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L^p-APPROXIMATION BY TRUNCATED MAX-PRODUCT SAMPLING OPERATORS OF KANTOROVICH-TYPE BASED ON FEJÉR KERNEL

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ABSTRACT. By use of the so-called max-product method, in this paper we associate to the truncated linear sampling operators based on the Fejér-type kernel, nonlinear sampling operators of Kantorovich type, for which we prove convergence results in the L^p-norm, $1 \leq p \leq +\infty$, with quantitative estimates.

1. Introduction. The sinc-approximation operators were first introduced and studied in [**5, 19, 25**] under the terms of cardinal and truncated cardinal functions. Later on, the properties of these linear approximation operators and their applications in signal theory were intensively studied in, e.g., [**1, 2, 6, 7, 8, 9, 12, 13, 17, 18, 20, 21, 22, 23, 24**] (and the references therein).

Based on Open Problem 5.5.4 [**16,** pages 324–326], in a series of papers we have introduced and studied the so called max-product operators attached to the Bernstein polynomials and to other linear Bernstein-type operators, like those of Favard-Szász-Mirakjan operators (truncated and nontruncated cases), Baskakov operators (truncated and nontruncated cases), Meyer-König and Zeller operators and Bleimann-Butzer-Hahn operators.

In [**10**], applying this idea to Whittaker's cardinal series, we obtained a Jackson-type estimate in uniform approximation of f by the max-

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product Whittaker sampling operator given by

$$S_{W,\varphi}^{(M)}(f)(t) = \frac{\bigvee_{k=-\infty}^{\infty} \varphi(Wt-k) f(k/W)}{\bigvee_{k=-\infty}^{\infty} \varphi(Wt-k)}, \quad t \in \mathbb{R}, \tag{1.1}$$

where $W > 0$, $f : \mathbb{R} \to \mathbb{R}_+$ and φ is a kernel given by the formula $\varphi(t) = \text{sinc}(t)$, where $\text{sinc}(t) = \sin(\pi t)/\pi t$, for $t \neq 0$ and at $t = 0$, $\text{sinc}(t)$ is defined to be the limiting value, that is, $\text{sinc}(0) = 1$.

Also, in **[11]**, a similar idea and study was applied to the sampling operator in (1.1) based on the Fejér-type kernel $\varphi(t) = (1/2) \cdot [\text{sinc}(t/2)]^2$.

In the same paper **[11]**, applying the max-product idea to the truncated sampling operator based on the Fejér's kernel and defined by

$$T_n(f)(x) = \sum_{k=0}^{n} \frac{\sin^2(nx-k\pi)}{(nx-k\pi)^2} \cdot f\left(\frac{k\pi}{n}\right), \quad x \in [0,\pi],$$

we have introduced and studied uniform approximation by the truncated max-product operator based on the Fejér kernel, given by

$$T_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} [\sin^2(nx-k\pi)]/[(nx-k\pi)^2] \cdot f(k\pi/n)}{\bigvee_{k=0}^{n} [\sin^2(nx-k\pi)]/[(nx-k\pi)^2]}, \; x \in [0,\pi], \tag{1.2}$$

where $f : [0,\pi] \to \mathbb{R}_+$. Here, since $\text{sinc}(0) = 1$, it means above that, for every $x = k\pi/n$, $k \in \{0, 1, \ldots, n\}$, we have $[\sin(nx-k\pi)]/[nx-k\pi] = 1$.

It is also worth mentioning here that qualitative L^p-approximation results and quantitative uniform approximation results for max-product neural networks have been obtained in very recent papers **[14, 15]**, respectively.

In the present paper, we study approximation properties with quantitative estimates in the L^p-norm, $1 \leq p \leq \infty$, for the Kantorovich variant of the above truncated max-product sampling operators $T_n^{(M)}(f)(x)$, defined for $x \in [0, \pi]$ and $n \in \mathbb{N}$ by

(1.3)

$$K_n^{(M)}(f)(x) = \frac{1}{\pi} \cdot \frac{\bigvee_{k=0}^{n} [\sin^2(nx - k\pi)]/[(nx - k\pi)^2] \cdot \left[(n+1) \int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} f(v)\, dv\right]}{\bigvee_{k=0}^{n} [\sin^2(nx - k\pi)]/[(nx - k\pi)^2]}$$

where $f : [0, \pi] \to \mathbb{R}_+$, $f \in L^p[0, \pi]$, $1 \le p \le \infty$.

2. Auxiliary results. Firstly, we present some properties of the operator $K_n^{(M)}$ which will be useful for proving the approximation results.

Lemma 2.1.

(i) *For any integrable function* $f : [0, \pi] \to \mathbb{R}$, $K_n^{(M)}(f)$ *is continuous on* $[0, \pi]$;
(ii) *If* $f \le g$, *then* $K_n^{(M)}(f) \le K_n^{(M)}(g)$;
(iii) $K_n^{(M)}(f + g) \le K_n^{(M)}(f) + K_n^{(M)}(g)$;
(iv) $|K_n^{(M)}(f) - K_n^{(M)}(g)| \le K_n^{(M)}(|f - g|)$;
(v) *If, in addition,* f *is positive on* $[0, \pi]$ *and* $\lambda \ge 0$, *then* $K_n^{(M)}(\lambda f) = \lambda K_n^{(M)}(f)$.

Proof. We omit the proofs of (i)–(ii) and (v), respectively, because they are immediate from the definition of $K_n^{(M)}$. As for the proof of (iv), we easily obtain the conclusion since $f \le |f - g| + g$ and $g \le |f - g| + f$; thus, applying (ii) and (iii), we obtain $K_n^{(M)}(f) \le K_n^{(M)}(|f - g|) + K_n^{(M)}(g)$ and $K_n^{(M)}(g) \le K_n^{(M)}(|f - g|) + K_n^{(M)}(f)$. □

For the next result, we need the first order modulus of continuity on $[0, \pi]$ defined for $f : [0, \pi] \to \mathbb{R}$ and $\delta \ge 0$ by

$$\omega_1(f; \delta) = \max\{|f(x) - f(y)| : x, y \in [0, \pi], |x - y| \le \delta\}.$$

Lemma 2.2. *For any continuous function $f : [0, \pi] \to \mathbb{R}_+$, we obtain*

$$\left|K_n^{(M)}(f)(x) - f(x)\right| \le \left[1 + \frac{1}{\delta} K_n^{(M)}(\varphi_x)(x)\right] \omega_1(f; \delta), \tag{2.1}$$

for any $x \in [0, \pi]$ and $\delta > 0$. Here, $\varphi_x(t) = |t - x|$, $t \in [0, \pi]$.

Proof. The proof is identical to the proof of [**3,** Corollary 2.4] (see also Corollary 2.3 in the same paper). Applying property (iv) of Lemma 2.1 and noting that $K_n^{(M)}$ preserves the constant functions, we obtain

$$\begin{aligned}\left|K_n^{(M)}(f)(x) - f(x)\right| &= \left|K_n^{(M)}(f)(x) - K_n^{(M)}(f(x))(x)\right| \\ &\le K_n^{(M)}\left(|f - f(x)|\right)(x).\end{aligned}$$

On the other hand, for any $t, x \in [0, \pi]$ and $\delta > 0$, we have

$$\begin{aligned}|f(x) - f(t)| \le \omega_1(f; |t - x|) &= \omega_1\left(f; \delta \cdot \frac{|t - x|}{\delta}\right) \\ &\le \left(1 + \frac{|t - x|}{\delta}\right) \omega_1\left(f; \delta\right).\end{aligned}$$

Now, applying properties (ii), (iii) and (v) of Lemma 2.1 and using again that $K_n^{(M)}$ preserves the constant functions, we easily obtain relation (2.1). □

3. Pointwise and uniform convergence results. Our first main result proves that $K_n^{(M)}(f)(x)$ converges to $f(x)$ at any point of continuity for f.

Theorem 3.1. *Suppose that $f : [0, \pi] \to \mathbb{R}_+$ is bounded on its domain and integrable on any subinterval of $[0, \pi]$. If f is continuous at $x_0 \in [0, \pi]$, then:*

$$\lim_{n \to \infty} K_n^{(M)}(f)(x_0) = f(x_0).$$

Proof. We use in the proof some ideas from [**14**]. We have

$$|K_n^{(M)}(f)(x_0) - f(x_0)| = |K_n^{(M)}(f)(x_0) - K_n^{(M)}(f(x_0))(x_0)|$$

$$\leq K_n^{(M)}(|f-f(x_0)|)(x_0)$$
$$= \frac{\bigvee_{k=0}^{n}[\sin^2(nx_0-k\pi)]/[(nx_0-k\pi)^2]\cdot\left[(n+1)\int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)}|f(v)-f(x_0)|dv\right]}{\pi\bigvee_{k=0}^{n}[\sin^2(nx_0-k\pi)]/[(nx_0-k\pi)^2]}.$$

Let $j\in\{0,\ldots,n-1\}$ be such that $x_0\in[(j\pi/n),[(j+1)\pi]/n]$. If

$$x_0\in\left[\frac{j\pi}{n},\frac{(j+1/2)\pi}{n}\right],$$

then

$$nx_0-j\pi\in\left[0,\frac{\pi}{2}\right].$$

By the well-known inequality $\sin t\geq(2/\pi)\cdot t$, $t\in[0,(\pi/2)]$, we obtain

$$\frac{\sin^2(nx_0-j\pi)}{(nx_0-j\pi)^2}\geq\frac{4}{\pi^2}.$$

If

$$x_0\in\left[\frac{(j+1/2)\,\pi}{n},\frac{(j+1)\pi}{n}\right],$$

then it follows that $nx_0-(j+1)\pi\in[-(\pi/2),0]$, which easily implies that

$$\frac{\sin^2(nx_0-(j+1)\,\pi)}{(nx_0-(j+1)\,\pi)^2}\geq\frac{4}{\pi^2}.$$

In conclusion, we obtain

$$\bigvee_{k=0}^{n}\frac{\sin^2(nx_0-k\pi)}{(nx_0-k\pi)^2}\geq\frac{4}{\pi^2},$$

and this implies

$$\left|K_n^{(M)}(f)(x_0)-f(x_0)\right|$$
$$\leq\frac{\pi}{4}\cdot\bigvee_{k=0}^{n}\frac{\sin^2(nx_0-k\pi)}{(nx_0-k\pi)^2}\cdot\left[(n+1)\int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)}|f(v)-f(x_0)|\,dv\right].$$

Now, let us choose arbitrary $\varepsilon>0$. Then there exists $\delta>0$ such that $|f(x_0)-f(y)|<(4\varepsilon/\pi)$ whenever $|x_0-y|<\delta$. Suppose that n is sufficiently large such that $\pi/(n+1)<\delta/4$. If $k\in\{0,\ldots,n\}$ is such that $|x_0 - (k\pi/n)| < \delta/2$, then, for any

$$v \in \left[\frac{k\pi}{n+1}, \frac{(k+1)\pi}{n+1}\right],$$

we have

$$\begin{aligned}|v - x_0| &\leq \left|v - \frac{k\pi}{n+1}\right| + \left|\frac{k\pi}{n+1} - \frac{k\pi}{n}\right| + \left|x_0 - \frac{k\pi}{n}\right| \\ &\leq \frac{2\pi}{n+1} + \frac{\delta}{2} < \delta.\end{aligned}$$

This implies

$$(n+1)\int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |f(v) - f(x_0)|\, dv \leq (n+1) \cdot \frac{4\varepsilon}{\pi(n+1)} = \frac{4\varepsilon}{\pi}$$

and hence, we get

$$\begin{aligned}\max_{|x_0-(k\pi/n)|<\delta/2} &\left\{\frac{\pi(n+1)\sin^2(nx_0 - k\pi)}{4(nx_0 - k\pi)^2}\right. \\ &\left.\cdot \left[\int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |f(v) - f(x_0)|\, dv\right]\right\} < \varepsilon. \end{aligned} \tag{3.1}$$

If $k \in \{0, \ldots, n\}$ is such that $|x_0 - (k\pi/n)| \geq \delta/2$, then it follows that $(nx_0 - k\pi)^2 \geq (n^2\delta^2)/4$, and this implies

$$\frac{\sin^2(nx_0 - k\pi)}{(nx_0 - k\pi)^2} \cdot \left[(n+1)\int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |f(v) - f(x_0)|\, dv\right] \leq \frac{8\pi\,\|f\|}{n^2\delta^2}.$$

Here, $\|f\| = \sup_{x\in[0,\pi]} |f(x)|$, and it is finite according to the hypotheses. Moreover, we used that

$$\int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |f(v) - f(x_0)|\, dv \leq 2\pi\,\|f\|.$$

Obviously, for sufficiently large n, we have

$$\frac{8\pi\,\|f\|}{n^2\delta^2} < \frac{4\varepsilon}{\pi}.$$

Therefore, we obtain

$$\max_{|x_0-k\pi/n|\geq\delta/2} \left\{\frac{\pi(n+1)\sin^2(nx_0 - k\pi)}{4(nx_0 - k\pi)^2}\right. \tag{3.2}$$

$$\cdot \left[\int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |f(v) - f(x_0)| \, dv \right] \Bigg\} < \varepsilon.$$

Combining relations (3.1) and (3.2), we easily obtain that, for sufficiently large n (depending only on ε), we have

$$\left| K_n^{(M)}(f)(x_0) - f(x_0) \right| < \varepsilon.$$

This implies the desired conclusion. □

In contrast to the qualitative type results in **[14]**, in the present paper we prove a quantitative result, as well, which follows.

Theorem 3.2. *Suppose that $f : [0, \pi] \to \mathbb{R}_+$ is continuous on $[0, \pi]$. Then for any $n \in \mathbb{N}$, $n \geq 1$, we have*

$$\left\| K_n^{(M)}(f) - f \right\| \leq 10\omega_1\left(f; \frac{1}{n}\right).$$

Proof. By Lemma 2.2, it suffices to estimate the following expression:

$$K_n^{(M)}(\varphi_x)(x) = \frac{1}{\pi} \cdot \frac{\bigvee_{k=0}^{n} [\sin^2(nx - k\pi)]/[(nx - k\pi)^2] \cdot \left[(n+1) \int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |v - x| \, dv\right]}{\bigvee_{k=0}^{n} [\sin^2(nx - k\pi)]/[(nx - k\pi)^2]}$$

for all $x \in [0, \pi]$. Obviously, $\sin^2(nx - k\pi)$ is constant for any $k \in \{0, 1, \dots, n\}$, and therefore, for all $x \in [0, \pi]$ we obtain:

$$K_n^{(M)}(\varphi_x)(x) = \frac{1}{\pi} \cdot \frac{\bigvee_{k=0}^{n} 1/(nx - k\pi)^2 \cdot \left[(n+1) \int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |v - x| \, dv\right]}{\bigvee_{k=0}^{n} 1/(nx - k\pi)^2}$$

For some arbitrary $x \in [0, \pi]$, let $j \in \{0, \ldots, n\}$ be such that

$$x \in \left[\frac{j\pi}{n}, \frac{(j+1)\pi}{n}\right].$$

At first, suppose that

$$x \in \left[\frac{j\pi}{n}, \frac{(j+1/2)\pi}{n}\right].$$

By simple calculations (or by applying [**11,** Lemma 4.3]) it is easily seen that

$$\bigvee_{k=0}^{n} \frac{1}{(nx - k\pi)^2} = \frac{1}{(nx - j\pi)^2},$$

and this implies

$$K_n^{(M)}(\varphi_x)(x) = \frac{1}{\pi} \cdot \bigvee_{k=0}^{n} \frac{(nx - j\pi)^2}{(nx - k\pi)^2} \cdot \left[(n+1) \int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |v - x| \, dv\right].$$

Applying the mean value theorem on each interval

$$\left[\frac{k\pi}{n+1}, \frac{(k+1)\pi}{n+1}\right],$$

there exists

$$v_k \in \left[\frac{k\pi}{n+1}, \frac{(k+1)\pi}{n+1}\right]$$

such that

$$\int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |v - x| \, dv = \frac{\pi}{n+1} \cdot |v_k - x| \,,$$

which means that

$$K_n^{(M)}(\varphi_x)(x) = \bigvee_{k=0}^{n} \frac{(nx - j\pi)^2}{(nx - k\pi)^2} \cdot |v_k - x| \,.$$

We have

$$\bigvee_{k=0}^{n} \frac{(nx - j\pi)^2}{(nx - k\pi)^2} \cdot |v_k - x| \leq \bigvee_{k=0}^{n} \frac{(nx - j\pi)^2}{(nx - k\pi)^2} \cdot \left(\left|v_k - \frac{k\pi}{n}\right| + \left|x - \frac{k\pi}{n}\right|\right).$$

Let us arbitrarily choose $k \in \{0, \dots, n\}$. Since $|v_k - (k\pi/n)| \le \pi/(n+1)$ and noting that $(nx - j\pi)^2/(nx - k\pi)^2 \le 1$, it results that

$$\frac{(nx - j\pi)^2}{(nx - k\pi)^2} \cdot \left| v_k - \frac{k\pi}{n} \right| \le \frac{\pi}{n+1}.$$

Then,

$$\frac{(nx - j\pi)^2}{(nx - k\pi)^2} \cdot \left| x - \frac{k\pi}{n} \right| = \frac{|nx - j\pi|}{|nx - k\pi|} \cdot \frac{|nx - j\pi|}{n}.$$

As $|nx - j\pi|/|nx - k\pi| \le 1$ and $|nx - j\pi| \le \pi/2$, we obtain

$$\frac{(nx - j\pi)^2}{(nx - k\pi)^2} \cdot \left| x - \frac{k\pi}{n} \right| \le \frac{\pi}{2n}.$$

All of these imply that

$$\frac{(nx - j\pi)^2}{(nx - k\pi)^2} \cdot \left(\left| v_k - \frac{k\pi}{n} \right| + \left| x - \frac{k\pi}{n} \right| \right) \le \frac{\pi}{n+1} + \frac{\pi}{2n} \le \frac{3\pi}{2n}$$

and, by the arbitrariness of k, it follows that

$$K_n^{(M)}(\varphi_x)(x) \le \frac{3\pi}{2n}. \tag{3.3}$$

The case

$$x \in \left[\frac{(j + 1/2)\pi}{n}, \frac{(j+1)\pi}{n} \right]$$

by absolutely similar reasonings leads to the same conclusion. Thus, we obtain $K_n^{(M)}(\varphi_x)(x) \le (3\pi)/(2n)$, for all $x \in [0, \pi]$. By relation (2.1), taking $\delta = (3\pi)/(2n)$ and noting that, in general, we have $\omega_1(f; \alpha\delta) \le ([\alpha] + 1)\omega_1(f; \delta)$ for any $\alpha > 0$ and $\delta > 0$ (here $[\alpha]$ means the integer part of α), we easily obtain the estimation from the conclusion. □

Remark 3.3. The estimate in the statement of Theorem 3.2 remains valid for lower bounded functions and of arbitrary sign. Indeed, if $c \in \mathbb{R}$ is such that $f(x) \ge c$ for all $x \in [0, \pi]$, then it is easy to see that defining the new max-product operator $\overline{K}^{(M)}(f)(x) = K_n^{(M)}(f - c)(x) + c$, we get $|f(x) - \overline{K}^{(M)}(f)(x)| \le 10\omega_1(f; 1/n)$, for all $x \in [0, \pi]$, $n \in \mathbb{N}$.

4. Convergence results in the L^p-norm. Let the L^p-norm,

$$\|f\|_p = \left(\int_0^\pi |f(t)|^p dt \right)^{1/p}, \quad \text{with } 1 \le p < +\infty.$$

In this section, we deal with the approximation by $K_n^{(M)}$ in the L^p-norm. For this purpose, firstly we need the following Lipschitz property of the operator $K_n^{(M)}$.

Theorem 4.1. *We have*

$$\left\| K_n^{(M)}(f) - K_n^{(M)}(g) \right\|_p \le 2^{(1-2p)/p} \pi^2 \cdot \|f - g\|_p,$$

for any $n \in \mathbb{N}$, $n \ge 1$, $f, g : [0, \pi] \to \mathbb{R}_+$, $f, g \in L^p[0, \pi]$ and $1 \le p < \infty$.

Proof. Applying the L^p norm, we get

$$\begin{aligned}
&\left\| K_n^{(M)}(f) - K_n^{(M)}(g) \right\|_p \\
&= \left(\int_0^\pi \left| K_n^{(M)}(f)(x) - K_n^{(M)}(g)(x) \right|^p dx \right)^{1/p} \\
&\le \left(\int_0^\pi \left(K_n^{(M)}(|f(x) - g(x)|) \right)^p dx \right)^{1/p} \\
&= \frac{1}{\pi} \Bigg(\int_0^\pi \\
&\left(\frac{\bigvee_{k=0}^n [\sin^2(nx - k\pi)]/[(nx - k\pi)^2](n+1) \int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |f(v) - g(v)|\, dv}{\bigvee_{k=0}^n [\sin^2(nx - k\pi)]/(nx - k\pi)^2} \right)^p dx \Bigg)^{1/p}.
\end{aligned}$$

As we already know from the previous section, for any $x \in [0, \pi]$, we have

$$\bigvee_{k=0}^n \frac{\sin^2(nx - k\pi)}{(nx - k\pi)^2} \ge \frac{4}{\pi^2},$$

which implies

$$\left\| K_n^{(M)}(f) - K_n^{(M)}(g) \right\|_p \le \frac{\pi}{4} \Bigg(\int_0^\pi \Bigg(\bigvee_{k=0}^n \frac{\sin^2(nx - k\pi)}{(nx - k\pi)^2}$$

$$\cdot (n+1) \int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |f(v) - g(v)| \, dv \Big)^p dx \Big)^{1/p}.$$

Since

$$0 \leq \bigvee_{k=0}^{n} \frac{\sin^2(nx - k\pi)}{(nx - k\pi)^2} \leq 1, \quad \text{for all } x \in [0, \pi],$$

it easily follows that

$$\left\| K_n^{(M)}(f) - K_n^{(M)}(g) \right\|_p \leq \frac{\pi}{4} \times \Bigg(\int_0^{\pi} \bigvee_{k=0}^{n} \frac{\sin^2(nx - k\pi)}{(nx - k\pi)^2}$$
$$\cdot \left[(n+1) \int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |f(v) - g(v)| \, dv \right]^p dx \Bigg)^{1/p}.$$

As the function $x \to x^p$ is convex, applying Jensen's inequality, we obtain

$$\left(\int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |f(v) - g(v)| \, dv \right)^p$$
$$\leq \frac{n+1}{\pi} \int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} \frac{\pi^p \, |f(v) - g(v)|^p}{(n+1)^p} \, dv,$$

and from here it follows that

$$\left\| K_n^{(M)}(f) - K_n^{(M)}(g) \right\|_p \leq \frac{\pi^{(2p-1)/p}}{4}$$
$$\times \left(\int_0^{\pi} \bigvee_{k=0}^{n} \left[(n+1) \frac{\sin^2(nx - k\pi)}{(nx - k\pi)^2} \int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |f(v) - g(v)|^p \, dv \right] dx \right)^{1/p}.$$

On the other hand, for some $k \in \{0, 1, \ldots, n\}$, using the substitution $y = nx - k\pi$, we obtain

$$\int_0^{\pi} (n+1) \frac{\sin^2(nx - k\pi)}{(nx - k\pi)^2} \, dx = \frac{n+1}{n} \cdot \int_{-k\pi}^{(n-k)\pi} \frac{\sin^2 y}{y^2} \, dy.$$

It is well-known that

$$\int_{-\infty}^{\infty} \frac{\sin^2 y}{y^2} \, dy = \pi,$$

which implies

$$\int_{-k\pi}^{(n-k)\pi} \frac{\sin^2 y}{y^2}\, dy \le \pi,$$

and hence,

$$\int_0^{\pi} (n+1) \frac{\sin^2 (nx - k\pi)}{(nx-k\pi)^2} dx \le 2\pi.$$

This implies

$$\begin{aligned} &\left\| K_n^{(M)}(f) - K_n^{(M)}(g) \right\|_p \\ &\le \frac{2^{1/p}\pi^2}{4} \cdot \left(\sum_{k=0}^{n} \int_{k\pi/(n+1)}^{(k+1)\pi/(n+1)} |f(v) - g(v)|^p \, dv \right)^{1/p} \\ &= \frac{2^{1/p}\pi^2}{4} \cdot \|f-g\|_p . \end{aligned}$$

The proof is complete. □

Now, let us define

$$C_+^1[0,\pi] = \{g : [0,\pi] \to \mathbb{R}_+; g \text{ is differentiable on } [0,\pi]\},$$

$\|\cdot\|_{C[0,\pi]}$ the uniform norm of continuous functions on $[0,\pi]$ and the Petree K-functional:

$$K(f;t)_p = \inf_{g \in C_+^1[0,\pi]} \{\|f-g\|_p + t\|g'\|_{C[0,\pi]}\}.$$

The second main result of this section is the following.

Theorem 4.2. *Let $1 \le p < \infty$. For all $f : [0,\pi] \to \mathbb{R}_+$, $f \in L^p[0,\pi]$ and $n \in \mathbb{N}$, we have*

$$\|f - K_n^{(M)}(f)\|_p \le c \cdot K\left(f; \frac{a}{n}\right)_p,$$

where

$$c = 1 + 2^{(1-2p)/p} \cdot \pi^2, \qquad a = \frac{3\pi^{1+1/p}}{2c}.$$

Proof. Let $g \in C_+^1[0,\pi]$ be fixed. Now, by Minkowski's inequality, we obtain

$$\begin{aligned}
&\|f - K_n^{(M)}(f)\|_p \\
&\quad = \|(f-g) + (g - K_n^{(M)}(g)) + (K_n^{(M)}(g) - K_n^{(M)}(f))\|_p \\
&\quad \le \|f-g\|_p + \|g - K_n^{(M)}(g)\|_p + \|K_n^{(M)}(g) - K_n^{(M)}(f)\|_p.
\end{aligned}$$

From Theorem 4.1 we have

$$\|K_n^{(M)}(g) - K_n^{(M)}(f)\|_p \le 2^{(1-2p)/p} \cdot \pi^2 \cdot \|f-g\|_p. \tag{4.1}$$

Now, let us estimate $\|g - K_n^{(M)}(g)\|_p$ for $g \in C_+^1[0,\pi]$. Thus, by $K_n^{(M)}(e_0)(x) = e_0(x) = 1$, we get

$$\begin{aligned}
|g(x) - K_n^{(M)}(g)(x)| &= |K_n^{(M)}(g(x))(x) - K_n^{(M)}(g(t))(x)| \\
&\le K_n^{(M)}(|g(x) - g(\cdot)|)(x).
\end{aligned}$$

Since, for $g \in C_+^1[0,\pi]$ and $x, t \in [0,\pi]$, we get

$$|g(x) - g(t)| \le \|g'\|_{C[0,\pi]} \cdot |x-t| = \|g'\|_{C[0,\pi]} \cdot \varphi_x(t),$$

and applying $K_n^{(M)}$, it easily follows that

$$K_n^{(M)}(|g(x) - g(\cdot)|)(x) \le \|g'\|_{C[0,\pi]} K_n^{(M)}(\varphi_x),$$

where $\varphi_x(t) = |x-t|$ for $x, t \in [0,\pi]$.

Therefore, rising at the power p and integrating above with respect to x, we immediately obtain

$$\|g - K_n^{(M)}(g)\|_p \le \|g'\|_{C[0,\pi]} \cdot \|K_n^{(M)}(\varphi_x)\|_p. \tag{4.2}$$

Concluding, from equations (4.1) and (4.2) and denoting $\Delta_{n,p} = \|K_n^{(M)}(\varphi_x)\|_p$ and $c = 1 + 2^{(1-2p)/p} \cdot \pi^2$, we obtain

$$\|f - K_n^{(M)}(f)\|_p \le (1 + 2^{(1-2p)/p} \cdot \pi^2) \left(\|f-g\|_p + \|g'\|_{C[0,\pi]} \cdot \Delta_{n,p}/c\right).$$

Passing the above to infimum with $g \in C_+^1[0,\pi]$, the right-hand side between parentheses becomes

$$K\left(f; \frac{\Delta_{n,p}}{c}\right)_p,$$

and we obtain

$$\|f - K_n^{(M)}(f)\|_p \leq c \cdot K\left(f; \frac{\Delta_{n,p}}{c}\right)_p. \tag{4.3}$$

But it is easy to see that $\Delta_{n,p} \leq \pi^{1/p} \cdot \|K_n^{(M)}(\varphi_x)\|$, which, by estimate (3.3) in the proof of Theorem 3.2, leads to

$$\Delta_{n,p} \leq \frac{3\pi^{1+1/p}}{2n}.$$

Finally, replacing this in estimate (4.3), we immediately get the estimate in Theorem 4.2. □

Remark 4.3. The statement of Theorem 4.2 can be restated for lower bounded functions and of arbitrary sign. Indeed, if $c \in \mathbb{R}$ is such that $f(x) \geq c$ for all $x \in [0, \pi]$, then it is easy to see that, defining the slightly modified max-product operator $\overline{K}^{(M)}(f)(x) = K_n^{(M)}(f - c)(x) + c$, we get

$$|f(x) - \overline{K}^{(M)}(f)(x)| = |(f(x) - c) - K_n^{(M)}(f - c)(x)|,$$

and, since we may consider here that $c < 0$, we immediately obtain the following relations:

$$\begin{aligned} K(f - c; t)_p &= \inf_{g \in C_+^1[0,\pi]} \{\|f - (g + c)\|_p + t\|g'\|_{C[0,\pi]}\} \\ &= \inf_{g \in C_+^1[0,\pi]} \{\|f - (g + c)\|_p + t\|(g + c)'\|_{C[0,\pi]}\} \\ &= \inf_{h \in C_+^1[0,\pi],\, h \geq c} \{\|f - h\|_p + t\|h'\|_{C[0,\pi]}\}. \end{aligned}$$

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