JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 29, Number 3, Fall 2017

NECESSARY FREDHOLM CONDITIONS FOR WEIGHTED SINGULAR INTEGRAL OPERATORS WITH SHIFTS AND SLOWLY OSCILLATING DATA

ALEXEI YU. KARLOVICH, YURI I. KARLOVICH AND AMARINO B. LEBRE

Communicated by Bernd Silbermann

To Professor Viktor G. Kravchenko on the occasion of his 75th birthday

ABSTRACT. We extend the main result of **[12]** to the case of more general weighted singular integral operators with two shifts of the form

$$(aI - bU_\alpha)P_\gamma^+ + (cI - dU_\beta)P_\gamma^-,$$

acting on the space $L^p(\mathbb{R}_+)$, $1 < p < \infty$, where

$$P_\gamma^\pm = (I \pm S_\gamma)/2$$

are operators associated with the weighted Cauchy singular integral operator S_γ, given by

$$(S_\gamma f)(t) = \frac{1}{\pi i}\int_{\mathbb{R}_+}\left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau - t}\,d\tau$$

with $\gamma \in \mathbb{C}$ satisfying $0 < 1/p + \Re\gamma < 1$, and U_α, U_β are the isometric shift operators given by

$$U_\alpha f = (\alpha')^{1/p}(f \circ \alpha), \qquad U_\beta f = (\beta')^{1/p}(f \circ \beta),$$

generated by diffeomorphisms α, β of $\mathbb{R}_+$ onto itself having only two fixed points at the endpoints 0 and ∞, under the assumptions that the coefficients a, b, c, d and the derivatives α', β' of the shifts are bounded and continuous on $\mathbb{R}_+$ and admit discontinuities of slowly oscillating type at 0 and ∞.

2010 AMS *Mathematics subject classification.* Primary 45E05, Secondary 47A53, 47B35, 47G10, 47G30.

Keywords and phrases. Orientation-preserving shift, weighted Cauchy singular integral operator, slowly oscillating function, Fredholmness.

This work was partially supported by the Portuguese Foundation for Science and Technology, project Nos. UID/MAT/00297/2013) (Centro de Matemática e Aplicações) and UID/MAT/04721/2013 (Centro de Análise Funcional, Estruturas Lineares e Aplicações). The second author was also supported by the SEP-CONACYT, project Nos. 168104 and 169496 (México).

Received by the editors on August 24, 2016, and in revised form on November 23, 2016.

DOI:10.1216/JIE-2017-29-3-365

1. Introduction. Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space X, and let $\mathcal{K}(X)$ be the ideal of all compact operators in $\mathcal{B}(X)$. An operator $A \in \mathcal{B}(X)$ is called *Fredholm* if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite dimensional. Then, the number

$$\operatorname{Ind} A := \dim \ker A - \dim \ker A^*$$

is referred to as the *index* of A, see, e.g., [**8,** Chapter 4]. For any pair $A, B \in \mathcal{B}(X)$, we will write $A \simeq B$ if $A - B \in \mathcal{K}(X)$.

Following Sarason [**33,** page 820], a bounded continuous function f on $\mathbb{R}_+ = (0,\infty)$ is called *slowly oscillating* (at 0 and ∞) if

$$\lim_{r\to s} \sup_{t,\tau\in[r,2r]} |f(t) - f(\tau)| = 0 \quad \text{for } s \in \{0,\infty\}.$$

The set $\mathrm{SO}(\mathbb{R}_+)$ of all slowly oscillating functions forms a C^*-algebra. This algebra properly contains $C(\overline{\mathbb{R}}_+)$, the C^*-algebra of all continuous functions on $\overline{\mathbb{R}}_+ := [0,+\infty]$.

Suppose that α is an orientation-preserving diffeomorphism of $\mathbb{R}_+$ onto itself, which has only two fixed points, 0 and ∞. We say that α is a *slowly oscillating shift* if $\log \alpha'$ is bounded and $\alpha' \in \mathrm{SO}(\mathbb{R}_+)$. The set of all slowly oscillating shifts is denoted by $\mathrm{SOS}(\mathbb{R}_+)$. By [**11,** Lemma 2.2], an orientation-preserving diffeomorphism $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $\mathrm{SOS}(\mathbb{R}_+)$ if and only if $\alpha(t) = te^{\omega(t)}$, $t \in \mathbb{R}_+$, for some real-valued function $\omega \in \mathrm{SO}(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ such that $\psi(t) := t\omega'(t)$ also belongs to $\mathrm{SO}(\mathbb{R}_+)$ and $\inf_{t\in\mathbb{R}_+}(1 + t\omega'(t)) > 0$. The real-valued slowly oscillating function

$$\omega(t) := \log[\alpha(t)/t], \quad t \in \mathbb{R}_+,$$

is called the *exponent function* of $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$.

Throughout the paper, we will suppose that $1 < p < \infty$. It is easily seen that, if $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$, then the shift operator W_α defined by $W_\alpha f = f \circ \alpha$ is bounded and invertible on all spaces $L^p(\mathbb{R}_+)$, and its inverse is given by $W_\alpha^{-1} = W_{\alpha_{-1}}$ where α_{-1} is the inverse function to α. Along with W_α, we consider the weighted shift operator

$$U_\alpha := (\alpha')^{1/p} W_\alpha,$$

an isometric isomorphism of the Lebesgue space $L^p(\mathbb{R}_+)$ onto itself. It is clear that $U_\alpha^{-1} = U_{\alpha_{-1}}$. Let $a, b \in \mathrm{SO}(\mathbb{R}_+)$. We say that a dominates

b and write $a \gg b$ if

$$\begin{gathered}\inf_{t\in\mathbb{R}_+} |a(t)| > 0,\\ \liminf_{t\to 0}(|a(t)| - |b(t)|) > 0,\\ \liminf_{t\to\infty}(|a(t)| - |b(t)|) > 0.\end{gathered}$$

The next theorem is a generalization of a pioneering result by Kravchenko **[25]** on invertibility of functional operators with shift and continuous data.

Theorem 1.1 ([**15,** Theorem 1.1])**.** *Suppose $a, b \in \mathrm{SO}(\mathbb{R}_+)$ and $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$. The binomial functional operator $aI - bU_\alpha$ is invertible on the Lebesgue space $L^p(\mathbb{R}_+)$ if and only if either $a \gg b$ or $b \gg a$.*

(a) *If $a \gg b$, then*

$$(aI - bU_\alpha)^{-1} = \sum_{n=0}^{\infty}(a^{-1}bU_\alpha)^n a^{-1} I.$$

(b) *If $b \gg a$, then*

$$(aI - bU_\alpha)^{-1} = -U_\alpha^{-1}\sum_{n=0}^{\infty}(b^{-1}aU_\alpha^{-1})^n b^{-1} I.$$

Let $\Re\gamma$ and $\Im\gamma$ denote the real and imaginary parts of $\gamma \in \mathbb{C}$, respectively. As usual, $\overline{\gamma} = \Re\gamma - i\Im\gamma$ denotes the complex conjugate of γ. If $\gamma \in \mathbb{C}$ satisfies

$$0 < 1/p + \Re\gamma < 1, \tag{1.1}$$

then the operators

$$\begin{aligned}(S_\gamma f)(t) &:= \frac{1}{\pi i}\int_{\mathbb{R}_+}\left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau - t}\,d\tau,\\ (R_\gamma f)(t) &:= \frac{1}{\pi i}\int_{\mathbb{R}_+}\left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau + t}\,d\tau,\end{aligned}$$

where the integrals are understood in the principal value sense, are bounded on the Lebesgue space $L^p(\mathbb{R}_+)$, see, e.g., [**4,** subsection 1.10.2], [**5**], [**9,** subsection 2.1.2], [**32,** Proposition 4.2.11]. Put

$$P_\gamma^{\pm} := (I \pm S_\gamma)/2.$$

This paper is a continuation of our work [**11, 12**] wherein we established a Fredholm criterion for the operator

$$N_1 := (aI - bW_\alpha)P_0^+ + (cI - dW_\alpha)P_0^-$$

on the Lebesgue spaces $L^p(\mathbb{R}_+)$ with slowly oscillating data, that is, under the assumptions $a, b, c, d \in \mathrm{SO}(\mathbb{R}_+)$ and $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$.

Unfortunately, the proof of the main result of [**12**] contains a gap in the proof of [**12,** Lemma 7.2]. That proof works only under the additional assumptions

$$\liminf_{t\to s} \alpha'(t) > 1 \quad \text{or} \quad \limsup_{t\to s} \alpha'(t) < 1 \tag{1.2}$$

for each fixed point $s \in \{0, \infty\}$ of the shift α (this condition is not mentioned in [**12**]).

As a matter of fact, since $fW_\alpha = f(\alpha')^{-1/p}U_\alpha$ and $f(\alpha')^{\pm 1/p}$ belong to $\mathrm{SO}(\mathbb{R}_+)$ whenever f belongs to $\mathrm{SO}(\mathbb{R}_+)$ and $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$, the Fredholm study of the operator N_1 is equivalent to the same problem for the operator

$$N_2 := (aI - bU_\alpha)P_0^+ + (cI - dU_\alpha)P_0^-.$$

The usage of U_α instead of W_α has several technical advantages. One of these is that the simplest singular operator with two isometric shifts of the form

$$F_0^{i,j} := U_\alpha^i P_0^+ + U_\beta^j P_0^-,$$

where $\alpha, \beta \in \mathrm{SOS}(\mathbb{R}_+)$, is Fredholm on $L^p(\mathbb{R}_+)$ for every $i, j \in \mathbb{Z}$, and its index is equal to 0 [**13,** Theorem 1.1]. This result was recently extended [**10,** Theorem 1.1] to the operators

$$F^{i,j} := U_\alpha^i P_\gamma^+ + U_\beta^j P_\gamma^-,$$

with $\gamma \in \mathbb{C}$ satisfying (1.1) and having small $|\Im\gamma|$. Another example of weighted singular integral operators with two slowly oscillating shifts, for which the index is available, is the operator

$$W := (I - fU_\alpha^{\varepsilon_1})P_{\gamma_*}^+ + (I - gU_\beta^{\varepsilon_2})P_{\gamma_*}^-,$$

where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ and $\gamma_* := 1/2 - 1/p$. We proved [**14,** Theorem 7.1] that, if $f, g \in \mathrm{SO}(\mathbb{R}_+)$ are such that $1 \gg f$, $1 \gg g$ and $\alpha, \beta \in \mathrm{SOS}(\mathbb{R}_+)$, then the operator W is Fredholm and $\operatorname{Ind} W = 0$.

The aim of this paper is two-fold: to fill in the gaps by removing the additional assumption (1.2) in the necessity portion of the Fredholm criteria for the operator N_1 (equivalently N_2) and to extend the main result of [**12**] to the setting of operators with two possibly different slowly oscillating shifts α, β and weighted operators $P_\gamma^\pm$ with $\gamma \in \mathbb{C}$ satisfying (1.1). More precisely, we will establish necessary conditions for the Fredholmness of the weighted singular integral operator with two slowly oscillating shifts of the form

$$N := (aI - bU_\alpha)P_\gamma^+ + (cI - dU_\beta)P_\gamma^-, \tag{1.3}$$

where $a, b, c, d \in \mathrm{SO}(\mathbb{R}_+)$, $\alpha, \beta \in \mathrm{SOS}(\mathbb{R}_+)$ and $\gamma \in \mathbb{C}$ satisfies (1.1).

We will achieve our aim combining the method of limit operators, see [**28, 31**], and a method of studying the Fredholmness of nonlocal bounded linear operators developed by the second author and Kravchenko [**19, 20, 21**] and further elaborated upon in [**17, 22**].

In order to formulate our main results, we need a bit more notation. By $M(\mathfrak{A})$, we denote the maximal ideal space of a unital commutative Banach algebra $\mathfrak{A}$. Identifying the points $t \in \overline{\mathbb{R}}_+$ with the evaluation functionals $t(f) = f(t)$ for $f \in C(\overline{\mathbb{R}}_+)$, we obtain $M(C(\overline{\mathbb{R}}_+)) = \overline{\mathbb{R}}_+$. Consider the fibers

$$M_s(\mathrm{SO}(\mathbb{R}_+)) := \left\{\xi \in M(\mathrm{SO}(\mathbb{R}_+)) : \xi|_{C(\overline{\mathbb{R}}_+)} = s\right\}$$

of the maximal ideal space $M(\mathrm{SO}(\mathbb{R}_+))$ over the points $s \in \{0, \infty\}$. By [**18,** Proposition 2.1], the set

$$\Delta := M_0(\mathrm{SO}(\mathbb{R}_+)) \cup M_\infty(\mathrm{SO}(\mathbb{R}_+))$$

coincides with $\operatorname{clos}_{\mathrm{SO}^*} \mathbb{R}_+ \setminus \mathbb{R}_+$, where $\operatorname{clos}_{\mathrm{SO}^*} \mathbb{R}_+$ is the weak-star closure of $\mathbb{R}_+$ in the dual space of $\mathrm{SO}(\mathbb{R}_+)$. Then,

$$M(\mathrm{SO}(\mathbb{R}_+)) = \Delta \cup \mathbb{R}_+.$$

In what follows, we write

$$a(\xi) := \xi(a)$$

for every $a \in \mathrm{SO}(\mathbb{R}_+)$ and every $\xi \in \Delta$. With the operator N given by (1.3), we associate the function n defined on $\mathbb{R}_+ \times \mathbb{R}$ by

$$n(t,x) = (a(t) - b(t)e^{i\omega(t)x})p_\gamma^+(x) + (c(t) - d(t)e^{i\eta(t)x})p_\gamma^-(x),$$

where $\omega, \eta \in \mathrm{SO}(\mathbb{R}_+)$ are the exponent functions of α, β, respectively, and

$$\begin{aligned} p_\gamma^\pm(x) &:= (1 \pm s_\gamma(x))/2, \\ s_\gamma(x) &:= \coth[\pi(x + i/p + i\gamma)], \quad x \in \mathbb{R}. \end{aligned} \tag{1.4}$$

Since $n(\cdot, x) \in \mathrm{SO}(\mathbb{R}_+)$ for every $x \in \mathbb{R}$, taking the Gelfand transform of $n(\cdot, x)$, we infer for $(\xi, x) \in (\Delta \cup \mathbb{R}_+) \times \mathbb{R}$ that

$$\begin{aligned} n(\xi, x) := (a(\xi) - b(\xi)&e^{i\omega(\xi)x})p_\gamma^+(x) \\ &+ (c(\xi) - d(\xi)e^{i\eta(\xi)x})p_\gamma^-(x), \end{aligned} \tag{1.5}$$

which gives extensions of the functions $n(\cdot, x)$ to $M(\mathrm{SO}(\mathbb{R}_+))$.

Theorem 1.2 (Main result). *Let* $1 < p < \infty$ *and* $\gamma \in \mathbb{C}$ *satisfy* (1.1). *Suppose that* a, b, c, d *belong to* $\mathrm{SO}(\mathbb{R}_+)$ *and* α, β *belong to* $\mathrm{SOS}(\mathbb{R}_+)$. *If the operator* N *given by* (1.3) *is Fredholm on the space* $L^p(\mathbb{R}_+)$, *then the following conditions are fulfilled*:

(i) *the binomial functional operators*

$$A_+ := aI - bU_\alpha, \qquad A_- := cI - dU_\beta$$

are invertible on the space $L^p(\mathbb{R}_+)$;

(ii) *for every* $\xi \in \Delta$, *the function* n *given by* (1.5) *satisfies*

$$\inf_{x \in \mathbb{R}} |n(\xi, x)| > 0.$$

This paper is organized as follows. We begin with some auxiliary results gathered in Section 2.

In Section 3, we collect known results on Mellin convolution operators with continuous and semi-almost periodic symbols.

Section 4 is devoted to singular integral operators with shifts. We start with the algebra $\mathcal{A}$ generated by the identity operator I and the singular integral operator S_0. Its elements are similar to Mellin convolution operators with continuous symbols. We mention several important relations involving operators $P_\gamma^\pm$ and R_γ. Further, we consider the algebra $\mathcal{FO}_{\alpha,\beta}$ of functional operators generated by the operators $U_\alpha^{\pm 1}$, $U_\beta^{\pm 1}$, and the operators of multiplication by slowly oscillating functions. We recall that $AB \simeq BA$ whenever $A \in \mathcal{FO}_{\alpha,\beta}$ and $B \in \mathcal{A}$. Finally, we show that the operator $P_0^+ P_0^- \chi_J I$ is compact for every closed interval $J \subset \mathbb{R}_+$.

In Section 5 we recall the notion of a limit operator, see e.g., [**28, 31**], and show that the limit operators of the operator N are:

$$N_\xi := (a(\xi) - b(\xi)U_{\alpha_\xi})P_\gamma^+ + (c(\xi) - d(\xi)U_{\beta_\xi})P_\gamma^-, \quad \xi \in \Delta,$$

where the shifts α_ξ and β_ξ are given for $t \in \mathbb{R}_+$ by $\alpha_\xi(t) := e^{\omega(\xi)}t$ and $\beta_\xi(t) := e^{\eta(\xi)}t$, respectively. Hence, the operators N_ξ are similar to Mellin convolution operators with semi-almost periodic symbols $n(\xi, \cdot)$. This allows us to prove the necessity of condition (ii) in Theorem 1.2.

Section 6 is devoted to the proof of the necessity of condition (i) in Theorem 1.2. We begin with a version of the general method for the study of Fredholmness of nonlocal operators [**19, 22**] adapted to the paired operator $M = A_+P_+ + A_-P_-$ with $P_+ + P_- = I$ on a Banach space X. We prove that, if M is Fredholm, then its coefficients A_+, A_- are invertible on X whenever the pairs (A_+, P_+) and (A_-, P_-) satisfy the so-called condition (A). Since the operator N given by (1.3) may be written as

$$N = A_+P_0^+ + C_-P_0^- = C_+P_0^+ + A_-P_0^-$$

with $A_+ = aI - bU_\alpha$, $A_- = cI - dU_\beta$ and some $C_+, C_- \in \mathcal{B}(L^p(\mathbb{R}_+))$, the rest of the section is devoted to the verification of condition (A) for the pairs (A_+, P_0^+) and (A_-, P_0^-). It is based on the recent, important result following from [**7,** Theorem 3.4] and states that the functional operator $A = aI - bU_\alpha$ with $a, b \in \mathrm{SO}(\mathbb{R}_+)$ and $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$ is invertible on the space $L^p(\mathbb{R}_+)$ if and only if each operator in the family of discretizations $\{A_\tau\}_{\tau\in\mathbb{R}_+}$ of A is invertible on the space $\ell^p(\mathbb{Z})$. A construction of sequences of functions required in condition (A) is based on the above result and is similar to that in [**23,** Theorem 5.2]. We would also like to mention that the Pilidi lemma [**30**] formulated in Section 2 plays an important role in our construction.

2. Auxiliary results.

2.1. Lower norms. Let X be a Banach space and $A \in \mathcal{B}(X)$. Following [**27,** subsection 1.3], see also [**29,** subsections B.3.1 and B.3.4], consider lower norms of the operator A defined by

$$|A|_+ = \inf\{\|Ax\|_X : \|x\|_X = 1\},$$
$$|A|_- = \sup\{k \geq 0 : kB_X \subset A(B_X)\},$$

where $B_X = \{x \in X : \|x\|_X \leq 1\}$. Fundamental properties of lower norms are collected in the following statements.

Theorem 2.1 (see, e.g., [**27,** Theorem 1.3.2]). *An operator $A \in \mathcal{B}(X)$ is invertible if and only if $|A|_+ > 0$ and $|A|_- > 0$. If A is invertible, then*

$$|A|_+ = |A|_- = \frac{1}{\|A^{-1}\|_{\mathcal{B}(X)}}.$$

Lemma 2.2 (see, e.g., [**29,** subsection B.3.8]). *If $A \in \mathcal{B}(X)$, then $|A^*|_- = |A|_+$ and $|A^*|_+ = |A|_-$.*

2.2. On convergence of operator sequences involving compact operators. Let us recall the following well-known fact.

Lemma 2.3 (see, e.g., [**32,** Lemma 1.4.7]). *Let X be a Banach space. Suppose that $A, B \in \mathcal{B}(X)$ and $A_n, B_n \in \mathcal{B}(X)$ for all $n \in \mathbb{N}$. If $K \in \mathcal{K}(X)$ and, if $A_n \to A$ and $B_n^* \to B^*$ strongly as $n \to \infty$, then $\|A_nKB_n - AKB\|_{\mathcal{B}(X)} \to 0$ as $n \to \infty$.*

2.3. Pilidi's lemma. Recall that a subset $\Gamma \subset \mathbb{C}$ is referred to as a *Jordan curve* if it is homeomorphic to the complex unit circle $\mathbb{T}$ and an *arc* if it is homeomorphic to a connected subset of the real line $\mathbb{R}$ which contains at least two distinct points. Following [**8,** page 15] or [**26,** page 15], a Jordan curve Γ is said to be *Lyapunov* if the tangent to Γ exists at every point $t \in \Gamma$ and it forms an angle $\theta(t)$ with the real line $\mathbb{R}$ that satisfies the Hölder condition

$$|\theta(t_1) - \theta(t_2)| \leq A|t_1 - t_2|^\mu$$

for all $t_1, t_2 \in \Gamma$, where constants $A > 0$ and $0 < \mu \leq 1$ are independent of t_1, t_2. Let D^+ denote the bounded domain with a Jordan Lyapunov boundary Γ and $D^- = \overline{\mathbb{C}} \setminus (D^+ \cup \Gamma)$. Let $C^+(\Gamma)$ denote the set of all functions analytic in D^+ and continuous in $D^+ \cup \Gamma$, and let $C_0^-(\Gamma)$ be the set of all functions analytic in D^-, continuous in $D^- \cup \Gamma$ and vanishing at infinity.

The next lemma was obtained by Pilidi [**30,** page 513].

Lemma 2.4. *Let Γ be a Jordan Lyapunov curve and $\gamma \subset \Gamma$ an arc. There exist sequences $\{\Phi_n^+\}_{n=1}^\infty \subset C^+(\Gamma)$ and $\{\Phi_n^-\}_{n=1}^\infty \subset C_0^-(\Gamma)$ such that $\|\Phi_n^\pm\|_{L^p(\Gamma)} = 1$ for all $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \|(1-\chi_\gamma)\Phi_n^\pm\|_{L^p(\Gamma)} = 0,$$

where χ_γ is the characteristic function of γ.

3. Mellin convolution operators.

3.1. Fourier and Mellin convolution operators. Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform,

$$(\mathcal{F}f)(x) := \int_{\mathbb{R}} f(y)e^{-ixy}dy, \quad x \in \mathbb{R},$$

and let $\mathcal{F}^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the inverse of $\mathcal{F}$. A function $a \in L^\infty(\mathbb{R})$ is called a *Fourier multiplier* on $L^p(\mathbb{R})$ if the mapping $f \mapsto \mathcal{F}^{-1}a\mathcal{F}f$ maps $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ into itself and extends to a bounded operator on $L^p(\mathbb{R})$. The latter operator is then denoted by $W^0(a)$. We let $\mathcal{M}_p(\mathbb{R})$ stand for the set of all Fourier multipliers on $L^p(\mathbb{R})$. Then, we can show that $\mathcal{M}_p(\mathbb{R})$ is a Banach algebra under the norm

$$\|a\|_{\mathcal{M}_p(\mathbb{R})} := \|W^0(a)\|_{\mathcal{B}(L^p(\mathbb{R}))}.$$

Let $d\mu(t) = dt/t$ be the (normalized) invariant measure on $\mathbb{R}_+$. Consider the Fourier transform on $L^2(\mathbb{R}_+, d\mu)$, which is usually referred to as the Mellin transform and is defined by

$$\mathcal{M} : L^2(\mathbb{R}_+, d\mu) \longrightarrow L^2(\mathbb{R}),$$
$$(\mathcal{M}f)(x) := \int_{\mathbb{R}_+} f(t)t^{-ix}\,\frac{dt}{t}.$$

Let $\mathcal{M}^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+, d\mu)$ be the inverse Mellin transform, and let E be the isometric isomorphism

$$E : L^p(\mathbb{R}_+, d\mu) \longrightarrow L^p(\mathbb{R}), \qquad (Ef)(x) := f(e^x), \quad x \in \mathbb{R}.$$

Then the map $A \mapsto E^{-1}AE$ transforms the Fourier convolution operator $W^0(a) = \mathcal{F}^{-1}a\mathcal{F}$ to the Mellin convolution operator

$$\mathrm{Co}(a) := \mathcal{M}^{-1}a\mathcal{M}$$

with the same symbol a. Hence, the class of Fourier multipliers on $L^p(\mathbb{R})$ coincides with the class of Mellin multipliers on $L^p(\mathbb{R}_+, d\mu)$.

3.2. Continuous and piecewise continuous multipliers. We denote by PC the C^*-algebra of all piecewise continuous functions on $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. By definition, $a \in PC$ if and only if $a \in L^\infty(\mathbb{R})$ and the one-sided limits

$$a(x_0 - 0) := \lim_{x \to x_0 - 0} a(x),$$
$$a(x_0 + 0) := \lim_{x \to x_0 + 0} a(x)$$

exist for each $x_0 \in \dot{\mathbb{R}}$. If a function a is given everywhere on $\mathbb{R}$, then its total variation is defined by

$$V(a) := \sup \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})|,$$

where the supremum is taken over $n \in \mathbb{N}$ and all partitions

$$-\infty < x_0 < x_1 < \cdots < x_n < +\infty.$$

If a has a finite total variation, then it has finite one-sided limits $a(x-0)$ and $a(x+0)$ for all $x \in \dot{\mathbb{R}}$, that is, $a \in PC$. By the Stechkin theorem, see e.g., [**2,** Theorem 17.1] or [**5,** Theorem 2.11], if $a \in PC$ has finite total variation $V(a)$, then $a \in \mathcal{M}_p(\mathbb{R})$ and

$$\|a\|_{\mathcal{M}_p(\mathbb{R})} \le \|S_\mathbb{R}\|_{\mathcal{B}(L^p(\mathbb{R}))}\big(\|a\|_{L^\infty(\mathbb{R})} + V(a)\big),$$

where $S_\mathbb{R}$ is the Cauchy singular integral operator on $\mathbb{R}$. According to [**5,** page 36], see also, [**2,** page 325], let PC_p be the closure in $\mathcal{M}_p(\mathbb{R})$ of the set of all functions $a \in PC$ with finite total variation on $\mathbb{R}$.

Following [**2,** page 331], put

$$C_p(\overline{\mathbb{R}}) := PC_p \cap C(\mathbb{R}),$$

where $\overline{\mathbb{R}} := [-\infty, +\infty]$.

3.3. Semi-almost periodic multipliers. Consider the isometric isomorphism

$$\begin{aligned} &\Phi : L^p(\mathbb{R}_+) \longrightarrow L^p(\mathbb{R}_+, d\mu), \\ &(\Phi f)(t) := t^{1/p} f(t), \quad t \in \mathbb{R}_+. \end{aligned} \tag{3.1}$$

The following simple statement motivates us to go beyond the class of continuous multipliers.

Lemma 3.1. *Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be a multiplicative shift given by $\alpha(t) = kt$ for all $t \in \mathbb{R}_+$ with some $k \in \mathbb{R}_+$. Then, $U_\alpha = \Phi^{-1} \operatorname{Co}(m)\Phi$ with $m(x) := e^{ix \log k}$ for $x \in \mathbb{R}$.*

Proof. The proof is a matter of a direct calculation. □

A function $p : \mathbb{R} \to \mathbb{C}$ of the form

$$p(x) = \sum_{\lambda \in \Omega} r_\lambda e^{i\lambda x},$$

where $r_\lambda \in \mathbb{C}$, $\lambda \in \mathbb{R}$, and Ω is a finite subset of $\mathbb{R}$, is called an *almost periodic polynomial*. The set of all almost periodic polynomials is denoted by AP_0. It follows from Lemma 3.1 that $AP_0 \subset \mathcal{M}_p(\mathbb{R})$. According to [**2,** page 372], AP_p denotes the closure of the set of all almost periodic polynomials in the norm of $\mathcal{M}_p(\mathbb{R})$, and SAP_p denotes the smallest closed subalgebra of $\mathcal{M}_p(\mathbb{R})$ that contains $C_p(\overline{\mathbb{R}})$ and AP_p.

Applying the inverse closedness of the algebra SAP_p in $L^\infty(\mathbb{R})$, see [**2,** Proposition 19.4], we immediately obtain the following.

Theorem 3.2. *Suppose $a \in SAP_p$. The Mellin convolution operator $\operatorname{Co}(a)$ is invertible on the space $L^p(\mathbb{R}_+, d\mu)$ if and only if*

$$\inf_{x \in \mathbb{R}} |a(x)| > 0.$$

4. Singular integral operators with shifts.

4.1. The algebra $\mathcal{A}$ of singular integral operators. Let $\mathcal{A}$ be the smallest closed subalgebra of $\mathcal{B}(L^p(\mathbb{R}_+))$ that contains the operators I and S_0. It is clear that $\mathcal{A}$ is commutative.

Consider the function

$$r_\gamma(x) := 1/\sinh[\pi(x + i/p + i\gamma)], \quad x \in \mathbb{R}. \tag{4.1}$$

The next statement is well known and goes back to Duduchava [**5, 6**] and Simonenko and Chin Ngok Minh [**34**], also see [**4,** subsection 1.10.2], [**9,** subsection 2.1.2], and [**32,** subsections 4.2.2–4.2.3]).

Theorem 4.1. *Let $1 < p < \infty$, $\gamma \in \mathbb{C}$, be such that $0 < 1/p + \Re\gamma < 1$. The functions s_γ and r_γ, given by* (1.4) *and* (4.1)*, respectively, belong to $C_p(\overline{\mathbb{R}})$; the operators S_γ and R_γ belong to $\mathcal{A}$; and*

$$S_\gamma = \Phi^{-1}\operatorname{Co}(s_\gamma)\Phi, \qquad R_\gamma = \Phi^{-1}\operatorname{Co}(r_\gamma)\Phi,$$

where Φ is given by (3.1)*.*

4.2. Some operator identities. Theorem 4.1 is a source of various identities involving the operators S_γ and R_γ.

Lemma 4.2. *Let $1 < p < \infty$ and $\gamma, \delta \in \mathbb{C}$ be such that $0 < 1/p + \Re\gamma < 1$ and $0 < 1/p + \Re\delta < 1$. Then,*

$$\begin{aligned} P_\delta^+ - P_\gamma^+ &= P_\gamma^- - P_\delta^- \\ &= \frac{1}{2}\sinh[\pi i(\gamma - \delta)]R_\gamma R_\delta. \end{aligned}$$

Proof. For $x \in \mathbb{R}$, set $x_\gamma := \pi(x + i/p + i\gamma)$ and $x_\delta := \pi(x + i/p + i\delta)$. Then

$$\begin{aligned} p_\delta^+(x) - p_\gamma^+(x) &= p_\gamma^-(x) - p_\delta^-(x) \\ &= \frac{1}{2}(s_\delta(x) - s_\gamma(x)) \\ &= \frac{1}{2}\frac{\sinh(x_\gamma)\cosh(x_\delta) - \cosh(x_\gamma)\sinh(x_\delta)}{\sinh(x_\gamma)\sinh(x_\delta)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sinh(x_\gamma - x_\delta)}{2\sinh(x_\gamma)\sinh(x_\delta)} \\
&= \frac{1}{2}\sinh[\pi i(\gamma-\delta)]r_\gamma(x)r_\delta(x).
\end{aligned}
$$

Combining this identity with Theorem 4.1, we arrive at the statement of the lemma. □

Lemma 4.3 ([**10,** Lemma 2.4])**.** *Let* $1 < p < \infty$ *and* $\gamma, \delta \in \mathbb{C}$ *be such that* $0 < 1/p + \Re\gamma < 1$ *and* $0 < 1/p + \Re\delta < 1$. *Then,*

$$P_\gamma^- P_\delta^+ = -\frac{e^{i\pi(\delta-\gamma)}}{4} R_\gamma R_\delta.$$

4.3. The adjoint of the operator S_γ.

Lemma 4.4. *Let* $1 < p < \infty$, $1/p + 1/q = 1$ *and* $\gamma \in \mathbb{C}$ *satisfy* (1.1). *Then, the adjoint operator to* S_γ *is given by* $(S_\gamma)^* = S_{-\overline{\gamma}}$ *on the space* $L^q(\mathbb{R}_+)$.

Proof. Since $0 < 1/p + \Re\gamma < 1$, the operator S_0 is bounded on the weighted Lebesgue space

$$L^p(\mathbb{R}_+, w_\gamma) := \{f \text{ is measurable:} f w_\gamma \in L^p(\mathbb{R}_+)\},$$

where $w_\gamma(t) := |t^\gamma| = t^{\Re\gamma}$ for $t \in \mathbb{R}_+$. It is clear that $L^q(\mathbb{R}_+, w_\gamma^{-1})$ is its dual space. Assume that $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$. Then,

$$f(t)t^{-\gamma} \in L^p(\mathbb{R}_+, w_\gamma), \qquad \overline{g(t)}t^\gamma \in L^q(\mathbb{R}_+, w_\gamma^{-1}).$$

Since $\overline{1/(\pi i)} = -1/(\pi i)$ and $\overline{(\tau/t)^{-\gamma}} = (\tau/t)^{-\overline{\gamma}}$, from [**24,** Chapter III, subsection 2.1, Corollary 1], it follows that

$$
\begin{aligned}
\int_{\mathbb{R}_+} (S_\gamma f)(t)\overline{g(t)}\,dt &= \frac{1}{\pi i}\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\frac{f(\tau)\tau^{-\gamma}}{\tau - t}\,d\tau\right)\overline{g(t)}t^\gamma\,dt \\
&= \frac{1}{\pi i}\int_{\mathbb{R}_+} f(\tau)\tau^{-\gamma}\left(\int_{\mathbb{R}_+}\frac{\overline{g(t)}t^\gamma}{\tau - t}\,dt\right)d\tau \\
&= \int_{\mathbb{R}_+} f(\tau)\overline{(S_{-\overline{\gamma}}g)(\tau)}\,d\tau
\end{aligned}
$$

for all $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$. This yields $(S_\gamma)^* = S_{-\overline{\gamma}}$. □

4.4. Compactness of some singular integral operators with shifts. Fix $\alpha, \beta \in \mathrm{SOS}(\mathbb{R}_+)$. Let $\mathcal{FO}_{\alpha,\beta}$ denote the smallest closed subalgebra of $\mathcal{B}(L^p(\mathbb{R}_+))$ that contains the operators $U_\alpha, U_\alpha^{-1}, U_\beta, U_\beta^{-1}$ and all operators cI with $c \in \mathrm{SO}(\mathbb{R}_+)$.

Lemma 4.5 ([**14,** Lemma 2.8]). *Let $\alpha, \beta \in \mathrm{SOS}(\mathbb{R}_+)$. If $A \in \mathcal{FO}_{\alpha,\beta}$ and $B \in \mathcal{A}$, then $AB \simeq BA$.*

The characteristic function of a measurable set $E \subset \mathbb{R}_+$ will be denoted by χ_E.

Lemma 4.6. *For any closed interval $J \subset \mathbb{R}_+$, the operator $P_0^+ P_0^- \chi_J I$ is compact on the space $L^p(\mathbb{R}_+)$.*

Proof. We know from Lemma 4.3 that

$$P_0^+ P_0^- = -\frac{1}{4} R_0^2. \tag{4.2}$$

The closed interval J is separated from 0 and ∞. Suppose that $\widetilde{\chi}_J \in C(\mathbb{R}_+)$ is a function which is equal to 1 on J and vanishes outside of a bounded open neighborhood of J. Then, applying Theorem 4.1 and Lemma 4.5, we obtain

$$R_0 \chi_J I = R_0 \widetilde{\chi}_J \chi_J \simeq \widetilde{\chi}_J R_0 \chi_J I =: A. \tag{4.3}$$

Since A is an integral operator with bounded kernel function of compact support, it follows from [**8,** Chapter 1, Theorem 4.2] that the operator A is compact on the space $L^p(\mathbb{R}_+)$. Hence, in view of (4.3) and (4.2), the operators $R_0 \chi_J I$ and $P_0^+ P_0^- \chi_J I$ are also compact on this space. $\square$

5. Limit operators and Theorem 1.2 (ii).

5.1. Limit operators: Abstract approach. Let X be a Banach space, and let X^* be its dual space. We say that a bounded linear operator U on X is a scaled isometry if U is invertible in $\mathcal{B}(X)$ and $\|U\|_{\mathcal{B}(X)} = 1/\|U^{-1}\|_{\mathcal{B}(X)}$. Every scaled isometry is of the form $k\widetilde{U}$, where $\widetilde{U}$ is an isometry and k is a positive constant.

Let $A \in \mathcal{B}(X)$ and $\mathcal{U} = \{U_n\}_{n=1}^{\infty}$ be a sequence of scaled isometries. If the strong limits

$$A_{\mathcal{U}} := \operatorname*{s\text{-}lim}_{n\to\infty}(U_n^{-1}AU_n) \quad \text{in } \mathcal{B}(X), \tag{5.1}$$

$$A_{\mathcal{U}^*} := \operatorname*{s\text{-}lim}_{n\to\infty}(U_n^{-1}AU_n)^* \quad \text{in } \mathcal{B}(X^*) \tag{5.2}$$

exist, then always $(A_{\mathcal{U}})^* = A_{\mathcal{U}^*}$, and we will refer to the operator $A_{\mathcal{U}}$ as the limit operator for operator A with respect to the sequence $\mathcal{U}$. Note that usually the limit operator $A_{\mathcal{U}}$ is defined independently of the existence of the strong limit $A_{\mathcal{U}^*}$, see e.g., [**31**], while we need the existence of both limits (5.1), (5.2) for our purposes. If the limit operator $A_{\mathcal{U}}$ exists, then it is uniquely determined by A and $\mathcal{U}$, which justifies the notation $A_{\mathcal{U}}$.

In the next statement, we collect basic properties of limit operators.

Lemma 5.1. *Let $\mathcal{U} = \{U_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ be a sequence of scaled isometries. If $A, B \in \mathcal{B}(X)$, $\alpha \in \mathbb{C}$, and if the limit operators $A_{\mathcal{U}}$, $B_{\mathcal{U}}$ exist, then the limit operators $(\alpha A)_{\mathcal{U}}$, $(A+B)_{\mathcal{U}}$, $(AB)_{\mathcal{U}}$ also exist and*

$$(\alpha A)_{\mathcal{U}} = \alpha A_{\mathcal{U}}, \qquad (A+B)_{\mathcal{U}} = A_{\mathcal{U}} + B_{\mathcal{U}}, \qquad (AB)_{\mathcal{U}} = A_{\mathcal{U}}B_{\mathcal{U}}.$$

The proof of the above result may be found in [**28,** Proposition 3.4] or [**31,** Proposition 1.2.2].

Theorem 5.2. *Let X be a Banach space, let $\mathfrak{A}$ be a closed subalgebra of $\mathcal{B}(X)$ and let $\mathfrak{J}$ be a closed two-sided ideal of $\mathfrak{A}$. Suppose that $A \in \mathfrak{A}$ and $\mathcal{U} = \{U_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ is a sequence of scaled isometries such that the limit operator $A_{\mathcal{U}}$ and the limit operators $J_{\mathcal{U}}$ exist and are equal to zero for all $J \in \mathfrak{J}$. If the coset $A+\mathfrak{J}$ is invertible in the quotient algebra $\mathfrak{A}/\mathfrak{J}$, then the limit operator $A_{\mathcal{U}}$ is invertible.*

The proof is obtained by analogy with [**31,** Proposition 1.2.9].

5.2. Realization with dilations. For $x \in \mathbb{R}_+$, consider the dilation operator V_x defined on $L^p(\mathbb{R}_+)$ by

$$(V_x f)(t) := f(t/x), \quad t \in \mathbb{R}_+.$$

It is easy to see that V_x is invertible on the spaces $L^p(\mathbb{R}_+)$ and $V_x^{-1} = V_{1/x}$. Moreover, $\|V_x\|_{\mathcal{B}(L^p(\mathbb{R}_+))} = x^{1/p}$, and hence, V_x is a scaled isometry for every $x \in \mathbb{R}_+$.

Fix $s \in \{0, \infty\}$. We say that a sequence $h := \{h_n\}_{n=1}^\infty \subset \mathbb{R}_+$ is a test sequence relative to the point s if

$$\lim_{n\to\infty} h_n = s.$$

With each test sequence h relative to the point s, we associate the sequence of scaled isometries $\mathcal{V}_h^s := \{V_{h_n}\}_{n=1}^\infty \subset \mathcal{B}(L^p(\mathbb{R}_+))$.

Lemma 5.3 ([**12,** Lemma 4.4])**.** *Let $h := \{h_n\}_{n=1}^\infty \subset \mathbb{R}_+$ be a test sequence relative to a point $s \in \{0, \infty\}$. For any operator $K \in \mathcal{K}(L^p(\mathbb{R}_+))$, the limit operator $K_{\mathcal{V}_h^s}$ with respect to the sequence of scaled isometries*

$$\mathcal{V}_h^s := \{V_{h_n}\}_{n=1}^\infty \subset \mathcal{B}(L^p(\mathbb{R}_+))$$

exists and is the zero operator.

Similarly to [**1,** Proposition 4.2, Corollary 4.3], we have the following important property of the maximal ideal space of the algebra $\mathrm{SO}(\mathbb{R}_+)$.

Lemma 5.4 ([**18,** Proposition 2.2])**.** *Suppose that $\{a_k\}_{k\in\mathbb{N}}$ is a countable subset of the space $\mathrm{SO}(\mathbb{R}_+)$ and $s \in \{0, \infty\}$. For each $\xi \in M_s(\mathrm{SO}(\mathbb{R}_+))$, there exists a sequence $\{t_n\}_{n=1}^\infty \subset \mathbb{R}_+$ such that $t_n \to s$ as $n \to \infty$ and*

$$\xi(a_k) = a_k(\xi) = \lim_{n\to\infty} a_k(t_n) \quad \textit{for all } k \in \mathbb{N}. \tag{5.3}$$

Conversely, if $\{t_n\}_{n=1}^\infty \subset \mathbb{R}_+$ is a sequence such that $t_n \to s$ as $n \to \infty$ and the limits $\lim_{n\to\infty} a_k(t_n)$ exist for all $k \in \mathbb{N}$, then there exists a functional $\xi \in M_s(\mathrm{SO}(\mathbb{R}_+))$ such that (5.3) *holds.*

Now, we are ready to calculate the limit operators of operator N given by (1.3).

Lemma 5.5. *Let $\gamma \in \mathbb{C}$ satisfy* (1.1)*. Suppose that $a, b, c, d \in \mathrm{SO}(\mathbb{R}_+)$, α and β are slowly oscillating shifts and $\omega, \eta \in \mathrm{SO}(\mathbb{R}_+)$ are the exponent functions of α, β, respectively. Let the operator N be given by* (1.3) *and $s \in \{0, \infty\}$. Then, for every $\xi \in M_s(\mathrm{SO}(\mathbb{R}_+))$, there exists* *a test sequence* $h^\xi = \{h_n^\xi\}_{n=1}^\infty \subset \mathbb{R}_+$ *relative to the point* s *such that the limit operator* $N_{\mathcal{V}_{h^\xi}^s}$ *with respect to the sequence of scaled isometries* $\mathcal{V}_{h^\xi}^s := \{V_{h_n^\xi}\}_{n=1}^\infty \subset \mathcal{B}(L^p(\mathbb{R}_+))$ *exists and coincides with*

$$N_\xi = \big(a(\xi)I - b(\xi)U_{\alpha_\xi}\big)P_\gamma^+ + \big(c(\xi)I - d(\xi)U_{\beta_\xi}\big)P_\gamma^-, \tag{5.4}$$

where $\alpha_\xi(t) = e^{\omega(\xi)}t$ *and* $\beta_\xi(t) = e^{\eta(\xi)}t$ *for* $t \in \mathbb{R}_+$.

Proof. Fix $s \in \{0, \infty\}$ and $\xi \in M_s(\mathrm{SO}(\mathbb{R}_+))$. It follows from **[11,** Lemma 2.4] that $\alpha_{-1}, \beta_{-1} \in \mathrm{SOS}(\mathbb{R}_+)$ along with $\alpha, \beta \in \mathrm{SOS}(\mathbb{R}_+)$. Therefore, the functions

$$\begin{aligned} &\omega(t) := \log[\alpha(t)/t], && \eta(t) := \log[\beta(t)/t], \\ &\zeta(t) := \log[\alpha_{-1}(t)/t], && \psi(t) := \log[\beta_{-1}(t)/t] \end{aligned}$$

are real-valued functions in $\mathrm{SO}(\mathbb{R}_+)$.

Consider the following set of slowly oscillating functions:

$$\mathcal{G} := \{a, b, c, d, \alpha', \beta', (\alpha_{-1})', (\beta_{-1})', \omega, \eta, \zeta, \psi, (\alpha')^{1/p}, (\beta')^{1/p}\}.$$

By Lemma 5.4, there exists a test sequence $h^\xi = \{h_n^\xi\}_{n=1}^\infty \subset \mathbb{R}_+$ relative to the point s such that the limit

$$g(\xi) = \xi(g) = \lim_{n\to\infty} g(h_n^\xi)$$

exists for every function $g \in \mathcal{G}$. It was shown in the proof of **[12,** Lemma 4.5] that

$$(gI)_{\mathcal{V}_{h^\xi}^s} = g(\xi)I \quad \text{for } g \in \{a, b, c, d, (\alpha')^{1/p}, (\beta')^{1/p}\} \tag{5.5}$$

and

$$\begin{aligned} (W_\alpha)_{\mathcal{V}_{h^\xi}^s} &= W_{\alpha_\xi}, \\ (W_\beta)_{\mathcal{V}_{h^\xi}^s} &= W_{\beta_\xi}. \end{aligned} \tag{5.6}$$

It follows from **[12,** Lemma 2.8] that

$$\begin{aligned} \alpha'(\xi) &= e^{\omega(\xi)} = \alpha'_\xi, \\ \beta'(\xi) &= e^{\eta(\xi)} = \beta'_\xi. \end{aligned} \tag{5.7}$$

We infer from (5.5)–(5.7) and Lemma 5.1 that

$$
\begin{aligned}
(U_\alpha)_{\mathcal{V}^s_{h^\xi}} &= ((\alpha')^{1/p} W_\alpha)_{\mathcal{V}^s_{h^\xi}} = ((\alpha')^{1/p} I)_{\mathcal{V}^s_{h^\xi}} (W_\alpha)_{\mathcal{V}^s_{h^\xi}} \\
&= (\alpha'(\xi))^{1/p} W_{\alpha_\xi} = (\alpha'_\xi)^{1/p} W_{\alpha_\xi} = U_{\alpha_\xi}
\end{aligned}
\tag{5.8}
$$

and, analogously,

$$
(U_\beta)_{\mathcal{V}^s_{h^\xi}} = U_{\beta_\xi}. \tag{5.9}
$$

By Lemma 4.4, $(S_\gamma)^* = S_{-\overline{\gamma}}$. Then, it is easy to see that, for every $n \in \mathbb{N}$,

$$
V^{-1}_{h^\xi_n} S_\gamma V_{h^\xi_n} = S_\gamma, \qquad (V^{-1}_{h^\xi_n} S_\gamma V_{h^\xi_n})^* = (S_\gamma)^*.
$$

Therefore,

$$
(P^+_\gamma)_{\mathcal{V}^s_{h^\xi}} = P^+_\gamma, \qquad (P^-_\gamma)_{\mathcal{V}^s_{h^\xi}} = P^-_\gamma. \tag{5.10}
$$

Combining (5.5) and (5.8)–(5.10) with Lemma 5.1, we see that the limit operator $N_\xi = N_{\mathcal{V}^s_{h^\xi}}$ exists and is calculated by (5.4). □

5.3. On the gap in the proof of [12, Lemma 7.2]. As already mentioned in the introduction, the proof of [**12,** Lemma 7.2] contains a gap. If condition (1.2) is not satisfied, then, in view of Lemma 5.4, there is a $\xi \in \Delta$ such that $\omega(\xi) = 0$. In this case, we see from the proof of Lemma 5.5 (and its predecessor [**12,** Lemma 4.5]) that the limit operators W_{α_ξ} and U_{α_ξ} of W_α and U_α, respectively, both collapse to the identity operator. Therefore, the theorems on invertibility of binomial functional operators for the study of $a(\xi)I - b(\xi)W_{\alpha_\xi}$ and $a(\xi)I - b(\xi)U_{\alpha_\xi}$ cannot be relied upon without the assumption of (1.2). This fact was overlooked when we wrote the proof of [**12,** Lemma 7.2]. In order to fill in this gap (in the more general setting of the present paper), we use another strategy of the proof of the necessity of condition (i) in Theorem 1.2, which is presented in Section 6.

5.4. Proof of the necessity of Theorem 1.2 (ii). The next lemma gives a useful necessary condition for the Fredholmness of N.

Lemma 5.6. *Let $\gamma \in \mathbb{C}$ satisfy* (1.1). *Suppose that $a, b, c, d \in \mathrm{SO}(\mathbb{R}_+)$, α and β are slowly oscillating shifts, and $\omega, \eta \in \mathrm{SO}(\mathbb{R}_+)$ are the exponent functions of α, β, respectively. If the operator N given by* (1.3) *is Fredholm on the space* $L^p(\mathbb{R}_+)$, *then, for all* $\xi \in \Delta$, *the operators* N_ξ *given by* (5.4) *are invertible on the space* $L^p(\mathbb{R}_+)$.

Proof. The argument is borrowed from the beginning of the proof of [**12,** Lemma 7.2]. Set $\xi \in \Delta$, that is, fix $s \in \{0, \infty\}$ and $\xi \in M_s(\mathrm{SO}(\mathbb{R}_+))$. By Lemma 5.5, there exists a test sequence

$$h^\xi = \{h_n^\xi\}_{n=1}^\infty \subset \mathbb{R}_+$$

relative to the point s such that the limit operator $N_{\mathcal{V}^s_{h^\xi}}$ with respect to the sequence of scaled isometries

$$\mathcal{V}^s_{h^\xi} := \{V_{h_n^\xi}\}_{n=1}^\infty \subset \mathcal{B}(L^p(\mathbb{R}_+))$$

exists and coincides with N_ξ given by (5.4). It follows from Lemma 5.3 that $K_{\mathcal{V}^s_{h^\xi}} = 0$ for every $K \in \mathcal{K}(L^p(\mathbb{R}_+))$. Since the operator N is Fredholm, the coset $N^\pi = N + \mathcal{K}(L^p(\mathbb{R}_+))$ is invertible in the quotient algebra $\mathcal{B}(L^p(\mathbb{R}_+))/\mathcal{K}(L^p(\mathbb{R}_+))$. Applying Theorem 5.2 with $\mathfrak{A} = \mathcal{B}(L^p(\mathbb{R}_+))$, $\mathfrak{J} = \mathcal{K}(L^p(\mathbb{R}_+))$, $A = N$ and $\mathcal{U} = \mathcal{V}^s_{h^\xi}$, we conclude that the operator $N_\xi = N_{\mathcal{V}^s_{h^\xi}}$ is invertible. □

Now, we are in a position to prove the necessity of condition (ii) in Theorem 1.2.

Theorem 5.7. *Let* $\gamma \in \mathbb{C}$ *satisfy* (1.1). *Suppose that* $a, b, c, d \in \mathrm{SO}(\mathbb{R}_+)$, $\alpha, \beta \in \mathrm{SOS}(\mathbb{R}_+)$ *and* ω *and* η *are the exponent functions of* α, β, *respectively. If the operator* N *given by* (1.3) *is Fredholm on the space* $L^p(\mathbb{R})$, *then, for every* $\xi \in \Delta$,

$$\inf_{x \in \mathbb{R}} |n(\xi, x)| > 0, \tag{5.11}$$

where the function n *is defined by* (1.5).

Proof. The proof is analogous to the proof of [**12,** Theorem 7.5]. By Lemma 5.6, for every $\xi \in \Delta$, the operator N_ξ defined by (5.4) is invertible on $L^p(\mathbb{R}_+)$. On the other hand, taking into account Theorem 4.1 and Lemma 3.1, we obtain $N_\xi = \Phi^{-1} \operatorname{Co}(n(\xi, \cdot))\Phi$, where $n(\xi, \cdot) \in SAP_p$ is given by (1.5). Hence, $\operatorname{Co}(n(\xi, \cdot))$ is invertible on the space $L^p(\mathbb{R}_+, d\mu)$. Then, we obtain (5.11) from Theorem 3.2. □

6. Necessity of Theorem 1.2 (i).

6.1. Condition (A). Let X be a Banach space. Recall that an operator $B_r \in \mathcal{B}(X)$, respectively $B_l \in \mathcal{B}(X)$, is said to be a right (resp. left) regularizer for A if

$$AB_r \simeq I, \quad \text{respectively, } B_l A \simeq I.$$

It is well known that an operator A is Fredholm on X if and only if it simultaneously admits a right and a left regularizer. Moreover, each right differs from each left regularizer by a compact operator, see e.g., [**8,** Chapter 4, Section 7]. Therefore, we may speak of a regularizer $B = B_r = B_l$ of a Fredholm operator A.

The ideas behind the next definition follow from [**17, 22**], also see [**26,** pages 249–251] and earlier approaches in [**19, 20, 21**]. Suppose that X is a Banach space and $A_+, A_-, P_+, P_- \in \mathcal{B}(X)$. We say that a pair $(A, P) \in \{(A_+, P_+), (A_-, P_-)\}$ satisfies condition (A) if the non-invertibility of the operator A implies that one of the following conditions holds:

(A-i) there exists a sequence $\{f_n\}_{n=1}^\infty$ of elements of X with norms $\|f_n\|_X = 1$ such that the sequences

$$\{Af_n\}_{n=1}^\infty \quad \text{and} \quad \{(P_+P_-)f_n\}_{n=1}^\infty$$

converge in X, but the sequence $\{Pf_n\}_{n=1}^\infty$ does not contain subsequences convergent in X;

(A-ii) there exists a sequence $\{g_n\}_{n=1}^\infty$ of elements of X^* with norms $\|g_n\|_{X^*} = 1$ such that the sequences

$$\{A^*g_n\}_{n=1}^\infty \quad \text{and} \quad \{(P_+P_-)^*g_n\}_{n=1}^\infty$$

converge in X^*, but the sequence $\{P^*g_n\}_{n=1}^\infty$ does not contain subsequences convergent in X^*.

Theorem 6.1. *Let $A_+, A_-, P_+, P_- \in \mathcal{B}(X)$ and*

$$P_+ + P_- = I, \qquad A_+P_+ \simeq P_+A_+, \qquad A_-P_- \simeq P_-A_-. \tag{6.1}$$

Suppose that the operator

$$M = A_+P_+ + A_-P_-$$

is Fredholm.

(a) *If the pair* (A_+, P_+) *satisfies condition* (A), *then the operator* A_+ *is invertible.*
(b) *If the pair* (A_-, P_-) *satisfies condition* (A), *then the operator* A_- *is invertible.*

Proof. The idea of the proof is borrowed from [**22,** Theorem 2.1]. It follows from (6.1) that

$$P_+P_- = P_-P_+, \qquad A_-P_+ \simeq P_+A_-, \qquad A_+P_- \simeq P_-A_+. \tag{6.2}$$

Let $M^{(-1)}$ be a regularizer of M. Then,

$$M^{(-1)}M \simeq MM^{(-1)} \simeq I. \tag{6.3}$$

(a) Assume that the pair (A_+, P_+) satisfies condition (A) and the operator A_+ is not invertible. Therefore, one of conditions (A-i) or (A-ii) is fulfilled for the pair (A_+, P_+). We infer from (6.1)–(6.3) that

$$\begin{aligned} P_+ &\simeq M^{(-1)}MP_+ = M^{(-1)}(A_+P_+ + (A_- - A_+)P_+P_-) \\ &\simeq M^{(-1)}P_+A_+ + M^{(-1)}(A_- - A_+)P_+P_- \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} P_+ &\simeq P_+MM^{(-1)} = (P_+A_+ + P_+(A_- - A_+)P_-)M^{(-1)} \\ &\simeq A_+P_+M^{(-1)} + P_+P_-(A_- - A_+)M^{(-1)}. \end{aligned} \tag{6.5}$$

It follows from relations (6.4)–(6.5) that there exist $K_1 \in \mathcal{K}(X)$ and $K_2 \in \mathcal{K}(X^*)$ such that

$$P_+ = M^{(-1)}P_+A_+ + M^{(-1)}(A_- - A_+)P_+P_- + K_1, \tag{6.6}$$

$$P_+^* = (M^{(-1)})^*P_+^*A_+^* + (M^{(-1)})^*(A_- - A_+)^*(P_+P_-)^* + K_2. \tag{6.7}$$

Let $\{f_n\}_{n=1}^\infty$ be a sequence in X as in condition (A-i). Since K_1 is compact and $\|f_n\|_X = 1$ for all $n \in \mathbb{N}$, we see that the sequence $\{K_1 f_n\}_{n=1}^\infty$ contains a convergent subsequence $\{K_1 f_{n_k}\}_{k=1}^\infty$. By condition (A-i), the subsequences $\{A_+ f_{n_k}\}_{k=1}^\infty$ and $\{P_+P_- f_{n_k}\}_{k=1}^\infty$ are also convergent. Thus, the subsequence $\{P_+ f_{n_k}\}_{k=1}^\infty$ is convergent in view of (6.6), but this contradicts condition (A-i), which says that $\{P_+ f_n\}_{n=1}^\infty$ does not contain convergent subsequences.

Analogously, if $\{g_n\}_{n=1}^{\infty}$ is a sequence in X^* as in condition (A-ii), then a contradiction may be reached in view of (6.7). This observation completes the proof of part (a).

(b) The proof of part (b) is analogous to the previous proof and, therefore, is omitted. □

6.2. Discretization of functional operators. Consider the isometric shift operator V defined on the space $\ell^p(\mathbb{Z})$ by

$$(Vf)(n) = f(n+1), \quad n \in \mathbb{Z},\ f \in \ell^p(\mathbb{Z}).$$

Obviously, the operator V is invertible on the space $\ell^p(\mathbb{Z})$, and

$$(V^{-1}f)(n) = f(n-1), \quad n \in \mathbb{Z},\ f \in \ell^p(\mathbb{Z}).$$

Suppose that $a, b \in \mathrm{SO}(\mathbb{R})$ and $\alpha \in \mathrm{SOS}(\mathbb{R})$. Let α_{-1} be the inverse function to α. For $t \in \mathbb{R}_+$ and $n \in \mathbb{Z}$, let $\alpha_0(t) := t$ and $\alpha_n(t) := \alpha[\alpha_{n-1}(t)]$. By analogy with **[7]**, **[18,** Section 6] and **[23,** subsection 3.1], we associate with every functional operator of the form $A = aI - bU_\alpha$ the family of discrete operators $A_\tau \in \mathcal{B}(\ell^p(\mathbb{Z}))$, $\tau \in \mathbb{R}_+$, defined by

$$A_\tau = a_\tau I - b_\tau V, \tag{6.8}$$

where

$$\begin{aligned} a_\tau(n) &= a[\alpha_n(\tau)], \\ b_\tau(n) &= b[\alpha_n(\tau)], \quad n \in \mathbb{Z}. \end{aligned} \tag{6.9}$$

Then $a_\tau, b_\tau \in \ell^\infty(\mathbb{Z})$.

The next theorem immediately follows from a more general result **[7,** Theorem 3.4].

Theorem 6.2. *Let $a, b \in \mathrm{SO}(\mathbb{R}_+)$ and $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$. The functional operator $A = aI - bU_\alpha$ is invertible on the space $L^p(\mathbb{R}_+)$ if and only if for all $\tau \in \mathbb{R}_+$ the discrete operators A_τ defined by* (6.8)–(6.9) *are invertible on the space $\ell^p(\mathbb{Z})$.*

6.3. Construction of special functions related to functional operators. In this subsection, we present a construction which will serve as a basis for the verification of conditions (A-i) and (A-ii).

For a finite set $Q \subset \mathbb{Z}$ and $z \in \mathbb{Z}$, put $Q + z := \{m + z : m \in Q\}$.

Lemma 6.3. *Let $1 < p < \infty$, $1/p + 1/q = 1$. Suppose that $a, b \in \mathrm{SO}(\mathbb{R}_+)$, $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$ and $A := aI - bU_\alpha$. Let $\tau \in \mathbb{R}_+$ and A_τ be the discrete operator associated to the functional operator A by* (6.8), (6.9)*.*

(a) *If $|A_\tau|_+ = 0$, then for every $\varepsilon > 0$ there exists a function $\varphi \in \ell^p(\mathbb{Z})$ with finite support $Q \subset \mathbb{Z}$ and a symmetric open neighborhood $\widetilde{u}$ of τ such that $\|\varphi\|_{\ell^p(\mathbb{Z})} = 1$, for all $k \in \widetilde{Q} := Q \cup (Q - 1)$ the closures $\overline{\alpha_k(\widetilde{u})}$ are pairwise disjoint and, for every symmetric open neighborhood u of τ satisfying $u \subset \widetilde{u}$ and every $g_u \in L^p(u)$, the function*

$$f_u(t) := \begin{cases} \varphi(k)(U_{\alpha_k}^{-1} g_u)(t) & \text{if } t \in \alpha_k(u), k \in Q, \\ 0 & \text{if } t \in \mathbb{R}_+ \setminus \left(\bigcup\limits_{k \in Q} \alpha_k(u) \right), \end{cases} \tag{6.10}$$

satisfies the relations

$$\|f_u\|_{L^p(\mathbb{R}_+)} = \|g_u\|_{L^p(u)}, \qquad \|Af_u\|_{L^p(\mathbb{R}_+)} \le \varepsilon \|g_u\|_{L^p(u)}. \tag{6.11}$$

(b) *If $|A^*_\tau|_+ = 0$, then for every $\varepsilon > 0$ there exists a function $\varphi^* \in \ell^q(\mathbb{Z})$ with finite support $Q \subset \mathbb{Z}$ and a symmetric open neighborhood $\widetilde{u}$ of τ such that $\|\varphi^*\|_{\ell^q(\mathbb{Z})} = 1$, for all $k \in \widetilde{Q} := Q \cup (Q + 1)$ the closures $\overline{\alpha_{-k}(\widetilde{u})}$ are pairwise disjoint and, for every symmetric open neighborhood u of τ satisfying $u \subset \widetilde{u}$ and every $g^*_u \in L^q(u)$, the function*

$$f^*_u(t) := \begin{cases} \varphi^*(-k)(U_{\alpha_k} g^*_u)(t) & \text{if } \ t \in \alpha_{-k}(u), \ k \in Q, \\ 0 & \text{if } t \in \mathbb{R}_+ \setminus \left(\bigcup\limits_{k \in Q} \alpha_{-k}(u) \right), \end{cases}$$

satisfies the relations

$$\|f^*_u\|_{L^q(\mathbb{R}_+)} = \|g^*_u\|_{L^q(u)}, \qquad \|A^* f^*_u\|_{L^q(\mathbb{R}_+)} \le \varepsilon \|g^*_u\|_{L^q(u)}.$$

Proof.

(a) Fix $\tau \in \mathbb{R}_+$. Since the set of functions of finite support is dense in $\ell^p(\mathbb{Z})$, it follows from the definition of the lower norm $|A_\tau|_+$ that, for every $\varepsilon > 0$, there exists a function $\varphi \in \ell^p(\mathbb{Z})$ of finite support $Q \subset \mathbb{Z}$

such that

$$\|\varphi\|_{\ell^p(\mathbb{Z})} = 1, \qquad \|A_\tau \varphi\|_{\ell^p(\mathbb{Z})} < \frac{\varepsilon}{2}. \tag{6.12}$$

For $\tau \in \mathbb{R}_+$, there is a point $\widetilde{\tau} \in \mathbb{R}_+$ such that τ lies in the open interval ℓ with the endpoints $\widetilde{\tau}$ and $\alpha(\widetilde{\tau})$. Then, $\alpha_i(\ell) \cap \alpha_j(\ell) = \emptyset$ for every $i, j \in \mathbb{Z}$ satisfying $i \neq j$. In view of the continuity of the coefficients a, b of the operator $A = aI - bU_\alpha$ on $\mathbb{R}_+$, there exists an open symmetric neighborhood $\widetilde{u} \subset \ell$ of the point $\tau \in \mathbb{R}_+$ such that, for all $k \in \widetilde{Q}$, the closures $\overline{\alpha_k(\widetilde{u})}$ are pairwise disjoint and, in view of (6.12), for any $t \in \widetilde{u}$,

$$\|A_t \varphi\|_{\ell^p(\mathbb{Z})} \le \|A_\tau \varphi\|_{\ell^p(\mathbb{Z})} + \frac{\varepsilon}{2} < \varepsilon. \tag{6.13}$$

Applying (6.10) and the first equality in (6.12), we obtain

$$\begin{aligned}
\|f_u\|^p_{L^p(\mathbb{R}_+)} &= \sum_{k \in Q} \int_{\alpha_k(u)} |\varphi(k)|^p \big|(U^{-1}_{\alpha_k} g_u)(t)\big|^p dt \\
&= \sum_{k \in Q} |\varphi(k)|^p \int_{\alpha_k(u)} \big|g_u[\alpha_{-k}(t)]\big|^p |\alpha'_{-k}(t)|\, dt \\
&= \Big(\sum_{k \in Q} |\varphi(k)|^p\Big) \int_u |g_u(t)|^p dt \\
&= \|\varphi\|^p_{\ell^p(\mathbb{Z})} \|g_u\|^p_{L^p(u)} \\
&= \|g_u\|^p_{L^p(u)},
\end{aligned}$$

which implies the first equality in (6.11).

It is easy to see that, for $t \in \alpha_k(u)$ with $k \in Q - 1$, we have

$$(U_\alpha f_u)(t) = \varphi(k+1)[U_\alpha(U^{-1}_{\alpha_{k+1}} g_u)](t) = (V\varphi)(k)(U^{-1}_{\alpha_k} g_u)(t),$$

and

$$(U_\alpha f_u)(t) = 0 \quad \text{for } t \in \mathbb{R}_+ \setminus \Big(\bigcup_{k \in Q-1} \alpha_k(u)\Big).$$

Hence, for $t \in \alpha_k(u)$ with $k \in \widetilde{Q}$, we obtain

$$(A f_u)(t) = [a(t)\varphi(k) - b(t)(V\varphi)(k)](U^{-1}_{\alpha_k} g_u)(t), \tag{6.14}$$

and

$$(Af_u)(t) = 0 \quad \text{for } t \in \mathbb{R}_+ \setminus \Bigg(\bigcup_{k \in \widetilde{Q}} \alpha_k(u) \Bigg). \tag{6.15}$$

Applying equalities (6.8), (6.9), (6.14), (6.15), and inequality (6.13), we get

$$\begin{aligned}
\|Af_u\|^p_{L^p(\mathbb{R}_+)} &= \sum_{k \in \widetilde{Q}} \int_{\alpha_k(u)} \big|a(t)\varphi(k) - b(t)(V\varphi)(k)\big|^p |(U^{-1}_{\alpha_k} g_u)(t)|^p dt \\
&= \sum_{k \in \widetilde{Q}} \int_u \big|a[\alpha_k(t)]\varphi(k) - b[\alpha_k(t)](V\varphi)(k)\big|^p |g_u(t)|^p dt \\
&= \sum_{k \in \widetilde{Q}} \int_u \big|(A_t\varphi)(k)\big|^p |g_u(t)|^p dt \\
&= \int_u \Bigg(\sum_{k \in \widetilde{Q}} \big|(A_t\varphi)(k)\big|^p \Bigg) |g_u(t)|^p dt \\
&= \int_u \big\|A_t\varphi\big\|^p_{\ell^p(\mathbb{Z})} |g_u(t)|^p dt \\
&\le (\varepsilon \|g_u\|_{L^p(u)})^p,
\end{aligned}$$

which implies the second inequality in (6.11) and completes the proof of part (a).

The proof of part (b) is analogous and based on the facts that $A^* = \overline{a}I - U^{-1}_\alpha \overline{b}I \in \mathcal{B}(L^q(\mathbb{R}_+))$ and $A^*_\tau = \overline{a_\tau}I - V^{-1}\overline{b_\tau}I \in \mathcal{B}(\ell^q(\mathbb{Z}))$. $\square$

6.4. Application of Pilidi's lemma. Pilidi's lemma plays a crucial role in the proof of the following important part of the necessity of condition (i) in Theorem 1.2 in our proof.

Lemma 6.4. *Let $\{u_n\}_{n=1}^\infty$ be a sequence of open neighborhoods of a point $\tau \in \mathbb{R}_+$, which are uniformly separated from 0 and ∞. For every $n \in \mathbb{N}$, there exist two functions $g_n, h_n \in L^p(u_n)$ such that*

$$\chi_{u_n} P_0^+ \chi_{u_n} g_n = g_n + h_n, \quad \textit{respectively,} \quad \chi_{u_n} P_0^- \chi_{u_n} g_n = g_n + h_n,$$

and

$$\|g_n\|_{L^p(u_n)} = 1, \qquad \|h_n\|_{L^p(u_n)} = O(1/n) \quad \textit{as } n \to \infty.$$

Proof. Since the neighborhoods u_n of τ are uniformly separated from 0 and ∞, there exists a Jordan Lyapunov curve Γ such that $u_n \subset \Gamma$ for all $n \in \mathbb{N}$. We prove the version for the "+" sign. The proof for the "−" sign is completely analogous.

Fix $n \in \mathbb{N}$. By Lemma 2.4, there exists a $\Phi_n^+ \in C^+(\Gamma)$ such that $\|\Phi_n^+\|_{L^p(\Gamma)} = 1$ and

$$\|(1-\chi_{u_n})\Phi_n^+\|_{L^p(\Gamma)} < \frac{1}{(2n)^p}.$$

It is clear that $\|\chi_{u_n}\Phi_n^+\|_{L^p(\Gamma)} \neq 0$. Set

$$\Psi_n^+ := \frac{\Phi_n^+}{\|\chi_{u_n}\Phi_n^+\|_{L^p(\Gamma)}}, \qquad g_n := \chi_{u_n}\Psi_n^+.$$

Then, $\|g_n\|_{L^p(u_n)} = 1$ and

(6.16)

$$\|\chi_{\Gamma\setminus u_n}\Psi_n^+\|_{L^p(\Gamma\setminus u_n)} \le \frac{\|(1-\chi_{u_n})\Phi_n^+\|_{L^p(\Gamma)}}{1-\|(1-\chi_{u_n})\Phi_n^+\|_{L^p(\Gamma)}} < \frac{1/(2n)^p}{1-1/(2n)^p} < \frac{1}{n}.$$

Let S_Γ be the Cauchy singular integral given on $L^p(\Gamma)$ by

$$(S_\Gamma g)(w) := \frac{1}{\pi i}\int_\Gamma \frac{g(z)}{z-w}\,dz,$$

where the integral is understood in the principal value sense. It is well known that $S_\Gamma \in \mathcal{B}(L^p(\Gamma))$, see e.g., [**8,** Chapter 1, Theorem 2.1]. Consider the operator $P_\Gamma^+ := (I_\Gamma + S_\Gamma)/2$. It is easy to see that

$$\chi_{u_n}P_0^+\chi_{u_n}I = \chi_{u_n}P_\Gamma^+\chi_{u_n}I.$$

Hence, taking into account that $P_\Gamma^+\Psi_n^+ = \Psi_n^+$, we obtain

$$\chi_{u_n}P_0^+\chi_{u_n}g_n = \chi_{u_n}P_\Gamma^+\Psi_n^+ - \chi_{u_n}P_\Gamma^+\chi_{\Gamma\setminus u_n}\Psi_n^+ = g_n + h_n,$$

where

$$h_n := -\chi_{u_n}P_\Gamma^+\chi_{\Gamma\setminus u_n}\Psi_n^+.$$

We see from (6.16) that

$$\begin{aligned}\|h_n\|_{L^p(u_n)} &\le \|P_\Gamma^+\chi_{\Gamma\setminus u_n}\Psi_n^+\|_{L^p(\Gamma)}\\ &\le \|P_\Gamma^+\|_{\mathcal{B}(L^p(\Gamma))}\|\chi_{\Gamma\setminus u_n}\Psi_n^+\|_{L^p(\Gamma)} < \frac{\|P_\Gamma^+\|_{\mathcal{B}(L^p(\Gamma))}}{n}. \qquad \square\end{aligned}$$

6.5. Proof of the necessity of Theorem 1.2 (i). Let $E \subset \mathbb{R}_+$ be a measurable subset. As usual, $|E|$ denotes the measure of $E \subset \mathbb{R}_+$. For the Lebesgue space $L^p(E)$ and a positive integer $m \geq 2$, let $L^p_m(E)$ denote the vector-valued Lebesgue space of columns with entries in $L^p(E)$ equipped with the norm

$$\|(f_1, \dots, f_m)^\top\|_{L^p_m(E)} = \left(\sum_{j=1}^{m} \|f_j\|^p_{L^p(E)} \right)^{1/p}.$$

Theorem 6.5. *Let $\gamma \in \mathbb{C}$ satisfy* (1.1). *Suppose that $a, b, c, d \in$ $\mathrm{SO}(\mathbb{R}_+)$ and $\alpha, \beta \in \mathrm{SOS}(\mathbb{R}_+)$. If the operator N given by* (1.3) *is Fredholm on the space $L^p(\mathbb{R})$, then the functional operators $A_+ :=$ $aI - bU_\alpha$ and $A_- := cI - dU_\beta$ are invertible on the space $L^p(\mathbb{R}_+)$.*

Proof. By Lemmas 4.2–4.3, the operator N is represented in each of the forms

$$\begin{aligned} N &= A_+P_0^+ + A_-P_0^- - \frac{1}{2}\sinh(\pi i\gamma)(A_+ - A_-)R_\gamma R_0 \\ &= A_+P_0^+ + C_-P_0^- = C_+P_0^+ + A_-P_0^-, \end{aligned} \tag{6.17}$$

where

$$\begin{aligned} C_+ &:= A_+ + 2\sinh(\pi i\gamma)e^{\pi i\gamma}(A_+ - A_-)P_\gamma^-, \\ C_- &:= A_- + 2\sinh(\pi i\gamma)e^{-\pi i\gamma}(A_+ - A_-)P_\gamma^+. \end{aligned}$$

We deduce from Lemma 4.5 and Theorem 4.1 that the commutators $DP_0^\pm - P_0^\pm D$ are compact for $D \in \{A_+, A_-, C_+, C_-\}$. It follows from this observation, representations (6.17) and Theorem 6.1 that, in order to obtain the invertibility of operators A_+ and A_- from the Fredholmness of operator N, it remains to show that the pairs (A_+, P_0^+) and (A_-, P_0^-) satisfy condition (A). This will be done following the scheme of the proof of [**23,** Theorem 5.2].

Now, we show that the pair (A_+, P_0^+) satisfies condition (A). If the operator A_+ is not invertible, then, from Theorem 6.2, it follows that a point $\tau \in \mathbb{R}_+$ exists such that the discrete operator A_τ given by (6.8), (6.9) is not invertible on the space $\ell^p(\mathbb{Z})$. We then deduce from Theorem 2.1 and Lemma 2.2 that one of the lower norms $|A_\tau|_+$ or $|A_\tau^*|_+$ is equal to 0.

Assume for definiteness that $|A_\tau|_+ = 0$. Then, by Lemma 6.3 (a), for every $n \in \mathbb{N}$, there exist a function $\varphi_n \in \ell^p(\mathbb{Z})$ of finite support $Q_n \subset \mathbb{Z}$ and a sequence $\{u_i^{(n)}\}_{i=1}^\infty$ of nested symmetric open neighborhoods of τ contained in the open interval l with endpoints $\widetilde{\tau}$ and $\alpha(\widetilde{\tau})$ such that $\|\varphi_n\|_{\ell^p(\mathbb{Z})} = 1$, for all $k \in \widetilde{Q}_n := Q_n \cup (Q_n - 1)$ the closures $\overline{\alpha_k(u_i^{(n)})}$ are pairwise disjoint, $|u_i^{(n)}| \to 0$ as $i \to \infty$ and, for every $i \in \mathbb{N}$ and every function $g_i^{(n)} \in L^p(u_i^{(n)})$, the function

(6.18)
$$f_i^{(n)}(t) := \begin{cases} \varphi_n(k)\big(U_{\alpha_k}^{-1} g_i^{(n)}\big)(t) & \text{if } t \in \alpha_k\big(u_i^{(n)}\big), k \in Q_n, \\ 0 & \text{if } t \in \mathbb{R}_+ \setminus \Big(\bigcup\limits_{k \in Q_n} \alpha_k\big(u_i^{(n)}\big)\Big), \end{cases}$$

satisfies the relations

$$\big\|f_i^{(n)}\big\|_{L^p(\mathbb{R}_+)} = \big\|g_i^{(n)}\big\|_{L^p\left(u_i^{(n)}\right)}, \tag{6.19}$$

$$\big\|A_+ f_i^{(n)}\big\|_{L^p(\mathbb{R}_+)} \le \frac{1}{n}\big\|g_i^{(n)}\big\|_{L^p\left(u_i^{(n)}\right)}. \tag{6.20}$$

Let
$$\mathcal{L}_n := \bigcup_{k \in Q_n} \alpha_k(l).$$

For every $i \in \mathbb{N}$, set
$$\Gamma_i^{(n)} := \bigcup_{k \in Q_n} \alpha_k\big(u_i^{(n)}\big).$$

Since $u_{i+1}^{(n)} \subset u_i^{(n)}$ for every $i \in \mathbb{N}$, we see that $\Gamma_{i+1}^{(n)} \subset \Gamma_i^{(n)}$.

Taking into account the fact that Q_n is finite, there exists a closed interval J_n such that $\Gamma_i^{(n)} \subset J_n$ for all $i \in \mathbb{N}$. It follows from Lemma 4.6 that the operator $P_0^+ P_0^- \chi_{J_n} I$ is compact on $L^p(\mathbb{R}_+)$.

Let m_n be the cardinality of the finite set Q_n. Consider the isometric isomorphism

$$\Sigma_n : L^p(\mathcal{L}_n) \to L^p_{m_n}(l),$$
$$(\Sigma_n f)(t) = \{(U_\alpha^k f)(t)\}_{k \in Q_n}, \quad t \in l.$$

Let $\operatorname{diag}_{m_n}\{T\}$ denote the diagonal operator $m_n \times m_n$-matrix, all of whose entries on the main diagonal are equal to T. It follows from

Lemma 4.5 that the operators $P_0^+ - (U_\alpha^k)^{-1} P_0^+ U_\alpha^k$ are compact on $L^p(\mathbb{R}_+)$ for all $k \in Q_n$. Therefore, the operator

$$\begin{aligned}\chi_{\alpha_k(l)} P_0^+ \chi_{\alpha_k(l)} I - \chi_{\alpha_k(l)} (U_\alpha^k)^{-1} P_0^+ U_\alpha^k \chi_{\alpha_k(l)} I \\ = \chi_{\alpha_k(l)} P_0^+ \chi_{\alpha_k(l)} I - \chi_{\alpha_k(l)} (U_\alpha^k)^{-1} \chi_l P_0^+ \chi_l U_\alpha^k \chi_{\alpha_k(l)} I\end{aligned}$$

is compact on the space $L^p(\alpha_k(l))$ for all $k \in Q_n$. Hence, the operator

$$K_n := \chi_{\mathcal{L}_n} P_0^+ \chi_{\mathcal{L}_n} I - \Sigma_n^{-1} \operatorname{diag}_{m_n} \{\chi_l P_0^+ \chi_l I\} \Sigma_n \tag{6.21}$$

is compact on the space $L^p(\mathcal{L}_n)$.

Since the set Q_n is finite and α preserves the orientation on $\mathbb{R}_+$ and $|u_i^{(n)}| \to 0$ as $i \to \infty$ for any fixed $n \in \mathbb{N}$, we see that

$$|\Gamma_i^{(n)}| = \sum_{k \in Q_n} |\alpha_k(u_i^{(n)})| \longrightarrow 0 \quad \text{as } i \to \infty.$$

Hence,

$$(\chi_{\Gamma_i^{(n)}} I)^* = \chi_{\Gamma_i^{(n)}} I \longrightarrow 0$$

strongly on $L^q(\mathbb{R}_+)$ as $i \to \infty$, where $q = p/(p-1)$. Thus, taking into account the facts that $P_0^+ P_0^- \chi_{J_n} I$ and K_n are compact and $\Gamma_i^{(n)} \subset J_n$ for all $i \in \mathbb{N}$, we deduce from Lemma 2.3 that there exists a number $i_n \in \mathbb{N}$ such that

$$\begin{aligned}\left|\Gamma_{i_n}^{(n)}\right| &< \frac{1}{2^n}, \\ \left\|P_0^+ P_0^- \chi_{\Gamma_{i_n}^{(n)}} I\right\|_{\mathcal{B}(L^p(\mathbb{R}_+))} &< \frac{1}{n}, \\ \left\|K_n \chi_{\Gamma_{i_n}^{(n)}} I\right\|_{\mathcal{B}(L^p(\mathbb{R}_+))} &< \frac{1}{n}.\end{aligned}$$

For all $n \in \mathbb{N}$, set $u_n := u_{i_n}^{(n)}$ and $\Gamma_n := \Gamma_{i_n}^{(n)}$. Thus,

$$\left\|P_0^+ P_0^- \chi_{\Gamma_n} I\right\|_{\mathcal{B}(L^p(\mathbb{R}_+))} < \frac{1}{n}, \tag{6.22}$$

$$\|\chi_{\Gamma_n} K_n \chi_{\Gamma_n} I\|_{\mathcal{B}(L^p(\mathbb{R}_+))} \le \|K_n \chi_{\Gamma_n} I\|_{\mathcal{B}(L^p(\mathbb{R}_+))} < \frac{1}{n}. \tag{6.23}$$

Since $|u_n| \le |\Gamma_n| < 1/2^n$, we see that the neighborhoods u_n are uniformly separated from 0 and ∞. We then deduce from Lemma 6.4

that there exist sequences $g_n, h_n \in L^p(u_n)$ such that

$$\|g_n\|_{L^p(u_n)} = 1, \qquad \|h_n\|_{L^p(u_n)} = O(1/n) \quad \text{as } n \to \infty, \tag{6.24}$$

and

$$\chi_{u_n} P_0^+ \chi_{u_n} g_n = g_n + h_n \quad \text{for all } n \in \mathbb{N}. \tag{6.25}$$

Let f_n be the function defined by (6.18) with $g_{i_n}^{(n)} = g_n$. It follows then from (6.18)–(6.20) and the first equality in (6.24) that

$$\|f_n\|_{L^p(\mathbb{R}_+)} = 1, \qquad \lim_{n\to\infty} A_+ f_n = 0. \tag{6.26}$$

By the construction of f_n, we have $\chi_{\Gamma_n} f_n = f_n$. Therefore, applying inequality (6.22) and the first equality in (6.26), we obtain

$$\begin{aligned}\|P_0^+ P_0^- f_n\|_{L^p(\mathbb{R}_+)} &= \|P_0^+ P_0^- \chi_{\Gamma_n} f_n\|_{L^p(\mathbb{R}_+)}\\ &\le \|P_0^+ P_0^- \chi_{\Gamma_n} I\|_{\mathcal{B}(L^p(\mathbb{R}_+))} \|f_n\|_{L^p(\mathbb{R}_+)} < \frac{1}{n},\end{aligned}$$

which implies that

$$\lim_{n\to\infty} P_0^+ P_0^- f_n = 0.$$

It remains to prove that the sequence $\{P_0^+ f_n\}_{n=1}^\infty$ does not contain convergent subsequences. Consider the isometric isomorphism

$$\sigma_n : L^p(\Gamma_n) \to L^p_{m_n}(u_n), \qquad (\sigma_n f)(t) = \{(U_\alpha^k f)(t)\}_{k\in Q_n}, \quad t \in u_n.$$

It is clear that $\chi_{u_n}\chi_l = \chi_{u_n}$, $\chi_{\Gamma_n}\chi_{\mathcal{L}_n} = \chi_{\Gamma_n}$ and

$$\begin{aligned}\Sigma_n \chi_{\Gamma_n} I &= \chi_{u_n} \Sigma_n, & \chi_{\Gamma_n} \Sigma_n^{-1} &= \Sigma_n^{-1} \chi_{u_n} I,\\ \sigma_n &= \chi_{u_n} \Sigma_n \chi_{\Gamma_n} I, & \sigma_n^{-1} &= \chi_{\Gamma_n} \Sigma_n^{-1} \chi_{u_n} I.\end{aligned}$$

From these identities and (6.21), we obtain

$$\begin{aligned}\chi_{\Gamma_n} P_0^+ \chi_{\Gamma_n} I &= \chi_{\Gamma_n} \Sigma_n^{-1} \operatorname{diag}_{m_n}\{\chi_l P_0^+ \chi_l I\} \Sigma_n \chi_{\Gamma_n} I + \chi_{\Gamma_n} K_n \chi_{\Gamma_n} I\\ &= \chi_{\Gamma_n} \sigma_n^{-1} \operatorname{diag}_{m_n}\{\chi_{u_n} P_0^+ \chi_{u_n} I\} \sigma_n \chi_{\Gamma_n} + \chi_{\Gamma_n} K_n \chi_{\Gamma_n} I.\end{aligned} \tag{6.27}$$

Assume the contrary, that is, that the sequence $\{P_0^+ f_n\}_{n=1}^\infty$ contains a convergent subsequence. Without loss of generality, we may assume that $\{P_0^+ f_n\}_{n=1}^\infty$ converges. Then, in view of the equality $f_n = \chi_{\Gamma_n} f_n$, we deduce that the sequence $\{\chi_{\Gamma_n} P_0^+ \chi_{\Gamma_n} f_n\}_{n=1}^\infty$ converges. Hence, taking into account (6.23) and (6.27), we deduce that the sequence

$$\{\sigma_n^{-1} \operatorname{diag}_{m_n}\{\chi_{u_n} P_0^+ \chi_{u_n} I\}\sigma_n f_n\}_{n=1}^\infty \tag{6.28}$$

converges as well.

Taking into account the definitions of f_n and σ_n, it is easy to see that, for all $n \in \mathbb{N}$,

$$\sigma_n f_n = \{\varphi_n(k) g_n\}_{k\in Q_n}.$$

Hence, from (6.25), we deduce that

$$\begin{aligned} \sigma_n^{-1} \operatorname{diag}_{m_n}\{\chi_{u_n} P_0^+ \chi_{u_n} I\}\sigma_n f_n &= \sigma_n^{-1}\left(\{\varphi_n(k)\chi_{u_n} P_0^+ \chi_{u_n} g_n\}_{k\in Q_n}\right) \\ &= \sigma_n^{-1}\left(\{\varphi_n(k)(g_n + h_n)\}_{k\in Q_n}\right) \\ &= f_n + \sigma_n^{-1}\left(\{\varphi_n(k) h_n\}_{k\in Q_n}\right). \end{aligned} \tag{6.29}$$

Since $\sigma_n^{-1} : L^p_{m_n}(u_n) \to L^p(\Gamma_n)$ is an isometry and $\|\varphi_n\|_{\ell^p(\mathbb{Z})} = 1$, from the second relation in (6.24), we get

$$\begin{aligned} \|\sigma_n^{-1}\left(\{\varphi_n(k) h_n\}_{k\in Q_n}\right)\|_{L^p(\Gamma_n)} &= \left(\sum_{k\in Q_n} \|\varphi_n(k) h_n\|^p_{L^p(u_n)}\right)^{1/p} \\ &= \|\varphi_n\|_{\ell^p(\mathbb{Z})}\|h_n\|_{L^p(u_n)} \\ &= O(1/n) \quad \text{as } n \to \infty. \end{aligned} \tag{6.30}$$

We then deduce from the convergence of the sequence (6.28), equality (6.29) and relation (6.30) that the sequence $\{f_n\}_{n=1}^\infty$ converges in the space $L^p(\mathbb{R}_+)$.

Set

$$f_\infty := \lim_{n\to\infty} f_n \in L^p(\mathbb{R}_+).$$

By this definition and the first equality in (6.26), we obtain

$$\|f_\infty\|_{L^p(\mathbb{R}_+)} = 1. \tag{6.31}$$

It follows from the estimate $|\Gamma_k| < 1/2^k$ that

$$\left|\bigcup_{k=n}^\infty \Gamma_k\right| \le \sum_{k=n}^\infty |\Gamma_k| \le \sum_{k=n}^\infty \frac{1}{2^k} = \frac{1}{2^{n-1}}.$$

Let χ_n be the characteristic function of the set

$$\bigcup_{k=n}^{\infty} \Gamma_k.$$

Hence, we deduce from the above estimate that $1-\chi_n \to 1$ everywhere on $\mathbb{R}_+$ as $n \to \infty$. By the Lebesgue dominated convergence theorem,

$$\|f_\infty\|_{L^p(\mathbb{R}_+)} = \lim_{n\to\infty} \|(1-\chi_n) f_\infty\|_{L^p(\mathbb{R}_+)}.$$

Since $f_n(t) = 0$ for $t \notin \Gamma_n$, we see that $(1-\chi_n(t))f_n(t) = 0$ for $t \in \mathbb{R}_+$, whence

$$\begin{aligned}\|f_\infty\|_{L^p(\mathbb{R}_+)} &= \lim_{n\to\infty} \|(1-\chi_n) f_\infty - (1-\chi_n) f_n\|_{L^p(\mathbb{R}_+)} \\ &\le \lim_{n\to\infty} 2\|f_n - f_\infty\|_{L^p(\mathbb{R}_+)} = 0,\end{aligned}$$

which contradicts (6.31). Therefore, the sequence $\{P_0^+ f_n\}_{n=1}^\infty$ does not contain convergent subsequences.

Thus, we have constructed a sequence $\{f_n\}_{n=1}^\infty$ of functions in $L^p(\mathbb{R}_+)$ such that $\|f_n\|_{L^p(\mathbb{R}_+)} = 1$ for all $n \in \mathbb{N}$. The sequences $\{A_+ f_n\}_{n=1}^\infty$ and $\{P_0^+ P_0^- f_n\}_{n=1}^\infty$ converge in $L^p(\mathbb{R}_+)$; however, the sequence $\{P_0^+ f_n\}_{n=1}^\infty$ does not contain subsequences convergent in $L^p(\mathbb{R}_+)$. This means that condition (A-i) is fulfilled if the operator A_+ is not invertible and $|A_\tau|_+ = 0$ for some $\tau \in \mathbb{R}_+$.

It can be shown analogously that condition (A-ii) is fulfilled if the operator A_+ is not invertible and $|A_\tau^*|_+ = 0$ for some $\tau \in \mathbb{R}_+$. This completes the proof of the assertion that the pair (A_+, P_0^+) satisfies condition (A). In the same way, it can be proved that the pair (A_-, P_0^-) also satisfies condition (A). $\square$

Finally, Theorems 5.7 and 6.5 imply Theorem 1.2.

Added in proof. In our subsequent paper [**16,** Theorem 1.3], we proved that conditions (i)–(ii) of Theorem 1.2 are also sufficient for the Fredholmness of the operator N. A formula for the index of the operator N is obtained in [**16,** Theorem 1.4].

Acknowledgments. We would like to thank the anonymous referee for the careful reading of the paper.

REFERENCES

1. A. Böttcher, Yu.I. Karlovich and V.S. Rabinovich, *The method of limit operators for one-dimensional singular integrals with slowly oscillating data*, J. Oper. Theory **43** (2000), 171–198.

2. A. Böttcher, Yu.I. Karlovich and I.M. Spitkovsky, *Convolution operators and factorization of almost periodic matrix functions*, Oper. Th. Adv. Appl. **131**, Birkhäuser, Basel, 2002.

3. A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators*, 2nd edition, Springer, Berlin, 2006.

4. V.D. Didenko and B. Silbermann, *Approximation of additive convolution-like operators*, *Real C^*-algebra approach*, Birkhäuser, Basel, 2008.

5. R. Duduchava, *Integral equations with fixed singularities*, Teubner Verlagsgesellschaft, Leipzig, 1979.

6. ———, *On algebras generated by convolutions and discontinuous functions*, Int. Eq. Oper. Th. **10** (1987), 505–530.

7. G. Fernández-Torres and Yu.I. Karlovich, *Two-sided and one-sided invertibility of the Wiener type functional operators with a shift and slowly oscillating data*, Banach J. Math. Anal., to appear, DOI:10.1215/17358787-2017-0006.

8. I. Gohberg and N. Krupnik, *One-dimensional linear singular integral equations*, I, *Introduction*, Oper. Th. Adv. Appl. **53**, Birkhäuser, Basel, 1992.

9. R. Hagen, S. Roch and B. Silbermann, *Spectral theory of approximation methods for convolution equations*, Oper. Th. Adv. Appl. **74**, Birkhäuser, Basel, 1994.

10. A.Yu. Karlovich, *Fredholmness and index of simplest weighted singular integral operators with two slowly oscillating shifts*, Banach J. Math. Anal. **9** (2015), 24–42.

11. A.Yu. Karlovich, Yu.I. Karlovich and A.B. Lebre, *Sufficient conditions for Fredholmness of singular integral operators with shifts and slowly oscillating data*, Int. Eq. Oper. Th. **70** (2011), 451–483.

12. ———, *Necessary conditions for Fredholmness of singular integral operators with shifts and slowly oscillating data*, Int. Eq. Oper. Th. **71** (2011), 29–53.

13. ———, *Fredholmness and index of simplest singular integral operators with two slowly oscillating shifts*, Operators and Matrices **8** (2014), 935–955.

14. ———, *On a weighted singular integral operator with shifts and slowly oscillating data*, Complex Anal. Oper. Th. **10** (2016), 1101–1131.

15. ———, *One-sided invertibility criteria for binomial functional operators with shift and slowly oscillating data*, Mediterranean J. Math. **13** (2016), 4413–4435.

16. ———, *The index of weighted singular integral operators with shifts and slowly oscillating data*, J. Math. Anal. Appl. **450** (2017), 606–630.

17. Yu.I. Karlovich, *On algebras of singular integral operators with discrete groups of shifts in L_p-spaces*, Soviet Math. Dokl. **39** (1989), 48–53.

18. Yu.I. Karlovich, *Nonlocal singular integral operators with slowly oscillating data*, Oper. Th. Adv. Appl. **181** (2008), 229–261.

19. Yu.I. Karlovich and V.G. Kravchenko, *On an algebra of singular integral operators with non-Carleman shift*, Soviet Math. Dokl. **19** (1978), 267–271.

20. ______, *On a general method of investigating operators of singular integral type with non-Carleman shift*, Soviet Math. Dokl. **22** (1980), 10–14.

21. ______, *An algebra of singular integral operators with piecewise-continuous coefficients and piecewsie-smooth shift on a composite contour*, Math. Izv. **23** (1984), 307–352.

22. Yu.I. Karlovich and B. Silbermann, *Local method for nonlocal operators on Banach spaces*, Oper. Th. Adv. Appl. **135** (2002), 235–247.

23. ______, *Fredholmness of singular integral operators with discrete subexponential groups of shifts on Lebesgue spaces*, Math. Nachr. **272** (2004), 55–94.

24. B.V. Khvedelidze, *The method of the Cauchy type integrals for discontinuous boundary value problems of the theory of holomorphic functions of one complex variable*, J. Soviet Math. **7** (1977), 309–414.

25. V.G. Kravchenko, *On a singular integral operator with a shift*, Soviet Math. Dokl. **15** (1974), 690–694.

26. V.G. Kravchenko and G.S. Litvinchuk, *Introduction to the theory of singular integral operators with shift*, Math. Appl. **289**, Kluwer Academic Publishers, Dordrecht, 1994.

27. V.G. Kurbatov, *Functional-differential operators and equations*, Math. Appl. **473**, Kluwer Academic Publishers, Dordrecht, 1999.

28. M. Lindner, *Infinite matrices and their finite sections*, *An introduction to the limit operator method*, Birkhäuser, Basel, 2006.

29. A. Pietsch, *Operator ideals*, North-Holland Math. Library **20**, North-Holland Publishing Co., Amsterdam, 1980.

30. V.S. Pilidi, *A priori estimates for one-dimensional singular integral operators with continuous coefficients*, Math. Notes **17** (1975), 512–515.

31. V.S. Rabinovich, S. Roch and B. Silbermann, *Limit operators and their applications in operator theory*, Oper. Th. Adv. Appl. **150**, Birkhäuser, Basel, 2004.

32. S. Roch, P.A. Santos and B. Silbermann, *Non-commutative Gelfand theories*, *A tool-kit for operator theorists and numerical analysts*, Universitext, Springer-Verlag, London, 2011.

33. D. Sarason, *Toeplitz operators with piecewise quasicontinuous symbols*, Indiana Univ. Math. J. **26** (1977), 817–838.

34. I.B. Simonenko and Chin Ngok Minh, *Local approach to the theory of one-dimensional singular integral equations with piecewise continuous coefficients*, *Noethericity*, Rostov University Press, Rostov-on-Don, 1986 (in Russian).

Universidade Nova de Lisboa, Centro de Matemática e Aplicações, Departamento de Matemática, Quinta da Torre, 2829–516 Caparica, Portugal
Email address: oyk@fct.unl.pt

Universidad Autónoma del Estado de Morelos, Centro de Investigación en Ciencias, Instituto de Investigación en Ciencias Básicas y Aplicadas, Av. Universidad 1001, Col. Chamilpa, C.P. 62209 Cuernavaca, Morelos, México
Email address: karlovich@uaem.mx

Universidade de Lisboa, Centro de Análise Funcional, Estruturas Lineares e Aplicações, Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049–001 Lisboa, Portugal
Email address: alebre@math.tecnico.ulisboa.pt