GENERATION OF NONLOCAL FRACTIONAL DYNAMICAL SYSTEMS BY FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We show that any two trajectories of solutions of a one-dimensional fractional differential equation (FDE) either coincide or do not intersect each other. However, in the higher-dimensional case, two different trajectories can meet. Furthermore, one-dimensional FDEs and triangular systems of FDEs generate nonlocal fractional dynamical systems, whereas a higher-dimensional FDE does not, in general, generate a nonlocal dynamical system.

1. Introduction. In recent years, fractional differential equations (FDEs) have attracted increasing interest due to the fact that they can model many mathematical problems in science and engineering [**11, 15, 21**]. In this paper, we consider a d-dimensional, fractional differential equation involving the *Caputo derivative* ${}^C D_{0+}^\alpha$ *of order* $\alpha \in (0, 1)$, for t in a finite interval $J := [0, T]$ or in the real half-line $J := [0, \infty)$:

$$(1.1) \qquad\qquad {}^C D_{0+}^\alpha x(t) = f(t, x(t)).$$

Here, $f : J \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous vector-valued function. A continuous function $x : J \to \mathbb{R}^d$ is called a *solution* of (1.1) if this equation is satisfied for all $t \in J \setminus \{0\}$, in which case $x(0)$ is called the *initial value* of the solution $x(\cdot)$.

We are interested in knowing whether (1.1) generates a dynamical system so that the tools and methods of the classical theory of dynamical systems are applicable in the investigation of FDEs. Another important problem in the theory of FDEs, which is closely related to the problem of generation of dynamical systems by FDEs, is the question of whether two different trajectories of an FDE can intersect. We solve both of these problems, namely, we show that a one-dimensional FDE or a triangular FDE generates a nonlocal fractional dynamical system, whereas, in general, a higher-dimensional FDE does not. Correspondingly, two different trajectories of a one-dimensional or a triangular FDE cannot meet, whereas different trajectories of a higher-dimensional FDE may intersect each other. As a byproduct of our investigation, we obtain lower bounds for the solutions of one-dimensional FDEs and of triangular linear FDEs.

The question of whether solutions of (1.1) can intersect was treated by Diethelm [**10, 11**], Diethelm and Ford [**12**], Agarwal et al. [**1**], Hayek et al. [**14**] and Bonilla, Rivero and Trujillo [**4**]. Note that, in the case of ordinary differential equations (ODEs), it is well known that two trajectories either coincide or they do not intersect; the authors mentioned above proved that similar results hold for fractional differential equations of order $\alpha \in (0, 1)$. The main difficulty of the problem for FDEs is the nonlocal nature (or history-dependence) of solutions of FDEs. The above authors used various tools to deal with the FDE case. However, several flaws make their proofs incomplete. We will present some discussion about this matter in Section 3.

The paper is organized as follows. Section 2 is a preparatory section, where we present some basic notions from fractional calculus and the theory of FDEs. Section 3 is devoted to results on separation of solutions of one-dimensional FDEs; we also discuss flaws in the proofs of results from the above-mentioned papers. In Section 4, we study the generation of nonlocal fractional dynamical systems by one-dimensional FDEs. Section 5 is devoted to high-dimensional triangular systems of FDEs, where, based on the results in Section 4, we show that a triangular system of FDEs does generate a nonlocal fractional dynamical system. In Section 6, we show that a higher-dimensional FDE does not, in general, generate a nonlocal dynamical system: two different trajectories of a high-dimensional FDE may intersect each other in finite time.

2. Preliminaries. We start this section by briefly recalling a framework of fractional calculus and fractional differential equations. For more details, we refer to the books of Diethelm [11] and Kilbas, Srivastava and Trujillo [15].

Let $\mathbb{R}^d$ be the standard d-dimensional Euclidean space equipped with usual Euclidean norm. We denote by $\mathbb{R}_+$ the set of all nonnegative real numbers, by $C([0, \infty); \mathbb{R}^d)$ the space of continuous functions from $[0, \infty)$ to $\mathbb{R}^d$, and by $(C_\infty(\mathbb{R}^d), \|\cdot\|_\infty) \subset C([0, \infty); \mathbb{R}^d)$ the space of all continuous functions $\xi : \mathbb{R}_+ \to \mathbb{R}^d$ that are uniformly bounded on $\mathbb{R}_+$, i.e.,

$$\|\xi\|_\infty := \sup_{t \in \mathbb{R}_+} \|\xi(t)\| < \infty.$$

It is well known that $(C_\infty(\mathbb{R}^d), \|\cdot\|_\infty)$ is a Banach space.

Let $\alpha > 0$, $[a, b] \subset \mathbb{R}$ and $x : [a, b] \to \mathbb{R}$, with $x \in L^1([a, b])$, i.e.,

$$\int_a^b |x(\tau)| \, d\tau < \infty.$$

Then, the Riemann-Liouville integral of order α of the function $x(\cdot)$ is defined by

$$I_{a+}^\alpha x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} x(\tau) \, d\tau \quad \text{for } t \in (a, b],$$

see, e.g., Diethelm [11], where the Gamma function is defined by

$$\Gamma(\alpha) := \int_0^\infty \tau^{\alpha-1} \exp(-\tau) \, d\tau \quad \text{for } \alpha > 0.$$

The corresponding Riemann-Liouville fractional derivative of order α of an absolutely continuous function $x(\cdot) : [a, b] \to \mathbb{R}$ is given by

$$^R D_{a+}^\alpha x(t) := (D^m I_{a+}^{m-\alpha} x)(t) \quad \text{for } t \in (a, b],$$

where $D = d/dt$ is the usual derivative and $m := \lceil \alpha \rceil$ is the smallest integer greater than or equal to α. On the other hand, the *Caputo fractional derivative* $^C D_{a+}^\alpha x$ of a function $x \in C^m([a, b])$ is defined by

$$^C D_{a+}^\alpha x(t) := (I_{a+}^{m-\alpha} D^m x)(t) \quad \text{for } t \in (a, b].$$

The Caputo fractional derivative of a d-dimensional vector function $x(t) = (x_1(t), \ldots, x_d(t))^{\mathrm{T}}$ is defined component-wise as

$$^{C}D_{0+}^{\alpha} x(t) = ({}^{C}D_{0+}^{\alpha} x_1(t), \ldots, {}^{C}D_{0+}^{\alpha} x_d(t))^{\mathrm{T}}.$$

It is well known that the initial-value problem of the FDE (1.1) is equivalent to a Volterra integral equation of the second kind. This leads to the following result.

Lemma 2.1. *A continuous function $x : J \to \mathbb{R}$ is a solution of the FDE (1.1) with the initial value condition $x(0) = x_0$ if and only if it is a solution of the Volterra integral equation of the second kind*

$$(2.1) \qquad x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, x(\tau)) \, d\tau.$$

Proof. For a proof when $d = 1$, we refer to Diethelm [**11,** page 86, Lemma 6.2] and Kilbas et al. [**15,** page 199, Theorem 3.24]; the multidimensional case $d > 1$ follows by componentwise application of the one-dimensional case. $\qquad \square$

3. Separation of trajectories of solutions of one-dimensional FDE.

3.1. Two different trajectories of a one-dimensional FDE do not meet. In this section, we consider the one-dimensional case of the system (1.1), i.e., the FDE

$$(3.1) \qquad\qquad {}^{C}D_{0+}^{\alpha} x(t) = f(t, x(t)),$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Assume that f satisfies the following Lipschitz condition on the second variable: there exists a nonnegative continuous function $L : J \to \mathbb{R}_+$ such that

$$(3.2) \qquad\qquad |f(t, x) - f(t, y)| \leq L(t)|x - y|$$
$$\text{for all } t \in J \text{ and all } x, y \in \mathbb{R}.$$

It is well known that, under the Lipschitz condition (3.2), the initial-value problem for (3.1) has a unique solution, defined on the whole interval J, for any given initial value; see, e.g., [**2,** Theorem 2], [**11,** Theorem 6.8], [**22,** Theorem 6.4]. We will show that any two solutions

of (3.1) either coincide or do not intersect on J. In order to do so, we need two useful technical tools for investigating FDEs: the variation-of-constants formula and the comparison principle. The interested reader is referred to [11, Chapter 7] and [15, Chapter 5] for a more detailed discussion about the use of Laplace transforms to obtain the variation-of-constants formula for solutions to FDEs.

Lemma 3.1 (Variation-of-constants formula for FDEs). *Consider the FDE* (3.1) *on the finite interval* $J = [0, T]$. *Assume that the function* $f(\cdot, \cdot)$ *in the equation* (3.1) *satisfies condition* (3.2). *If the function* $f(\cdot, \cdot)$ *is of the form*

$$f(t, x) = Mx + g(t, x)$$

for some fixed $M \in \mathbb{R}$ *and all* $t \in J$ *and* $x \in \mathbb{R}$, *then the solution* $x(\cdot)$ *of* (3.1) *with the initial value* $x(0) = x_0$ *satisfies, for all* $t \in J$, *the formula*

$$x(t) = E_\alpha(Mt^\alpha)x_0 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(M(t-\tau)^\alpha) g(\tau, x(\tau)) \, d\tau,$$

where, for $z \in \mathbb{C}$, *the Mittag-Leffler functions are defined by*

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad and \quad E_\alpha(z) := E_{\alpha,1}(z).$$

Proof. We define a function $\widehat{g}(t, x)$ on $\mathbb{R}_{\geq 0} \times \mathbb{R}$ by

$$\widehat{g}(t, x) := \begin{cases} g(t, x) & \text{if } t \in J \text{ and } x \in \mathbb{R}, \\ g(T, x) & \text{if } t \geq T \text{ and } x \in \mathbb{R}. \end{cases}$$

Then, there exists a positive constant $\widehat{L}$ such that

$$|\widehat{g}(t, x) - \widehat{g}(t, y)| \leq \widehat{L}|x - y| \quad \text{and} \quad |\widehat{g}(t, 0)| \leq \widehat{L}$$

for all $x, y \in \mathbb{R}$ and all $t \in \mathbb{R}_{\geq 0}$.

We now consider, on the half real line $[0, \infty)$, the FDE

$$\begin{aligned} {}^C D_{0+}^\alpha \widehat{x}(t) &= M\widehat{x}(t) + \widehat{g}(t, \widehat{x}(t)) \\ &= M\widehat{x}(t) + \widehat{g}(t, \widehat{x}(t)) - \widehat{g}(t, 0) + \widehat{g}(t, 0). \end{aligned} \tag{3.3}$$

It is obvious that the function $\widehat{g}(t, x) - \widehat{g}(t, 0)$ is Lipschitz continuous with respect to the second variable and satisfies the condition

$$|\widehat{g}(t, x) - \widehat{g}(t, 0)| \leq \widehat{L}|x|, \quad \text{for all } t \in \mathbb{R}_{\geq 0} \text{ and } x \in \mathbb{R}.$$

Moreover, $\widehat{g}(t, 0)$ is a bounded continuous function on $\mathbb{R}_{\geq 0}$. By virtue of Lemma 2.1 and the Lipschitz property of $\widehat{g}$, for any $x_0 \in \mathbb{R}$, the equation (3.3) with the initial condition $\widehat{x}(0) = x_0$ has a unique solution which satisfies the Volterra integral equation

$$\widehat{x}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \left[M\widehat{x}(\tau) + \widehat{g}(\tau, \widehat{x}(\tau)) - \widehat{g}(\tau, 0) \right] d\tau$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \widehat{g}(\tau, 0) \, d\tau,$$

for all $t \geq 0$. Therefore, for any $t \geq 0$, we have

$$|\widehat{x}(t)| \leq |x_0| + \frac{\widehat{L}t^\alpha}{\Gamma(\alpha + 1)} + \frac{M + \widehat{L}}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} |\widehat{x}(\tau)| \, d\tau,$$

and consequently,
(3.4)
$$\frac{|\widehat{x}(t)|}{\exp(t)} \leq \frac{1}{\exp(t)} \left(|x_0| + \frac{\widehat{L}t^\alpha}{\Gamma(\alpha + 1)} \right) + \frac{M + \widehat{L}}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{|\widehat{x}(\tau)|}{\exp(\tau)} \, d\tau.$$

Put $v(t) = \widehat{x}(t) / \exp(t)$ for $t \geq 0$, and set

$$K := \sup_{t \geq 0} \frac{1}{\exp(t)} \left(|x_0| + \frac{\widehat{L}t^\alpha}{\Gamma(\alpha + 1)} \right) < \infty;$$

from (3.4), we obtain the estimate

$$v(t) \leq K + \frac{M + \widehat{L}}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} v(\tau) \, d\tau, \quad t \geq 0,$$

which, by a Gronwall-type inequality [11, page 111, Lemma 6.19], implies that $v(t) \leq K E_\alpha((M + \widehat{L})t^\alpha)$ for $t \geq 0$. Thus,

$$|\widehat{x}(t)| \leq K \exp(t) E_\alpha((M + \widehat{L})t^\alpha) \quad \text{for } t \geq 0,$$

which shows that, for any $x_0 \in \mathbb{R}$, the solution $\widehat{x}(\cdot)$ of the equation (3.3) with the initial value $\widehat{x}(0) = x_0$ is exponentially bounded on $[0, \infty)$. Hence, we can apply the Laplace transform to both sides of equation

(3.3) and obtain, for some positive constant c,

$$s^\alpha \mathcal{L}\{\widehat{x}(t)\}(s) - s^{\alpha-1}x_0 = M\mathcal{L}\{\widehat{x}(t)\}(s) + \mathcal{L}\{\widehat{g}(t, \widehat{x}(t))\}(s), \quad \Re(s) > c;$$

see [**11,** page 134, Theorem 7.1]. Therefore,

$$\mathcal{L}\{\widehat{x}(t)\}(s) = \frac{s^{\alpha-1}}{s^\alpha - M}x_0 + \frac{1}{s^\alpha - M}\mathcal{L}\{\widehat{g}(t, \widehat{x}(t))\}(s),$$

assuming that $\Re(s) > \max\{c, |M|^{1/\alpha}\}$; thus, applying the inverse Laplace transform to both sides gives

$$\widehat{x}(t) = \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha - M}\right\}(t)x_0 + \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha - M}\mathcal{L}\{\widehat{g}(t, \widehat{x}(t))\}(s)\right\}(t),$$

for all $t \geq 0$. Using the well-known formula [**21,** page 21, formula (1.80),],

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(Mt^\alpha)\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha - M}, \quad \Re(s) > |M|^{1/\alpha},$$

with $\beta = \alpha$ or $\beta = 1$ and, using the properties of the Laplace transform [**11,** page 231, Theorem D.11], we conclude that, for all $t \geq 0$,

$$\widehat{x}(t) = E_\alpha(Mt^\alpha)x_0 + \int_0^t (t - \tau)^{\alpha-1}E_{\alpha,\alpha}(M(t - \tau)^\alpha)\widehat{g}(\tau, \widehat{x}(\tau))\, d\tau.$$

For a given initial value x_0, equation (3.1) has a unique solution on J, and likewise, equation (3.3) has a unique solution on $\mathbb{R}_{\geq 0}$. These two solutions coincide on J since $\widehat{g}(t, x) = g(t, x)$ for all $x \in \mathbb{R}$ and $t \in J$, and therefore, $x(\cdot)$ satisfies

$$x(t) = E_\alpha(Mt^\alpha)x_0 + \int_0^t (t - \tau)^{\alpha-1}E_{\alpha,\alpha}(M(t - \tau)^\alpha)g(\tau, x(\tau))\, d\tau,$$

for all $t \in J$. $\qquad\qquad\square$

Remark 3.2. It is easily seen that, in the setting of Lemma 3.1, if J is not compact but is instead the real half-line $[0, \infty)$, then the variation-of-constants formula holds true on the whole of $[0, \infty)$.

Remark 3.3. It can be seen that the proof of Lemma 3.1 can be easily carried out for the higher-dimensional case with M changed to a constant matrix and with the functions $x(\cdot)$, f, g changed to vector functions, cf., [**11,** page 135, Remark 7.1] for both the one-dimensional

and the higher-dimensional cases, where f and g depend only upon t but not upon x.

In order to prove the main result in this section, we need a preparatory lemma that is a small modification of a result of Lakshmikantham [16, Theorem 2.1]. For the reader's convenience, we present a proof here.

Lemma 3.4 (Comparison principle). *Let $0 < q < 1$, and assume that the continuous functions v, v_1, w ,$w_1 \in C(J, \mathbb{R})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$ satisfy*

$$v(t) \leq v_1(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, v(s)) \, ds$$

and

$$w(t) \geq w_1(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, w(s)) \, ds$$

for all $t \in J$. Suppose further that $g(t, x)$ is nondecreasing in x for each $t \in J$. If

(3.5) $v_1(t) < w_1(t) \quad \text{for all } t \in J,$

then

(3.6) $v(t) < w(t) \quad \text{for all } t \in J.$

Proof. Suppose, for contradiction, that (3.6) is not true. Then, due to the continuity of $v(\cdot)$ and $w(\cdot)$, there exists a $t_1 \in J \setminus \{0\}$ such that

(3.7) $v(t_1) = w(t_1) \quad \text{and} \quad v(t) < w(t) \text{ for } 0 \leq t < t_1.$

Using (3.5), (3.7) and the nondecreasing nature of g, we find that

$$v(t_1) \leq v_1(t_1) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} g(s, v(s)) \, ds$$

$$\leq v_1(t_1) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} g(s, w(s)) \, ds$$

$$< w_1(t_1) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} g(s, w(s)) \, ds \leq w(t_1),$$

contradicting the condition $v(t_1) = w(t_1)$ in (3.7). Hence, (3.6) holds, and the proof is complete. $\qquad\square$

Now we are in a position to prove our main theorem of this section.

Theorem 3.5 (Different trajectories do not meet). *Assume that f satisfies the Lipschitz condition (3.2). Then, for any two different initial values $x_{10} \neq x_{20}$ in $\mathbb{R}$, the trajectories of the corresponding solutions of the FDE (3.1) do not meet on J, i.e., the solutions $x_1(\cdot)$ and $x_2(\cdot)$ of (3.1) starting from $x_{10} = x_1(0)$ and $x_{20} = x_2(0)$ satisfy $x_1(t) \neq x_2(t)$ for all $t \in J$.*

Proof. For definiteness, we assume that

$$(3.8) \qquad\qquad x_1(0) = x_{10} < x_{20} = x_2(0).$$

In order to prove the theorem, we show that $x_1(t) < x_2(t)$ for all $t \in J$. Suppose that this is not true. Then there is a $T_1 \in J \setminus \{0\}$ such that $x_1(T_1) \geq x_2(T_1)$. By the continuity of $x_1(\cdot)$ and $x_2(\cdot)$, and by (3.8), there is a $T_2 > 0$ such that

$$x_1(T_2) = x_2(T_2)$$

and

$$x_1(t) < x_2(t)$$

for all $0 \leq t < T_2$. Set $M := \max_{0 \leq t \leq T_2} L(t)$, and define

$$(3.9) \qquad g(t, x) := f(t, x) + Mx \quad \text{for all } 0 \leq t \leq T_2 \text{ and } x \in \mathbb{R}.$$

Then, $g : [0, T_2] \times \mathbb{R} \to \mathbb{R}$ is nondecreasing in x for each $t \in [0, T_2]$. In fact, by (3.2), (3.9) and the choice of M, if $t \in [0, T_2]$ and $x \leq y$, then

$$\begin{aligned} g(t, y) &- g(t, x) \\ &= M(y - x) + f(t, y) - f(t, x) \geq (M - L(t))(y - x) \geq 0. \end{aligned}$$

By virtue of Lemma 3.1, since $x_1(\cdot)$ and $x_2(\cdot)$ are solutions of (3.1), on the interval $[0, T_2]$ we have

$$\begin{aligned} x_1(t) &= E_\alpha(-Mt^\alpha)x_{10} \\ &\qquad + \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-M(t - \tau)^\alpha)\, g(\tau, x_1(\tau))\, d\tau, \end{aligned}$$

(3.10)

and

$$x_2(t) = E_\alpha(-Mt^\alpha)x_{20}$$

(3.11)
$$+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-M(t-\tau)^\alpha)\, g(\tau, x_2(\tau))\, d\tau.$$

Since $E_{\alpha,\alpha}(s) > 0$ for all $s \in \mathbb{R}$ (see, e.g., Cong, et al. [**5,** Lemma 2]), the function $E_{\alpha,\alpha}(-M(t-\tau)^\alpha)g(\tau, x)$ is nondecreasing in the variable x for $0 \le \tau \le t \le T_2$. Therefore, Lemma 3.4 is applicable to the pair of integral equations (3.10) and (3.11) on $[0, T_2]$ and gives $x_1(T_2) < x_2(T_2)$. Thus, we arrive at a contradiction, and consequently, the conclusion of the theorem is true. □

Remark 3.6. Theorem 3.5 provides a full solution to [**10,** Conjecture 1.2]. In that paper, Diethelm provided a partial solution to the problem of separation of trajectories; he proved Conjecture 1.2 under some restrictive conditions [**10,** Theorem 2.2].

3.2. Discussion on the problem of separation of solutions of a general FDE. Diethelm [**11,** Theorem 6.12] formulated and proved a theorem on separation of solutions of an FDE that is the same as our Theorem 3.5. For the proof, he used a fixed point theorem on a short interval of time and then used induction to extend the result to the whole (long) interval of time under consideration. However, his proof contains a flaw in the induction part: the passage to the next step from $N = 1$ to $N = 2$ does not work since the argument leading to a contractive mapping on the first subinterval of time fails on the second subinterval, due to the fact that the FDE is history dependent.

In a subsequent joint work with Ford [**12,** Theorems 3.1, 4.1], Diethelm gave an alternative proof of his result [**11,** Theorem 6.12] on separation of solutions of an FDE. Instead of induction forward in time from one subinterval to the next, they used induction backward in time from one subinterval to the foregoing interval. However, their argument in the first step of this induction does not work. Using the numbering and notation of Diethelm and Ford [**12**], their equation (13) is equivalent to their equation (8) only if we consider (8) and (13) in the whole interval $[0, b]$. Actually, the solution $y(t)$, considered as the solution of the terminal problem (13), depends on the future value $y(b) = c$, and hence, we cannot say that the function $g(t)$ in their

work [**12,** pages 29, 30] is independent of the restriction of function y to $[T_{N-1}, T_N]$. Therefore, the contraction property of the map $\mathcal{F}$ in their formula (14) does not lead to the existence and uniqueness of the solution of (14), as claimed. Consequently, their proof of Theorem 3.1 [**12**], and hence also of Theorem 4.1, is incomplete.

Note that the earlier proof [**11,** Theorem 6.12] is correct for a first "short" interval of time: the smallness of time combined with the bounded Lipschitz condition for f makes a certain operator contractive, and hence, two different solutions cannot meet. In addition, we note that continuity alone can also assure the non-intersection of different trajectories in a short time interval.

Hayek, et al. [**14,** Theorem 3.1] have proved the separation theorem on a "short" interval of time. However, Section 4 of that paper is invalid due to the historical dependence of solutions of the FDE, which prevents the application of the "usual method of prolongation" (for ODEs) as claimed by the authors.

Bonilla, et al. [**4**] treated higher-dimensional linear systems of FDEs and, in Section 3 of that paper, they relied on the above-mentioned result [**14,** Theorem 3.1]. Hence, there are gaps in their proofs of some results [**4,** page 71, Theorem 1 and page 72, Propositions 1 and 2]. For a counterexample, see Section 6, which, coincidentally, also shows that the arguments of Diethelm and Ford [**11, 12**] cannot work; otherwise, as is easily seen, those arguments would work for the higher-dimensional case as well, leading to a contradiction.

4. One-dimensional FDEs generate nonlocal dynamical systems. In this section, based on the results regarding separation of trajectories presented in Section 3, we show that one-dimensional FDEs generate nonlocal dynamical systems. Hence, tools and methods from the classical theory of dynamical systems are applicable.

4.1. Bounds for solutions of FDEs. First, we formulate and prove a lower bound for solutions of a one-dimensional FDE, which provides us with a better understanding of the geometry of the solutions.

Theorem 4.1 (Convergence rate for solutions of one-dimensional FDEs)**.** *Assume that f satisfies the Lipschitz condition* (3.2)*, and put*

$$(4.1) \qquad\qquad L^*(t) = \max_{0 \leq \tau \leq t} L(\tau).$$

Then, for any two solutions $x_1(\cdot)$ and $x_2(\cdot)$ of the FDE (3.1) *and, for any $t \in J$,*

$$|x_2(t) - x_1(t)| \geq |x_2(0) - x_1(0)|E_\alpha(-L^*(t)t^\alpha).$$

Proof. For definiteness, we assume that $x_2(0) \geq x_1(0)$. Then, by Theorem 3.5, we have $x_2(t) \geq x_1(t)$ for any $t \in J$. For an arbitrary but fixed $t \geq 0$, we repeat the arguments in the proof of Theorem 3.5 on the interval $[0, t]$ to conclude that

$$\begin{aligned}
x_2(s) - x_1(s) &= E_\alpha(-L^*(t)s^\alpha)(x_2(0) - x_1(0)) \\
&\quad + \int_0^s (s - \tau)^{\alpha-1}E_{\alpha,\alpha}(-L^*(t)(s - \tau)^\alpha) \\
&\quad \times (g(\tau, x_2(\tau)) - g(\tau, x_1(\tau)))\, d\tau \\
&\geq E_\alpha(-L^*(t)s^\alpha)(x_2(0) - x_1(0)).
\end{aligned}$$

Now, take $s = t$ to complete the proof. $\qquad\square$

Corollary 4.2 (Lower bound for solutions of one-dimensional FDEs)**.** *Assume that f satisfies the Lipschitz condition (3.2). Assume additionally that $f(t, 0) = 0$ for all $t \in J$. Then, for any solution $x(\cdot)$ of the* FDE (3.1) *and any $t \in J$,*

$$|x(t)| \geq |x(0)|E_\alpha(-L^*(t)t^\alpha).$$

Proof. Since $f(t, 0) = 0$, the FDE (3.1) has the trivial solution. Apply Theorem 4.1 to the pair consisting of $x(\cdot)$ and the trivial solution of (3.1). $\qquad\square$

For the divergence rate and upper bound for solutions of the FDEs, the following statements are easy modifications of well-known results.

Theorem 4.3 (Divergence rate for solutions of one-dimensional FDEs)**.** *Assume that f satisfies the Lipschitz condition (3.2), and recall the notation (4.1). Then, for any two solutions $x_1(\cdot)$ and $x_2(\cdot)$ of the* FDE (3.1), *and for any $t \in J$,*

$$|x_2(t) - x_1(t)| \leq |x_2(0) - x_1(0)|E_\alpha(L^*(t)t^\alpha).$$

Proof. By Lemma 2.1, for all $t \in J$, we have

$$x_1(t) = x_1(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, x_1(\tau)) \, d\tau$$

and

$$x_2(t) = x_2(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, x_2(\tau)) \, d\tau.$$

Therefore, if $0 \le s \le t$, then

$$
\begin{aligned}
|x_2(s) - x_1(s)| &\le |x_2(0) - x_1(0)| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^s (s - \tau)^{\alpha-1} |f(\tau, x_2(\tau)) - f(\tau, x_1(\tau))| \, d\tau \\
&\le |x_2(0) - x_1(0)| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^s (s - \tau)^{\alpha-1} L^*(t) |x_2(\tau) - x_1(\tau)| \, d\tau.
\end{aligned}
$$

By virtue of the Gronwall-type inequality for FDEs (see [**11,** page 111, Theorem 6.19] and [**22,** page 288, Lemma 3.1]), this estimate implies that

$$|x_2(s) - x_1(s)| \le |x_2(0) - x_1(0)| E_\alpha(L^*(t)s^\alpha).$$

Since $t \in J$ is arbitrary, the theorem follows by setting $s = t$. $\qquad\square$

Corollary 4.4 (Upper bound for solutions of one-dimensional FDEs). *Assume that f satisfies the Lipschitz condition (3.2). Assume additionally that $f(t,0) = 0$ for all $t \in J$. Then, for any solution $x(\cdot)$ of the FDE (3.1), and for any $t \in J$,*

$$|x(t)| \le |x(0)| E_\alpha(L^*(t)t^\alpha).$$

Proof. Since $f(t,0) = 0$, the FDE (3.1) has the trivial solution. Apply Theorem 4.3 to the pair consisting of $x(\cdot)$ and the trivial solution of (3.1). $\qquad\square$

Remark 4.5. It is easily seen that Theorem 4.3 and Corollary 4.4 also hold true for the case of a higher-dimensional system of FDEs.

4.2. One-dimensional FDEs generate two-parameter flows.
Now, we are in a position to show that one-dimensional FDEs generate two-parameter flows. First, we define the evolution mappings of (3.1).

Definition 4.6. The *evolution mapping* of (3.1) is given by

$$\Phi_{0,T_1} : \mathbb{R} \longrightarrow \mathbb{R}, \quad x_0 \mapsto x(T_1),$$

where $x_0 \in \mathbb{R}$ is an arbitrary initial value of (3.1), $x(\cdot)$ is the solution of (3.1) starting from $x(0) = x_0$, and $x(T_1)$ is the evaluation of $x(\cdot)$ at T_1.

Definition 4.7. A two-parameter family of mappings

$$\phi_{s,t}(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}, \quad s, t \in J,$$

is called a *two-parameter flow* in $\mathbb{R}$ if it has the following three properties:

   (i) $\phi_{s,t}(x)$ is continuous as a function of the three variables $s, t \in J$ and $x \in \mathbb{R}$;
  (ii) for any fixed $s, t \in J$, the mapping $\phi_{s,t}$ is a homeomorphism of $\mathbb{R}$; and
 (iii) the flow property $\phi_{s,t} \circ \phi_{u,s} = \phi_{u,t}$ holds for all $u, s, t \in J$.

Theorem 4.8 (One-dimensional FDEs generate two-parameter flows in $\mathbb{R}$)**.** *The following statements hold for the one-dimensional* FDE (3.1).

  (i) *The evolution mapping $\Phi_{0,t}$ generated by (3.1) is a bijection for each $t \in J$.*
 (ii) *The* FDE (3.1) *generates a two-parameter family of bijections on J by its evolution mappings, as follows:*

(4.2) $$\Phi_{s,t} := \Phi_{0,t} \circ \Phi_{0,s}^{-1} \quad \text{for all } s, t \in J,$$

     where $\Phi_{0,\cdot}$ is the evolution mapping of (3.1) from Definition 4.6.
(iii) *The family $\Phi_{s,t}$, for $s, t \in J$, generated by the* FDE (3.1), *is a two-parameter flow in $\mathbb{R}$.*
 (iv) *If f is linear in x, then the two-parameter flow generated by the* FDE (3.1) *is a flow of linear operators.*

Proof.

(i) Fix an arbitrary $T_1 \in J$. By Theorem 3.5, Φ_{0,T_1} is injective. In order to show that Φ_{0,T_1} is surjective, it suffices to show that, for an arbitrary $x^* \in \mathbb{R}$, the terminal-value problem

$$(4.3) \qquad {}^C D_{0+}^\alpha x(t) = f(t, x(t)) \quad \text{for } t \in [0, T_1],$$
$$(4.4) \qquad x(T_1) = x^*,$$

has a continuous solution, assuming that the function f is continuous and satisfies the Lipschitz condition (3.2). Set

$$M_1 := L^*(T_1) = \max_{0 \le t \le T_1} L(t),$$

where $L(t)$ is determined from (3.2). We denote by $\widehat{x}(\cdot)$ the solution of the FDE (4.3) satisfying the initial condition $\widehat{x}(0) = 0$. Put

$$M_2 := \max_{0 \le t \le T_1} |\widehat{x}(t)|, \qquad M_3 := |x^* - \widehat{x}(T_1)| + M_2,$$

$$M_4 := \frac{M_3}{E_\alpha(-M_1 T_1^\alpha)}, \qquad M_5 := M_2 + M_4 E_\alpha(M_1 T_1^\alpha) + 1.$$

Clearly, $M_2 < M_3 < M_4 < M_5$. Define the function $\widehat{f}$ on $[0, T_1] \times \mathbb{R}$ by

$$(4.5) \qquad \widehat{f}(t, x) = \begin{cases} f(t, x) & \text{if } |x| \le M_5, \\ f(t, M_5 x / |x|) & \text{if } |x| > M_5, \end{cases}$$

and consider the terminal value problem

$$(4.6) \qquad {}^C D_{0+}^\alpha x(t) = \widehat{f}(t, x(t)),$$
$$(4.7) \qquad x(T_1) = x^*.$$

Clearly $\widehat{f}$ is Lipschitz continuous and bounded on $[0, T_1] \times \mathbb{R}$ with Lipschitz constant $L(t)$. Hence, the problem (4.6)–(4.7) has at least one solution, say $x_1(\cdot)$ [**3**, Theorem 8]. We show that $x_1(\cdot)$ is the required solution of (4.3)–(4.4). To this end, notice that, for all $t \in [0, T_1]$, we have

$$|\widehat{x}(t)| \le M_2 < M_5,$$

and hence,

$$f(t, \widehat{x}(t)) = \widehat{f}(t, \widehat{x}(t)).$$

Therefore, $\widehat{x}(\cdot)$ is the solution of the FDE (4.6) satisfying the initial condition $\widehat{x}(0) = 0$. By applying Theorem 4.1 to the solutions $\widehat{x}(\cdot)$ and $x_1(\cdot)$ of the FDE (4.6), we see that, for any $t \in [0, T_1]$,

$$|\widehat{x}(t) - x_1(t)| \geq |\widehat{x}(0) - x_1(0)| E_\alpha(-L^*(t)t^\alpha)$$
$$\geq |x_1(0)| E_\alpha(-M_1 T_1^\alpha).$$

Substituting $t = T_1$, we obtain

$$|\widehat{x}(T_1) - x_1(T_1)| \geq |x_1(0)| E_\alpha(-M_1 T_1^\alpha),$$

and hence,

$$|x_1(0)| \leq \frac{|\widehat{x}(T_1) - x^*|}{E_\alpha(-M_1 T_1^\alpha)} \leq M_4.$$

Applying Theorem 4.3 to the solutions $\widehat{x}(\cdot)$ and $x_1(\cdot)$ of the FDE (4.6) shows that, for any $t \in [0, T_1]$,

$$|x_1(t)| \leq |\widehat{x}(t)| + |x_1(0)| E_\alpha(M_1 T_1^\alpha) \leq M_5,$$

and hence, $f(t, x_1(t)) = \widehat{f}(t, x_1(t))$. Therefore, $x_1(\cdot)$ is a solution of the FDE (4.3), and (i) is proved.

(ii) By (i), the evolution mappings of (3.1) are bijective, and hence $\Phi_{s,t}$ is well defined by (4.2). The flow property is easily verified.

(iii) By (ii), the FDE (3.1) generates a two-parameter family of bijections $\Phi_{s,t}$ of $\mathbb{R}$ for all $s, t \in J$. From Theorems 4.1 and 4.3, it follows that the bijections $\Phi_{s,t}$ are homeomorphisms and Φ continuously depends on three variables s, t and x.

(iv) Obvious. $\square$

Definition 4.9. The two-parameter flow $\Phi_{s,t}$, specified in Theorem 4.8 and generated by the FDE (3.1), is called the *nonlocal dynamical system* generated by (3.1).

Remark 4.10. Two distinguished features of the two-parameter flow generated by the FDE (3.1) are as follows.

(i) The flow has an historical memory. Although the past has impact on the behavior of the solutions, the solutions form a two-parameter flow of homeomorphisms.

(ii) The flow is, in general, α-Hölder, but it is not C^1.

Remark 4.11. Li and Ma [**18,** Theorem 2] claimed that they constructed a dynamical system from an FDE. However, their construction is false. For a counterexample, see Cong, et al. [**6,** Remark 12].

5. Triangular systems of FDEs generate nonlocal dynamical systems.
In this section, using the results of Section 4, we show that a higher-dimensional, triangular system of FDEs also generates a nonlocal dynamical system.

Now, we consider a d-dimensional triangular system of (not necessarily linear) FDEs,

(5.1)
$$
\begin{aligned}
{}^{C}D_{0+}^{\alpha} x_1(t) &= f_1(t, x_1(t)), \\
{}^{C}D_{0+}^{\alpha} x_2(t) &= f_2(t, x_1(t), x_2(t)), \\
&\cdots \\
{}^{C}D_{0+}^{\alpha} x_d(t) &= f_d(t, x_1(t), x_2(t), \ldots, x_d(t)),
\end{aligned}
$$

for $t \in J$, where $x(\cdot) = (x_1(\cdot), \ldots, x_d(\cdot))^{\mathrm{T}} \in \mathbb{R}^d$, and where the vector-valued function $f = (f_1, \ldots, f_d)^{\mathrm{T}}$ is Lipschitz in the x variables, i.e., there exists a continuous function $L : J \to [0, \infty)$ such that, for all $i = 1, \ldots, d$ and all $t \in J$, we have

(5.2) $\quad |f_i(t, x_1, \ldots, x_i) - f_i(t, y_1, \ldots, y_i)|$
$$
\leq L(t)\sqrt{(x_1 - y_1)^2 + \cdots + (x_i - y_i)^2}.
$$

This triangular system has a distinguished property: it can be solved successively coordinate-wise and, each time, we must solve only a one-dimensional FDE. Hence, the triangular system inherits many features of the one-dimensional FDEs.

Proposition 5.1 (Convergence rate for solutions of a triangular system of FDEs). *Assume that the Lipschitz condition* (5.2) *is satisfied, and define* $L^*(t)$ *as before in* (4.1). *Then, for any two solutions* $x(\cdot)$ *and* $y(\cdot)$ *of the triangular FDE* (5.1), *and, for any* $t \in J$,
$$
\|x(t) - y(t)\| \geq \|x(0) - y(0)\| E_\alpha(-L^*(t)t^\alpha).
$$

Proof. Let $x(\cdot) = (x_1(\cdot), \ldots, x_d(\cdot))^{\mathrm{T}}$ and $y(\cdot) = (y_1(\cdot), \ldots, y_d(\cdot))^{\mathrm{T}}$ be two arbitrary solutions of the triangular FDE (5.1). Consider the first

equation in (5.1): it is a one-dimensional FDE for the first coordinate. Applying Theorem 4.1 to this equation, we obtain

$$|x_1(t) - y_1(t)| \geq |x_1(0) - y_1(0)|E_\alpha(-L^*(t)\,t^\alpha).$$

Since the first coordinate is solvable from the first equation, we can substitute it into the second equation of system ((5.1)) and get a one-dimensional FDE for the second coordinate,

$$^CD_{0+}^\alpha u(t) = f_2(t, x_1(t), u(t)) =: \widehat{f}_2(t, u(t)),$$

where, due to (5.2), the function

$$\widehat{f}_2(\cdot, \cdot) : J \times \mathbb{R} \longrightarrow \mathbb{R}$$

is $L(t)$-Lipschitz continuous with respect to the second variable. Applying Theorem 4.1 to the solutions $x_2(\cdot)$ and $y_2(\cdot)$ of this one-dimensional FDE, we obtain

$$|x_2(t) - y_2(t)| \geq |x_2(0) - y_2(0)|E_\alpha(-L^*(t)\,t^\alpha).$$

Continuing this process, we get, for $i = 1, \ldots, d$ and $t \in J$, the inequality

$$(5.3) \qquad |x_i(t) - y_i(t)| \geq |x_i(0) - y_i(0)|E_\alpha(-L^*(t)\,t^\alpha).$$

The conclusion of the proposition follows at once.   □

An important particular case of the triangular system of FDEs (5.1) is a *linear* triangular system,

$$(5.4) \qquad\qquad ^CD_{0+}^\alpha x(t) = A(t)x(t),$$

where $t \in J$, $x \in \mathbb{R}^d$ and

$$A : J \longrightarrow \mathbb{R}^{d \times d}$$

is a bounded continuous triangular $(d \times d)$ matrix-valued function. Thus, $A(\cdot) = [a_{ij}(\cdot)]_{1 \leq i,j \leq d}$ with either $a_{ij} = 0$ for all $i > j$ (upper triangular) or else $a_{ij} = 0$ for all $i < j$ (lower triangular), and there exists a continuous function $L : J \to [0, \infty)$ such that

$$(5.5) \qquad\qquad \|A(t)\| \leq L(t) \quad \text{for all } t \in J.$$

Clearly, Proposition 5.1 is applicable to the linear triangular system (5.4). Moreover, we also have a lower bound for solutions of (5.4).

Proposition 5.2 (Lower bound for solutions of a linear triangular system of FDEs). *Assume that the triangular matrix function A satisfies (5.5), and recall the notation (4.1). Then, for any solution $x(\cdot)$ of the FDE (5.4), and for any $t \in J$,*

$$\|x(t)\| \geq \|x(0)\| E_\alpha(-L^*(t)\, t^\alpha).$$

Proof. Since system (5.4) has the trivial solution 0, we can apply Proposition 5.1 to the two solutions $x(\cdot)$ and 0 of (5.4) and arrive at the desired conclusion. $\qquad\square$

Similar to the one-dimensional case, for any $T_1 \in J$, the *evolution mapping* of (5.1) is given by

$$(5.6) \qquad \phi_{0,T_1} : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad x_0 \mapsto x(T_1),$$

where $x_0 \in \mathbb{R}^d$ is an arbitrary initial value of (5.1), $x(\cdot)$ is the solution of (5.1) starting from $x(0) = x_0$, and $x(T_1)$ is the evaluation of $x(\cdot)$ at T_1. By the same arguments as in Section 4, we can show that the evolution mapping $\phi_{0,t}$ of the triangular FDE (5.1) is a bijection for any $t \in J$.

Definition 5.3. A two-parameter family of mappings

$$\phi_{s,t}(\cdot) : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad s, t \in J,$$

is called a *two-parameter flow* in $\mathbb{R}^d$ if it has the following three properties:

(i) $\phi_{s,t}(x)$ is continuous as a function of the three variables $s, t \in J$ and $x \in \mathbb{R}^d$;
(ii) for any fixed $s, t \in J$, the mapping $\varphi_{s,t}(\cdot)$ is a homeomorphism of $\mathbb{R}^d$; and
(iii) the flow property $\phi_{s,t} \circ \phi_{u,s} = \phi_{u,t}$ holds for all $u, s, t \in J$.

Theorem 5.4 (Triangular systems of FDEs generate nonlocal dynamical systems). *The following statements hold.*

(i) *The triangular system of* FDEs (5.1) *generates a two-parameter flow in $\mathbb{R}^d$, namely,*

$$\phi_{s,t} := \phi_{0,t} \circ \phi_{0,s}^{-1} : \mathbb{R}^d \longrightarrow \mathbb{R}^d \quad \text{for } s,\, t \in J,$$

where $\phi_{0,t}$ is the evolution mapping of (5.1).

(ii) *The linear triangular system of* FDEs (5.4) *generates a two-parameter flow of d-dimensional linear nonsingular operators.*

Proof. Similar to Theorem 4.8.   □

Definition 5.5. The two-parameter flow $\phi_{s,t}$ generated by the triangular FDE (5.1) is called the *nonlocal dynamical system* generated by (5.1).

6. A general high-dimensional system of FDEs does not generate a dynamical system. Finally, we show that, in the high-dimensional case, different trajectories of an FDE *can* intersect each other. Thus, a high-dimensional system of FDEs does *not*, in general, generate a dynamical system. In order for a high-dimensional FDE to generate a nonlocal dynamical system, we need an additional property of the FDE, such as triangularity. The results of this section also provide a counterexample to the assertions of Bonilla, et al. [**4,** Theorem 1, Propositions 1, 2].

Theorem 6.1 (Different trajectories of a high dimensional system of FDEs can meet). *For any $d \geq 2$, there exists a system of type (1.1) with the property that it has two different solutions, $x_1(\cdot)$ and $x_2(\cdot)$, with $x_1(0) \neq x_2(0)$, but which intersect each other at some finite time moment $T \in (0, \infty)$, i.e., $x_1(T) = x_2(T)$.*

Proof. It suffices to construct a two-dimensional system of type (1.1) having the desired property. In fact, the system we construct will also be linear and autonomous.

Since $\alpha \in (0, 1)$, the complex valued Mittag-Leffler function $E_\alpha(\cdot)$ has infinitely many zeros in $\mathbb{C}$ [**13,** page 30, Corollary 3.10]. Fix $z^* \in \mathbb{C}$ such that $E_\alpha(z^*) = 0$. Let

$$\phi := \arg(z^*) \in (-\pi, \pi] \quad \text{and} \quad \lambda := \cos\phi + i\sin\phi,$$

where $i = \sqrt{-1} \in \mathbb{C}$. Note that, since $\alpha \in \mathbb{R}$, we have $E_\alpha(\overline{z^*}) = \overline{E_\alpha(z^*)} = 0$, where $\overline{w}$ denotes the complex conjugate of the complex number w. Since $\alpha \in (0, 1)$, we have $z^* \notin \mathbb{R}$, and hence, $\lambda \notin \mathbb{R}$.

Consider the matrix

$$A := \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix},$$

which has two (complex) eigenvalues, namely, λ and $\bar{\lambda}$. We show that the associated linear and autonomous FDE,

$$(6.1) \qquad {}^{C}D_{0+}^{\alpha} x(t) = Ax(t) \quad \text{for } t \in \mathbb{R}_+,$$

has the desired property. Indeed, it is known [**11**, page 152, Theorem 7.15] that this FDE has two linearly independent solutions of the form

$$x_1(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \quad \text{and} \quad x_2(t) = \begin{bmatrix} -v(t) \\ u(t) \end{bmatrix},$$

where $u, v : \mathbb{R}_+ \to \mathbb{R}$ are given by

$$u(t) := E_\alpha(\lambda t^\alpha) + E_\alpha(\bar{\lambda} t^\alpha)$$

and

$$v(t) := i(E_\alpha(\lambda t^\alpha) - E_\alpha(\bar{\lambda} t^\alpha)).$$

Since $u(0) = 2$ and $v(0) = 0$, it follows that

$$x_1(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

The general solution of (6.1) is

$$(6.2) \qquad x(t) = ax_1(t) + bx_2(t),$$

where $a, b \in \mathbb{R}$ are arbitrary real constants. Let $T > 0$ be the unique finite positive number satisfying

$$\lambda T^\alpha = z^*;$$

a unique such T exists due to the definitions of z^* and λ. Clearly, $u(T) = v(T) = 0$, and hence, $x_1(T) = x_2(T) = (0,0)^{\mathrm{T}}$. From (6.2), it follows that for *any* solution $x(t)$ of (6.1), we have $x(T) = 0$. □

From Theorem 6.1 we immediately obtain the next corollary.

Corollary 6.2 (A higher dimensional FDE does not generate a nonlocal dynamical system)**.** *For $d \geq 2$, the FDE (1.1) does not, in general,*

generate a two-parameter flow in $\mathbb{R}^d$. Hence, it does not, in general, generate a nonlocal dynamical system.

Remark 6.3. We note the following.

(i) Actually, for the FDE (6.1), *all* of the solutions are equal to $(0,0)^{\mathrm{T}}$ at time T; hence, *all* of them meet each other.

(ii) Theorem 6.1 shows that, in contrast to the initial-value problem, the terminal-value problem for FDEs is not always solvable.

(iii) Theorem 6.1 allows us to better understand the dynamics of FDEs by revealing a distinguished feature in comparison with ODEs: different trajectories of an FDE may meet, whereas for an ODE they cannot.

(iv) By a small modification of A in the proof of Theorem 6.1, we can make the time of intersection T small.

(v) By a small modification of the proof, it can be shown that Theorem 6.1 also holds for any positive real $\alpha \neq 1$.

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