/K) - \rho_2)}{\int_0^t g(s)\, ds} \int_0^L \varphi_t \int_0^t g(t-s)(\psi_{xt}(t) - \psi_{xt}(s))\, ds\, dx \le \frac{\varepsilon}{2} \rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon}(g \circ \psi_{xt}).$$

On the other hand, by integration by parts and using Lemma 2.4 (for $-g'$ and ψ_x) and the fact that E is non-increasing, we obtain

$$\frac{((\rho_1 b/K) - \rho_2)}{\int_0^t g(s)\, ds} \int_0^L \varphi_t \int_0^t g(t-s)\psi_{xt}(s)\, ds\, dx = \frac{((\rho_1 b/K) - \rho_2)}{\int_0^t g(s)\, ds}$$

$$
\begin{aligned}
&\cdot \int_0^L \varphi_t \Big(g(0)\psi_x - g(t)\psi_{0x} + \int_0^t g'(t-s)\psi_x(s)\,ds \Big)\,dx \\
&= \frac{((\rho_1 b/K) - \rho_2)}{\int_0^t g(s)\,ds} \\
&\quad \cdot \int_0^L \varphi_t \Big(g(t)(\psi_x - \psi_{0x}) - \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))\,ds \Big)\,dx \\
&\le \frac{\varepsilon}{2}\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon} g(t) \int_0^L (\psi_x^2 + \psi_{0x}^2)\,dx - \frac{c}{\varepsilon} g' \circ \psi_x \\
&\le \frac{\varepsilon}{2}\rho_1 \int_0^L \varphi_t^2 dx + \frac{c}{\varepsilon} E(0) g(t) - \frac{c}{\varepsilon} g' \circ \psi_x.
\end{aligned}
$$

Inserting the last two inequalities into (5.8), we get (5.7). □

Lemma 5.5. *Let* (φ, ψ) *be the strong solution of* (P)*. Then, for any* $t \ge t_0$*, we have*

$$
\begin{aligned}
\mathscr{L}'(t) &\le -kE(t) + c(g \circ \psi_x + g \circ \psi_{xt}) \\
&\quad + c\Big(E(0) + E_*(0) + \int_0^L \psi_{0xx}^2 dx \Big) g(t),
\end{aligned}
\tag{5.9}
$$

for some $k > 0$.

Proof. It follows from Young's inequality and (5.3) that

$$
\begin{aligned}
-\int_0^L \psi_{tt}\psi_{0xx}\,dx &\le \frac{1}{2}\int_0^L (\psi_{tt}^2 + \psi_{0xx}^2)\,dx \\
&\le c\Big(E_*(t) + \int_0^L \psi_{0xx}^2 dx \Big) \\
&\le c\Big(E_*(0) + \int_0^L \psi_{0xx}^2 dx \Big).
\end{aligned}
\tag{5.10}
$$

Then, plugging (5.7) and (5.10) into (5.6), we obtain

$$
\begin{aligned}
\mathscr{L}'(t) \le -\frac{K}{4}\int_0^L (\varphi_x + \psi)^2 dx - \Big(\frac{lN_3}{2} - \frac{5}{4}c \Big) \int_0^L \psi_x^2 dx \\
- \big(\tau - (N_3 + 1)\varepsilon_0 \big) \int_0^L \rho_1 \varphi_t^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&- \left(N_2 g_0 - \frac{1}{4} - \frac{cN_3}{\varepsilon_0} - c\right)\rho_2 \int_0^L \psi_t^2 dx \\
&+ c\left(4N_2^2 + N_3 + \frac{5}{4}\right) g \circ \psi_x + \frac{c}{\varepsilon_0} g \circ \psi_{xt} \\
&+ \left(\frac{N_1}{2} - 4cN_2^2 - c - \frac{c}{\varepsilon_0}\right) g' \circ \psi_x + \frac{c}{\varepsilon_0} E(0) g(t) \\
&+ c\left(E_*(0) + \int_0^L \psi_{0xx}^2 dx\right) g(t).
\end{aligned}
$$

At this point, we choose $N_3,\ \varepsilon_0,\ N_2$ and N_1 as in (4.5) to get (5.9). □

Theorem 5.6. *Let* $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in (H^2(0,L) \cap H_0^1(0,L)) \times H_0^1(0,L)$. *Assume that conditions* (H1) *and* (H2) *hold and the coefficients of the problem* (P) *satisfy*

$$\frac{\rho_1}{K} \neq \frac{\rho_2}{b}.$$

Then, for any $t_0 > 0$*, there exists a positive constant* C*, for which the strong solution of* (P) *satisfies, for* $t > t_0$*,*

$$E(t) \le C\left(\frac{1}{\int_{t_0}^t \xi^{2p-1}(s)\, ds}\right)^{1/(2p-1)} \quad \text{for } 1 \le p < \frac{3}{2}. \tag{5.11}$$

Proof. Multiplying both sides of (5.9) by $\xi(t)$ and using Corollaries 2.6 and 5.3, we obtain

$$
\begin{aligned}
\xi(t)\mathscr{L}'(t) &\le -k\xi(t)E(t) + c\xi(t)(g \circ \psi_x + g \circ \psi_{xt}) + c\xi(t)g(t) \\
&\le -k\xi(t)E(t) + c\xi(t)g(t) \\
&\quad + c\left[\big(-E'(t)\big)^{1/(2p-1)} + \big(-E_*'(t) + c_1 g(t)\big)^{1/(2p-1)}\right].
\end{aligned}
$$

Set $\alpha = 2p - 2$, then multiply both sides of the above inequality by $(\xi E)^\alpha(t)$ and exploit Young's inequality with

$$q = \frac{\alpha+1}{\alpha} \quad \text{and} \quad q' = \alpha + 1,$$

to obtain

$$\xi^{\alpha+1}(t)E^\alpha(t)\mathscr{L}'(t) \le -(k - c\gamma)(\xi E)^{\alpha+1}(t) - \frac{c}{\gamma}E'(t) - \frac{c}{\gamma}E_*'(t)$$

$$+ \frac{cc_1}{\gamma} g(t) + c\xi^{\alpha+1}(t)E^{\alpha}(t)g(t), \quad \text{for all } \gamma > 0.$$

We choose $\gamma > 0$ small enough such that $\lambda_2 := k - c\gamma > 0$ and use the non-increasing property of ξ and g to get

$$(\xi^{\alpha+1}E^{\alpha}\mathscr{L}+cE+cE_*)'(t) \leq -\lambda_2(\xi E)^{\alpha+1}(t)+c\xi^{\alpha+1}(t)E^{\alpha}(t)g(t)+cg(t),$$

which implies that

$$\lambda_2(\xi E)^{\alpha+1}(t) \leq -(\xi^{\alpha+1}E^{\alpha}\mathscr{L}+cE+cE_*)'(t)+c\xi^{\alpha+1}(t)E^{\alpha}(t)g(t)+cg(t).$$

We choose N_1 even larger (if needed) in the proof of Lemma 5.5 so that $\mathscr{L} \geq cE$. Then, integration over (t_0, t), together with the non-increasing property of E and ξ, and the hypothesis $(H1)$ yield, for $t \geq t_0$,

$$\begin{aligned} &\lambda_2 E^{\alpha+1}(t) \int_{t_0}^{t} \xi^{\alpha+1}(s)\, ds \\ &\quad \leq \lambda_2 \int_{t_0}^{t} (\xi E)^{\alpha+1}(s)\, ds \leq (\xi^{\alpha+1}E^{\alpha}\mathscr{L} + cE + cE_*)(t) \\ &\qquad + (\xi^{\alpha+1}E^{\alpha}\mathscr{L} + cE + cE_*)(t_0) + (c\xi^{\alpha+1}(0)E^{\alpha}(0) + c) \int_{t_0}^{t} g(s)\, ds \\ &\quad \leq (\xi^{\alpha+1}E^{\alpha}\mathscr{L} + cE + cE_*)(0) \\ &\qquad + \int_0^L \psi_{0xx}^2(x)\, dx + (c\xi^{\alpha+1}(0)E^{\alpha}(0) + c)(b - l). \end{aligned}$$

This entails that

$$E(t) \leq C\left(\frac{1}{\int_{t_0}^{t} \xi^{2p-1}(s)\, ds}\right)^{1/(2p-1)} \quad \text{for all } t > t_0.$$

This completes the proof of Theorem 5.6. □

Example 5.7. Let $g(t) = e^{-at}$, where $a > 0$. Then $g'(t) = -\xi(t)g(t)$ with $\xi(t) = a$. It follows from (5.11) that, for any fixed $t_0 > 0$, there exists a $C > 0$ such that

$$E(t) \leq \frac{C}{t - t_0} \quad \text{for all } t > t_0.$$

Example 5.8. Consider the same function g as in Example 4.3, and write g' as in (4.9). Then, it follows from (5.11) that, for any $t_0 > 0$, there exists $C > 0$ such that

$$E(t) \le C\left(\frac{1}{\int_{t_0}^{t} \xi^{2p-1}(s)\,ds}\right)^{1/(2p-1)} = \frac{c}{(1+t)^{q/(q+2)}}, \quad \text{for } t \text{ large.}$$

For more examples, see **[11, 18]**.

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