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GLOBAL EXISTENCE AND ASYMPTOTIC STABILITY OF MILD SOLUTIONS FOR STOCHASTIC EVOLUTION EQUATIONS WITH NONLOCAL INITIAL CONDITIONS

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ABSTRACT. The aim of this paper is to discuss the global existence, uniqueness and asymptotic stability of mild solutions for a class of semilinear evolution equations with nonlocal initial conditions on infinite interval. A sufficient condition is given for judging the relative compactness of a class of abstract continuous family of functions on infinite intervals. With the aid of this criteria the compactness of the solution operator for the problem studied on the half line is obtained. The theorems proved in this paper improve and extend some related results in this direction. Discussions are based on stochastic analysis theory, analytic semigroup theory, relevant fixed point theory and the well known Gronwall-Bellman type inequality. An example to illustrate the feasibility of our main results is also given.

1. Introduction. In the past two decades, stochastic differential and integro-differential systems have attracted great interest because of their practical applications in many areas, such as physics, economics, population dynamics, chemistry, medicine biology, social sciences and other areas of science and engineering. For more details, we refer the reader to the books by Sobczyk [**23**], Grecksch and Tudor [**14**], Da Prato and Zabczyk [**9**] and Liu [**17**]. One of the branches of sto-

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chastic differential equations is the theory of stochastic evolution equations. Because semi-linear stochastic evolution equations are abstract formulations for many problems arising in the domains of engineering technology, economic systems and biology, etc., stochastic evolution equations have attracted increasing attention in recent years. There are many interesting results on the existence, uniqueness and asymptotic behavior of mild solutions to stochastic evolution equations, see [**1, 5, 12, 18, 21, 24**] and the references therein. Taniguchi, Liu and Truman [**24**] discussed the existence, uniqueness, pth moment and almost sure Lyapunov exponents of mild solutions to a class of stochastic partial functional differential equations with finite delays by using semigroup methods. El-Borai, Mostafa and Ahmed [**12**] studied exponentially asymptotical stability of stochastic differential equation in a real separable Hilbert space. Luo [**18**] discussed the exponential stability of mild solutions for stochastic partial differential equations by using the contraction mapping principle and stochastic integral techniques. By introducing a suitable metric between the transition probability functions of mild solutions, Bao, Hou and Yuan [**1**] obtained the exponential stability of mild solutions to a class of stochastic partial differential equations. Ren, Zhou and Chen [**21**] established the existence, uniqueness and stability of mild solutions for a class of time-dependent stochastic evolution equations with Poisson jumps and infinite delay under the non-Lipschitz condition. Recently, Chen, Li and Zhang [**5**] discussed the existence of saturated mild solutions and global mild solutions, the continuous dependence of mild solutions on initial values and orders as well as asymptotical stability in the pth moment for the initial value problem of a class of fractional stochastic evolution equations in real separable Hilbert spaces by using α-order fractional resolvent operator theory, the Schauder fixed point theorem and the piecewise extension method.

In addition, the theory of nonlocal evolution equations has become an important area of investigation in recent years due to their application to various problems arising in physics, biology, aerospace and medicine. It has been demonstrated that nonlocal problems have better affects in applications than classical Cauchy problems. For example, it has been used to represent mathematical models for evolution of various phenomena, such as nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion, see McKibben [**19**].

For this reason, differential equations with nonlocal initial conditions were studied by many authors, and some basic results on nonlocal problems have been obtained, see [**2, 3, 11, 13, 16, 25, 26, 27**] and the references therein. In particular, Byszewski [**3**] obtained the existence and uniqueness of classical solutions to a class of abstract functional differential equations with nonlocal conditions of the form

$$u'(t) = f(t, u(t), u(\xi(t))), \quad t \in I = [t_0, t_0 + b], \tag{1.1}$$

$$u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = u_0, \tag{1.2}$$

where $b > 0$ is a constant, $t_0 < t_1 < \cdots < t_p \leq t_0 + b$; $f : I \times E \times E \to E$ and $\xi : I \to I$ are given functions satisfying some assumptions; E is a Banach space, $u_0 \in E$, $c_k \neq 0$, $k = 1, 2, \ldots, p$, and $p \in \mathbb{N}$. The author pointed out that, if $c_k \neq 0$, $k = 1, 2, \ldots, p$, then the results of the paper can be applied to kinematics to determine the location evolution $t \to u(t)$ of a physical object for which we do not know the positions $u(0), u(t_1), \ldots, u(t_p)$, but we know that the nonlocal condition (1.2) holds. The nonlocal condition of type (1.2) has also been used by Deng [**11**] to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, condition (1.2) allows for the additional measurements at t_k, $k = 1, 2, \ldots, p$, which is more precise than the measurement merely at $t = t_0$. Consequently, in order to describe some physical phenomena, the nonlocal condition can be more useful than the classical initial condition $u(t_0) = u_0$.

Very recently, Cui, et al. [**8**] studied the existence results of mild solutions for a class of stochastic integro-differential evolution equations with nonlocal initial conditions in Hilbert spaces assuming that the nonlocal item is only continuous but without imposing compactness and convexity. By using a new strategy which relies on the compactness of the operator semigroup, Schauder fixed point theorem and approximating techniques, Chen and Li [**4**] obtained the existence of α-mild solutions for a class of fractional stochastic integro-differential evolution equations with nonlocal initial conditions in a real separable Hilbert space. Sakthivel et al. [**22**] investigated the approximate controllability of fractional stochastic differential inclusions with nonlocal conditions with the help of the fixed point theorem for multi-valued operators and fractional calculus. Chen, et al. [**6**] obtained the existence of mild solutions for a class of fractional stochastic evolution equations with nonlocal initial conditions in Hilbert spaces under the situation that the nonlinear term satisfies some appropriate growth conditions by using fractional calculations, Schauder fixed point theorem, stochastic analysis theory, α-order fractional solution operator theory and α-resolvent family theory.

However, the following two aspects should be considered. Firstly, to the best of the authors' knowledge, all existing articles (such as **[4, 6, 8, 22]**) are devoted only to the study of the local existence of mild solutions for stochastic evolution equations with nonlocal conditions on compact interval; there are not yet any results on the existence of global mild solutions for stochastic evolution equations with nonlocal initial conditions on infinite interval. Secondly, we have not yet seen any research on the globally asymptotical stability of mild solutions for stochastic evolution equations with nonlocal conditions on $[0, +\infty)$. Motivated by all of the above-mentioned aspects, in this article we discuss the global existence, uniqueness and asymptotic stability of mild solutions for a class of semilinear stochastic evolution equations with nonlocal initial conditions on the infinite interval

$$du(t) + Au(t)\,dt = f(t, u(t))\,d\mathbb{W}(t), \quad t \in J, \tag{1.3}$$

$$u(0) = \sum_{k=1}^{\infty} c_k u(t_k), \tag{1.4}$$

where the state $u(\cdot)$ takes values in the real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot,\cdot)$, and norm $\|\cdot\|$, $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a positive definite self-adjoint operator.

Let $\mathbb{K}$ be another separable Hilbert space with inner product $(\cdot,\cdot)_{\mathbb{K}}$ and norm $\|\cdot\|_{\mathbb{K}}$. Assume that $\{\mathbb{W}(t) : t \geq 0\}$ is a cylindrical $\mathbb{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. We shall also employ the same notation $\|\cdot\|$ for the norm of $\mathcal{L}(\mathbb{K}, \mathbb{H})$, which denotes the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. We set $\mathcal{L}(\mathbb{H}) = \mathcal{L}(\mathbb{H}, \mathbb{H})$, and $f : J \times \mathbb{H} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ is a continuous nonlinear mapping $J = [0, +\infty)$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to +\infty\,(k \to +\infty)$, c_k are real numbers, $c_k \neq 0$, $k = 1, 2, \cdots$. As we can easily see, the general nonlocal initial condition (1.4) contains as particular case (1.2). By using the theory of operator semigroups, we can transform nonlocal problem (1.3)–(1.4) into an equivalent integral equation, see Definition 2.1, and then, applying Schauder's fixed point theorem, the theory of analytic semigroups and stochastic analysis theory to discuss the global existence of mild solutions for nonlocal problem (1.3)–(1.4). The asymptotic stability of mild solutions for nonlocal problem (1.3)–(1.4) have also been discussed, in which the Gronwall-Bellman type inequalities paly an important role.

In the next section, we first introduce some notation and preliminaries, which are used throughout this paper, as well as the definition of the mild solution for stochastic evolution equation nonlocal problem (1.3)–(1.4) is also given. In Section 3, we state and prove the global existence, uniqueness and asymptotic stability of mild solutions for semilinear stochastic evolution equation nonlocal problem (1.3)–(1.4). In the last paragraph, we give an example to illustrate the feasibility of our abstract results.

2. Preliminaries. Let $\mathbb{H}$ and $\mathbb{K}$ be two real, separable Hilbert spaces and $\mathcal{L}(\mathbb{K},\mathbb{H})$ the space of bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathbb{H}$, $\mathbb{K}$ and $\mathcal{L}(\mathbb{K},\mathbb{H})$, and use $(\cdot,\cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$ without any confusion. Throughout this paper, we assume that $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ is a complete filtered probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. In general, we just write $u(t)$ instead of $u(t,\mathbb{W})$ and $u(t):J\to\mathbb{H}$. Let $\{e_k, k\in\mathbb{N}\}$ be a complete orthonormal basis of $\mathbb{K}$. Suppose that $\{\mathbb{W}(t): t\geq 0\}$ is a cylindrical $\mathbb{K}$-valued Wiener process defined on the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ with a finite trace nuclear covariance operator $Q\geq 0$. Denote $\mathrm{Tr}(Q)=\sum_{k=1}^{\infty}\lambda_k=\lambda<\infty$, which satisfies that $Qe_k=\lambda_k e_k$, $k\in\mathbb{N}$. Let $\{\mathbb{W}_k(t), k\in\mathbb{N}\}$ be a sequence of one-dimensional standard Wiener processes mutually independent on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ such that

$$\mathbb{W}(t)=\sum_{k=1}^{\infty}\sqrt{\lambda_k}\,\mathbb{W}_k(t)e_k. \tag{2.1}$$

We further assume that $\mathcal{F}_t=\sigma\{\mathbb{W}(s),\ 0\leq s\leq t\}$ is the σ-algebra generated by $\mathbb{W}$ and $\mathcal{F}$.

For φ, $\psi \in \mathcal{L}(\mathbb{K}, \mathbb{H})$, we define $\mathcal{L}_Q = \mathcal{L}_2(Q^{1/2}\mathbb{K}, \mathbb{H})$ as the space of all Q Hilbert-Schmidt operators from $Q^{1/2}\mathbb{K}$ to $\mathbb{H}$ with the inner product $(\varphi, \psi)_Q = \mathrm{Tr}(\varphi Q \psi^*)$, where ψ^* is the adjoint of the operator ψ. Clearly, for any bounded operator $\psi \in \mathcal{L}(\mathbb{K}, \mathbb{H})$,

$$\|\psi\|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{k=1}^{\infty} \|\sqrt{\lambda_k}\psi e_k\|. \tag{2.2}$$

Let $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ be a positive definite self-adjoint operator in Hilbert space $\mathbb{H}$, and let it be compact resolvent. By the spectral resolution theorem of positive definite self-adjoint operators, the spectrum $\sigma(A)$ only consists of real eigenvalues, and it can be arrayed in a sequence as

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lambda_n \to \infty \; n \to \infty. \tag{2.3}$$

From **[15, 20]**, we know that $-A$ generates an analytic operator semigroup $T(t)(t \geq 0)$ on H, which is exponentially stable and satisfies

$$\|T(t)\| \leq e^{-\lambda_1 t}, \quad \text{for all } t \geq 0. \tag{2.4}$$

Since the positive definite self-adjoint operator A has compact resolvent, the embedding $D(A) \hookrightarrow H$ is compact, and therefore, $T(t)$, $t \geq 0$, is also a compact semigroup.

Throughout this paper, we assume that the constant c_k, $k = 1, 2, \ldots$, satisfies the condition

$$\text{(P1)} \quad \sum_{k=1}^{\infty} |c_k| < e^{\lambda_1 t_1}.$$

By assumption (P1), we have

$$\left\| \sum_{k=1}^{\infty} c_k T(t_k) \right\| \leq \sum_{k=1}^{\infty} |c_k| e^{-\lambda_1 t_1} < 1. \tag{2.5}$$

By the operator spectrum theorem, we know that the operator

$$B := \left(I - \sum_{k=1}^{\infty} c_k T(t_k) \right)^{-1} \tag{2.6}$$

exists and is bounded. Furthermore, by Neumann expression, B can be expressed by

$$B = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} c_k T(t_k) \right)^n. \tag{2.7}$$

Therefore,

$$\|B\| \le \sum_{n=0}^{\infty} \left\| \sum_{k=1}^{\infty} c_k T(t_k) \right\|^n = \frac{1}{1 - \left\| \sum_{k=1}^{\infty} c_k T(t_k) \right\|} \le \frac{1}{1 - e^{-\lambda_1 t_1} \sum_{k=1}^{\infty} |c_k|}. \tag{2.8}$$

By the above discussion, we can obtain the definition of mild solutions for nonlocal problem (1.3)–(1.4).

Definition 2.1. An $\mathcal{F}_t$-adapted stochastic process $u : J \to \mathbb{H}$ is called a *mild solution* of nonlocal problem (1.3)–(1.4) if $u(t) \in \mathbb{H}$ has cádlág paths on $t \in J$ almost surely and, for each $t \in J$, $u(t)$ satisfies the integral equation

$$\begin{aligned} u(t) = {} & \sum_{k=1}^{\infty} c_k T(t) B \int_0^{t_k} T(t_k - s) f(s, u(s))\, d\mathbb{W}(s) \\ & + \int_0^t T(t-s) f(s, u(s))\, d\mathbb{W}(s), \quad \mathbb{P}\text{-almost surely.} \end{aligned} \tag{2.9}$$

The collection of all strongly measurable, square integrable $\mathbb{H}$-valued random variables, denoted $L^2(\Omega, \mathbb{H})$, is a Banach space equipped with the norm

$$\|u(\cdot)\|_{L^2} = (\mathbb{E}\|u(\cdot, \mathbb{W})\|^2)^{1/2},$$

where $\mathbb{E}(\cdot)$ denotes the expectation over $(\Omega, \mathbb{P})$. An important subspace of $L^2(\Omega, \mathbb{H})$ is given by

$$L_0^2(\Omega, \mathbb{H}) = \{u \in L^2(\Omega, \mathbb{H}) \mid u \text{ is } \mathcal{F}_0\text{-measurable}\}. \tag{2.10}$$

Denote the continuous $\mathcal{F}_t$-adapted progressively measurable processes satisfying $\sup_{t \in J} \mathbb{E}\|u(t)\|^2 < +\infty$ by

$$C_e(J, L^2(\Omega, \mathbb{H})).$$

Then it is easy to see that $C_e(J, L^2(\Omega, \mathbb{H}))$ is a Banach space endowed with the norm

$$\|u\|_e = \left(\sup_{t\in J} e^{-t}\mathbb{E}\|u(t)\|^2 \right)^{1/2}. \tag{2.11}$$

For any constant $r > 0$, let

$$B_r = \{u \in C_e(J, L^2(\Omega, \mathbb{H})) : \|u\|_e^2 \leq r\}. \tag{2.12}$$

Clearly, B_r is a bounded closed, convex set in $C_e(J, L^2(\Omega, \mathbb{H}))$.

The next result plays an important role in the proof of our main result.

Lemma 2.2. *A set $H \subset C_e(J, L^2(\Omega, \mathbb{H}))$ is relatively compact if*

(i) *the set $H(t) = \{u(t) \mid u \in H\}$ is relatively compact in $L^2(\Omega, \mathbb{H})$ for every $t \in J$;*
(ii) $\lim_{t\to+\infty} e^{-t}\mathbb{E}\|u(t)\|^2 = 0$ *uniformly for any $u \in H$;*
(iii) *H is a locally equicontinuous family of functions, i.e., for any constant $a > 0$, the functions in H are equicontinuous in $[0, a]$.*

Proof. From condition (ii), we know that, for all $\epsilon > 0$, a constant $a > 0$ exists big enough such that, when $t > a$,

$$e^{-t}\mathbb{E}\|u(t)\|^2 < \frac{\epsilon^2}{5}, \quad \text{for all } u \in H. \tag{2.13}$$

Denote

$$J_a = [0, a], \qquad u|_{J_a} = \{u(t) \mid t \in J_a\}, \qquad H_a = \{u|_{J_a} \mid u \in H\}. \tag{2.14}$$

Then it is easy to see that $H_a \subset C(J_a, L^2(\Omega, \mathbb{H}))$. By conditions (i) and (iii), we know that $H_a(t)$ is relatively compact for every $t \in J_a$, and H_a is an equicontinuous family of functions in $C(J_a, L^2(\Omega, \mathbb{H}))$. By means of the well known Ascoli-Arzela theorem in the finite closed interval one obtains that H_a is a relatively compact set. From Hausdorff's theorem, we know that finite ϵ-networks $\{u_1|_{J_a}, u_2|_{J_a}, \ldots, u_k|_{J_a}\} \subset H_a$ exist.

Next, we prove that $\{u_1, u_2, \dots, u_k\}$ is a finite ϵ-network of H. For arbitrary $u \in H$, some u_i, $i = 1, 2, \dots, k$, exists such that

$$\sup_{t\in J_a} e^{-t}\mathbb{E}\|u(t) - u_i(t)\|^2 < \epsilon^2. \tag{2.15}$$

Therefore,

$$\begin{aligned} \|u - u_i\|_e^2 &= \sup_{t\in J} e^{-t}\mathbb{E}\|u(t) - u_i(t)\|^2 \\ &= \sup_{t\in[0,a]\cup(a,+\infty)} e^{-t}\mathbb{E}\|u(t) - u_i(t)\|^2. \end{aligned} \tag{2.16}$$

By (2.13), we obtain

$$\sup_{t\in(a,+\infty)} e^{-t}\mathbb{E}\|u(t) - u_i(t)\|^2 \le \frac{4\epsilon^2}{5}. \tag{2.17}$$

By (2.15)–(2.17) and the fact

$$\begin{aligned} &\sup_{t\in[0,a]\cup(a,+\infty)} e^{-t}\mathbb{E}\|u(t) - u_i(t)\|^2 \\ &\le \max\left\{\sup_{t\in[0,a]} e^{-t}\mathbb{E}\|u(t) - u_i(t)\|^2, \sup_{t\in(a,+\infty)} e^{-t}\mathbb{E}\|u(t) - u_i(t)\|^2\right\}, \end{aligned}$$

we obtain that

$$\|u - u_i\|_e = \left(\sup_{t\in J} e^{-t}\mathbb{E}\|u(t) - u_i(t)\|^2\right)^{1/2} < \epsilon, \quad i = 1, 2, \dots, k. \tag{2.18}$$

Therefore, $\{u_1, u_2, \dots, u_k\}$ is an ϵ-network of H. Combining this fact with completeness of $C_e(J, L^2(\Omega, \mathbb{H}))$ as well as Hausdorff's theorem, one obtains that H is a relative set in $C_e(J, L^2(\Omega, \mathbb{H}))$. $\square$

Next, let us state the following well-known lemmas which will be used in the sequel of this paper.

Lemma 2.3 ([**9,** Lemma 7.7]). *Let $p \ge 2$, and let Φ be an $\mathcal{L}(\mathbb{K}, \mathbb{H})$-valued, predictable process such that*

$$\mathbb{E}\int_0^t \|\Phi(s)\|^p ds < \infty \quad \textit{for every } t > 0.$$

Then

$$\sup_{0\le s\le t} \mathbb{E}\left\| \int_0^s \Phi(u)\, d\mathbb{W}(u)\right\|^p \le \left(\frac{p(p-1)}{2}\right)^{p/2} \left(\int_0^t \left(\mathbb{E}\|\Phi(s)\|^p\right)^{2/p} ds\right)^{p/2}. \tag{2.19}$$

Lemma 2.4 ([**7**]). *If*

$$m(t) \le g(t) + \int_{t_0}^t k(s)m(s)\, ds, \quad t \in [t_0, K), \tag{2.20}$$

where all the functions involved are continuous on $[t_0, K)$, $K \le +\infty$, *and* $k(t) \ge 0$ *and* $g(t)$ *are nondecreasing, then*

$$m(t) \le g(t) \exp\left(\int_{t_0}^t k(s)\, ds\right), \quad t \in [t_0, K). \tag{2.21}$$

3. Main results. In this section, we state and prove the global existence, uniqueness and asymptotic stability of mild solutions for semilinear evolution equations with nonlocal initial conditions on infinite interval (1.3)–(1.4).

Theorem 3.1. *Let* A *be a positive definite self-adjoint operator in Hilbert space* $\mathbb{H}$, *and let it have a compact resolvent, the function* $f : J \times \mathbb{H} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ *continuous. If the condition* (P1) *and*

(P2) *there exist constants*

$$0 < \alpha < \frac{\lambda_1 \left(1 - e^{-\lambda_1 t_1} \sum\limits_{k=1}^{\infty} |c_k|\right)^2}{\left(\sum\limits_{k=1}^{\infty} |c_k|\right)^2 + \left(1 - e^{-\lambda_1 t_1} \sum\limits_{k=1}^{\infty} |c_k|\right)^2}$$

and $\beta > 0$ *such that*

$$\mathbb{E}\|f(t, u(t))\|^2 \le \alpha\, \mathbb{E}\|u(t)\|^2 + \beta,$$

for all $t \in J$, $u(t) \in \mathbb{H}$, *hold, then, the nonlocal problem* (1.3)–(1.4) *has at least one global mild solution on* $[0, +\infty)$.

Proof. We consider the operator $\mathbb{F}$ on $C_e(J, L^2(\Omega, \mathbb{H}))$ defined by

$$
\begin{aligned}
(\mathbb{F}u)(t) = {} & \sum_{k=1}^{\infty} c_k T(t) B \int_0^{t_k} T(t_k - s) f(s, u(s)) d\mathbb{W}(s) \\
& + \int_0^t T(t-s) f(s, u(s))\, d\mathbb{W}(s), \quad t \in J.
\end{aligned} \tag{3.1}
$$

By condition (P1) and Definition 2.1, it is easy to see that the mild solution of nonlocal problem (1.3)–(1.4) is equivalent to the fixed point of the operator $\mathbb{F}$ defined by (3.1).

In the following, we will prove that operator $\mathbb{F}$ has a fixed point by applying the famous Schauder fixed point theorem.

First, we prove that the operator $\mathbb{F}$ maps the functions in $C_e(J, L^2(\Omega, \mathbb{H}))$ to $C_e(J, L^2(\Omega, \mathbb{H}))$, and it is continuous. For any $u \in C_e(J, L^2(\Omega, \mathbb{H}))$, by the definition of the space $C_e(J, L^2(\Omega, \mathbb{H}))$, we know that a constant $\gamma > 0$ exists such that

$$
\gamma := \sup_{t \in J} \mathbb{E}\|u(t)\|^2 < +\infty. \tag{3.2}
$$

By (2.4), (2.8), (3.1), (3.2), conditions (P1), (P2) and Lemma 2.3, we obtain that

$$
\begin{aligned}
\mathbb{E}\|(\mathbb{F}u)(t)\|^2 \leq {} & 2\mathbb{E}\bigg\| \sum_{k=1}^{\infty} c_k T(t) B \int_0^{t_k} T(t_k - s) f(s, u(s))\, d\mathbb{W}(s) \bigg\|^2 \\
& + 2\mathbb{E}\bigg\| \int_0^t T(t-s) f(s, u(s))\, d\mathbb{W}(s) \bigg\|^2 \\
\leq {} & 2\bigg(\sum_{k=1}^{\infty} |c_k| e^{-\lambda_1 t} \|B\| \bigg)^2 \int_0^{t_k} e^{-2\lambda_1 (t_k - s)} [\alpha \mathbb{E}\|u(s)\|^2 + \beta]\, ds \\
& + 2 \int_0^t e^{-2\lambda_1 (t-s)} [\alpha \mathbb{E}\|u(s)\|^2 + \beta]\, ds \\
\leq {} & \Lambda(\alpha\gamma + \beta),
\end{aligned} \tag{3.3}
$$

where

$$\Lambda = \frac{\left(\sum_{k=1}^{\infty} |c_k|\right)^2 + \left(1 - e^{-\lambda_1 t_1} \sum_{k=1}^{\infty} |c_k|\right)^2}{\lambda_1 \left(1 - e^{-\lambda_1 t_1} \sum_{k=1}^{\infty} |c_k|\right)^2}. \tag{3.4}$$

From (3.3), we know that

$$\lim_{t \to +\infty} e^{-t} \mathbb{E}\|(\mathbb{F}u)(t)\|^2 = 0, \tag{3.5}$$

and therefore $\mathbb{F}u \in C_e(J, L^2(\Omega, \mathbb{H}))$.

On the other hand, let $\{u_n\}_{n=1}^{\infty} \subset C_e(J,\ L^2(\Omega, \mathbb{H}))$ be a sequence such that $\lim_{n \to +\infty} u_n = u$ in $C_e(J, L^2(\Omega, \mathbb{H}))$. By the continuity of the second variable of the nonlinear term f, we have

$$\lim_{n \to +\infty} f(s, u_n(s)) = f(s, u(s)), \quad \text{almost everywhere } s \in J. \tag{3.6}$$

Let

$$\eta = \sup_{t \in J} \mathbb{E}\|u_n(t)\|^2 < +\infty, \quad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

From condition (P2), we obtain that

$$\begin{aligned} &e^{-2\lambda_1(t-s)} \mathbb{E}\|f(s, u_n(s)) - f(s, u(s))\|^2 \\ &\quad \le e^{-2\lambda_1(t-s)} \Big(2\mathbb{E}\|f(s, u_n(s))\|^2 + 2\mathbb{E}\|f(s, u(s))\|^2\Big) \\ &\quad \le 4e^{-2\lambda_1(t-s)} [\alpha(\gamma + \eta) + \beta] \quad \text{for all } t \in J. \end{aligned} \tag{3.8}$$

By (3.7), we know that, for each $s \in [0, t]$, $t \in J$,

$$\begin{aligned} &\int_0^t e^{-2\lambda_1(t-s)} \mathbb{E}\|f(s, u_n(s)) - f(s, u(s))\|^2 ds \\ &\quad \le 4 \int_0^t e^{-2\lambda_1(t-s)} [\alpha(\gamma + \eta) + \beta]\, ds \le \frac{2[\alpha(\gamma + \eta) + \beta]}{\lambda_1}. \end{aligned} \tag{3.9}$$

Therefore, by (2.4), (2.8), (3.1), (3.6)–(3.9), Lemma 2.3 and the Lebesgue dominated convergence theorem, we obtain

$$
\begin{aligned}
&\mathbb{E}\|(\mathbb{F}u_n)(t) - (\mathbb{F}u)(t)\|^2 \\
&\quad \le 2\mathbb{E}\bigg\| \sum_{k=1}^{\infty} c_k T(t) B \int_0^{t_k} T(t_k - s)[f(s, u_n(s)) - f(s, u(s))]\, d\mathbb{W}(s) \bigg\|^2 \\
&\qquad + 2\mathbb{E}\bigg\| \int_0^t T(t-s)[f(s, u_n(s)) - f(s, u(s))]\, d\mathbb{W}(s) \bigg\|^2 \\
&\quad \le \frac{2\left(\sum_{k=1}^{\infty} |c_k|\right)^2}{\left(1 - e^{-\lambda_1 t_1} \sum_{k=1}^{\infty} |c_k|\right)^2} \int_0^{t_k} e^{-2\lambda_1 (t_k - s)} \mathbb{E}\|f(s, u_n(s)) - f(s, u(s))\|^2 ds \\
&\qquad + 2\int_0^t e^{-2\lambda_1 (t-s)} \mathbb{E}\|f(s, u_n(s)) - f(s, u(s))\|^2 ds \\
&\qquad\qquad \longrightarrow 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
\tag{3.10}
$$

Hence, by (3.10), we obtain that

$$
\|(\mathbb{F}u_n) - (\mathbb{F}u)\|_e = \left(\sup_{t\in J} e^{-t}\mathbb{E}\|(\mathbb{F}u_n)(t) - (\mathbb{F}u)(t)\|^2\right)^{1/2} \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.11}
$$

Therefore, we have proved that $\mathbb{F} : C_e(J, L^2(\Omega, \mathbb{H})) \to C_e(J, L^2(\Omega, \mathbb{H}))$ is a continuous operator.

Subsequently, we prove that there exists a positive constant R such that $\mathbb{F}(B_R) \subset B_R$. In fact, if we choose

$$
R \ge \frac{\beta}{1 - \alpha\Lambda}, \tag{3.12}
$$

where Λ is defined by (3.4), then, for every $u \in B_R$, it follows from (2.4), (2.8), (3.1), (3.12), condition (P2) and Lemma 2.3 that

$$
\begin{aligned}
e^{-t}\mathbb{E}\|(\mathbb{F}u)(t)\|^2 &\le \mathbb{E}\|(\mathbb{F}u)(t)\|^2 \\
&\le 2\mathbb{E}\bigg\| \sum_{k=1}^{\infty} c_k T(t) B \int_0^{t_k} T(t_k - s) f(s, u(s))\, d\mathbb{W}(s) \bigg\|^2 \\
&\quad + 2\mathbb{E}\bigg\| \int_0^t T(t-s) f(s, u(s))\, d\mathbb{W}(s) \bigg\|^2
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
&\le 2\Big[\sum_{k=1}^{\infty}|c_k|\|B\|\Big]^2\int_0^{t_k} e^{-2\lambda_1(t_k-s)}(\alpha R+\beta)\,ds\\
&\quad + 2\int_0^{t} e^{-2\lambda_1(t-s)}(\alpha R+\beta)\,ds\\
&\le \Lambda(\alpha R+\beta)\le R,
\end{aligned}
$$

which means that $\|\mathbb{F}u\|_e^2 = \sup_{t\in J} e^{-t}\mathbb{E}\|(\mathbb{F}u)(t)\|^2 \le R$. Therefore, $\mathbb{F}: B_R \to B_R$ is a continuous operator.

Next, we prove that $\mathbb{F}: B_R\to B_R$ is a compact operator. In order to prove this fact, we first show that $\{(\mathbb{F}u)(t) \mid u\in B_R\}$ is relatively compact in $L^2(\Omega,\mathbb{H})$ for every $t\in J$. It is easy to see that, for every $u\in B_R$,

$$
(\mathbb{F}u)(0) = \sum_{k=1}^{\infty} c_k B\int_0^{t_k} T(t_k-s)f(s,u(s))\,d\mathbb{W}(s). \tag{3.14}
$$

For any $0<\epsilon<t_1$ and $u\in B_R$, we define the operator $\mathbb{F}_0^{\epsilon}$ by

$$
\begin{aligned}
(\mathbb{F}_0^{\epsilon}u)(0) &= \sum_{k=1}^{\infty} c_k B\int_0^{t_k-\epsilon} T(t_k-s)f(s,u(s))\,d\mathbb{W}(s)\\
&= T(\epsilon)\sum_{k=1}^{\infty} c_k B\int_0^{t_k-\epsilon} T(t_k-s-\epsilon)f(s,u(s))\,d\mathbb{W}(s).
\end{aligned}
\tag{3.15}
$$

Since $T(t)$ is compact for every $t>0$, the set $\{(\mathbb{F}_0^{\epsilon}u)(0) : u\in B_R\}$ is relatively compact in $L^2(\Omega,\mathbb{H})$ for every $\epsilon\in(0,t_1)$. Moreover, for every $u\in B_R$, by (2.4), (2.8), (3.4), (3.14), (3.15), condition (P2) and Lemma 2.3, we obtain that

$$
\begin{aligned}
&\mathbb{E}\|(\mathbb{F}u)(0)-(\mathbb{F}_0^{\epsilon}u)(0)\|^2\\
&\quad = \mathbb{E}\Big\|\sum_{k=1}^{\infty} c_k B\int_{t_k-\epsilon}^{t_k} T(t_k-s)f(s,u(s))\,d\mathbb{W}(s)\Big\|^2\\
&\quad \le \frac{\Big(\sum_{k=1}^{\infty}|c_k|\Big)^2}{\Big(1-e^{-\lambda_1 t_1}\sum_{k=1}^{\infty}|c_k|\Big)^2}\cdot\int_{t_k-\epsilon}^{t_k} e^{-2\lambda_1(t_k-s)}[\alpha\mathbb{E}\|u(s)\|^2+\beta]\,ds
\end{aligned}
\tag{3.16}
$$

$$\leq \frac{\left(\sum\limits_{k=1}^{\infty}|c_k|\right)^2(\alpha R+\beta)}{\left(1-e^{-\lambda_1 t_1}\sum\limits_{k=1}^{\infty}|c_k|\right)^2} \quad \text{as} \quad \epsilon \to 0.$$

Therefore, we have proved that there are relatively compact sets $\{(\mathbb{F}_0^\epsilon u)(0) : u \in B_R\}$ arbitrarily close to the set $\{(\mathbb{F}u)(0) : u \in B_R\}$. This means that the set $\{(\mathbb{F}u)(0) : u \in B_R\}$ is relatively compact in $L^2(\Omega, \mathbb{H})$. Let $0 < t < +\infty$ be given, $0 < \epsilon < t$ and $u \in B_R$. We define the operator $(\mathbb{F}^\epsilon u)$ by

$$\begin{aligned}(\mathbb{F}^\epsilon u)(t) = \sum_{k=1}^{\infty} c_k T(t) B \int_0^{t_k} T(t_k - s) f(s, u(s))\, d\mathbb{W}(s) \\ + \int_0^{t-\epsilon} T(t-s) f(s, u(s))\, d\mathbb{W}(s).\end{aligned} \tag{3.17}$$

Since the operator $T(t)$ is compact for every $t > 0$, the set $\{(\mathbb{F}^\epsilon u)(t) : u \in B_R\}$ is relatively compact in $\mathbb{H}$ for every $\epsilon \in (0, t)$. By applying a similar method which was used in (3.16), we can prove that there is a relatively compact set $\{(\mathbb{F}^\epsilon u)(t) : u \in B_R\}$ arbitrarily close to the set $\{(\mathbb{F}u)(t) : u \in B_R\}$ in $L^2(\Omega, \mathbb{H})$ for $0 < t < +\infty$. Therefore, the set $\{(\mathbb{F}u)(t) : u \in B_R\}$ is also relatively compact in $L^2(\Omega, \mathbb{H})$ for $0 < t < +\infty$.

Further, we prove that, for every $u \in B_R$, the following equality

$$\lim_{t \to +\infty} e^{-t}\mathbb{E}\|(\mathbb{F}u)(t)\|^2 = 0 \tag{3.18}$$

is satisfied. In fact, by (3.13), we know that, for every $u \in B_R$, $\mathbb{E}\|(\mathbb{F}u)(t)\|^2 \leq R$. Combining this fact with the property of exponential functions, one can easily see that (3.18) is satisfied.

Finally, we demonstrate that $\mathbb{F}(B_R)$ is a locally equicontinuous family of functions in $C_e(J, L^2(\Omega,\ \mathbb{H}))$. Suppose that $0 < a < +\infty$ is an arbitrary constant. For any $u \in B_R$ and $0 \leq t' < t'' \leq a$, by means of (2.8) and Lemma 2.3 we obtain that

$$\begin{aligned}&\mathbb{E}\|(\mathbb{F}u)(t'') - (\mathbb{F}u)(t')\|^2 \\ &\quad \leq 3\mathbb{E}\left\|(T(t'') - T(t'))\sum_{k=1}^{\infty} c_k B \int_0^{t_k} T(t_k - s) f(s, u(s))\, d\mathbb{W}(s)\right\|^2\end{aligned}$$

$$
\begin{aligned}
&+3\mathbb{E}\left\|\int_{t'}^{t''} T(t''-s)f(s,u(s))\,d\mathbb{W}(s)\right\|^2 \\
&+3\mathbb{E}\left\|\int_0^{t'} (T(t''-s)-T(t'-s))f(s,u(s))d\mathbb{W}(s)\right\|^2 \\
\le\ & 3\mathbb{E}\left\|(T(t'')-T(t'))\sum_{k=1}^{\infty} c_k B\int_0^{t_k} T(t_k-s)f(s,u(s))d\mathbb{W}(s)\right\|^2 \\
&+3\int_{t'}^{t''} e^{-2\lambda_1(t''-s)}[\alpha\mathbb{E}\|u(s)\|^2+\beta]\,ds \\
&+3\int_0^{t'} \|T(t''-s)-T(t'-s)\|^2[\alpha\mathbb{E}\|u(s)\|^2+\beta]\,ds \\
:=\ & I_1+I_2+I_3,
\end{aligned}
\tag{3.19}
$$

where

$$
I_1 = 3\mathbb{E}\left\|(T(t'')-T(t'))\sum_{k=1}^{\infty} c_k B\int_0^{t_k} T(t_k-s)f(s,u(s))\,d\mathbb{W}(s)\right\|^2,
\tag{3.20}
$$

$$
I_2 = 3\int_{t'}^{t''} e^{-2\lambda_1(t''-s)}[\alpha\mathbb{E}\|u(s)\|^2+\beta]\,ds,
\tag{3.21}
$$

$$
I_3 = 3\int_0^{t'} \|T(t''-s)-T(t'-s)\|^2[\alpha\mathbb{E}\|u(s)\|^2+\beta]\,ds.
\tag{3.22}
$$

Therefore, we only need to check that I_i tends to 0 independently of $u \in B_R$ when $t'' \to t'$, $i = 1,2,3$. For I_1, by (2.4), (2.8), condition (P2) and Lemma 2.3, we have

$$
\begin{aligned}
&\mathbb{E}\left\|\sum_{k=1}^{p} c_k B\int_0^{t_k} T(t_k-s)f(s,u(s))\,d\mathbb{W}(s)\right\|^2 \\
&\le \left(\sum_{k=1}^{p}|c_k|\right)^2\|B\|^2\int_0^{t_k} e^{-2\lambda_1(t_k-s)}[\alpha\mathbb{E}\|u(s)\|^2+\beta]\,ds
\end{aligned}
\tag{3.23}
$$

$$\leq \frac{\left(\sum\limits_{k=1}^{\infty}|c_k|\right)^2(\alpha R+\beta)}{2\lambda_1\left(1-e^{-\lambda_1 t_1}\sum\limits_{k=1}^{\infty}|c_k|\right)^2}.$$

Therefore, from the strong continuity of the semigroup $T(t)$, $t \geq 0$, (3.20) and (3.23), we can easily obtain that $I_1 \to 0$ as $t'' \to t'$. For I_2, we can get by direct calculus that

$$I_2 \leq 3(\alpha R+\beta)(t''-t') \longrightarrow 0 \quad \text{as } t'' \to t'. \tag{3.24}$$

For I_3, by (3.22) and the property of Lebesgue integral, we obtain

$$\begin{aligned} I_3 &\leq 3(\alpha R+\beta)\int_0^{t'}\|T(t''-s)-T(t'-s)\|^2 ds \\ &= 3(\alpha R+\beta)\int_0^{t'}\|T(t''-t'+s)-T(s)\|^2 ds \\ &\quad \longrightarrow 0 \quad \text{as } t'' \to t'. \end{aligned} \tag{3.25}$$

As a result, $\mathbb{E}\|(\mathbb{F}u)(t'')-(\mathbb{F}u)(t')\|^2$ tends to 0 independently of $u \in B_R$ as $t'' \to t'$, which means that the operator $\mathbb{F} : B_R \to B_R$ is equicontinuous in $[0,a]$ for arbitrary constant $0 < a < +\infty$, namely, the operator $\mathbb{F} : B_R \to B_R$ is locally equicontinuous. Hence, by Lemma 2.2, one gets that $\mathbb{F} : B_R \to B_R$ is a compact operator. Therefore, by Schauder's fixed point theorem, see **[10]**, we obtain that $\mathbb{F}$ has at least one fixed point $u \in B_R$, which is in turn is a mild solution of nonlocal problem (1.3)–(1.4) on $[0,+\infty)$. □

Theorem 3.2. *Let A be a positive definite self-adjoint operator in Hilbert space $\mathbb{H}$, and let it have a compact resolvent, the function $f : J \times \mathbb{H} \to \mathcal{L}(\mathbb{K},\mathbb{H})$ continuous and $\sup_{t\in J}\mathbb{E}\|f(t,\theta)\|^2 < +\infty$ (θ denotes the 0 element in $\mathbb{H}$). If condition* (P1) *and the following condition*

(P3) *there exists a constant*

$$0 < \vartheta < \frac{\lambda_1\left(1-e^{-\lambda_1 t_1}\sum\limits_{k=1}^{\infty}|c_k|\right)^2}{\left(\sum\limits_{k=1}^{\infty}|c_k|\right)^2+\left(1-e^{-\lambda_1 t_1}\sum\limits_{k=1}^{\infty}|c_k|\right)^2}$$

such that

$$\mathbb{E}\|f(t,u(t)) - f(t,v(t))\|^2 \le \vartheta\, \mathbb{E}\|u(t) - v(t)\|^2,$$

for all $t \in J$, $u(t), v(t) \in \mathbb{H}$,

hold, then nonlocal problem (1.3)–(1.4) *has a unique global mild solution* $\widehat{u}$ *in* $C_e(J, L^2(\Omega,\mathbb{H}))$, *and it is globally asymptotically stable.*

Proof. The proof is divided into two steps.

Step 1. *Uniqueness of the global mild solution.* From Theorem 3.1, we know that the mild solution of problem (1.3)–(1.4) is equivalent to the fixed point of the operator $\mathbb{F}$ defined by (3.1). For any $u \in C_e(J, L^2(\Omega,\mathbb{H}))$, by condition (P3), definition of the space $C_e(J, L^2(\Omega,\mathbb{H}))$ and the fact that $\sup_{t\in J}\mathbb{E}\|f(t,\theta)\|^2 < \infty$, we know that, for any $t \in J$,

$$\mathbb{E}\|f(t,u(t))\|^2 \le \mathbb{E}\|f(t,\theta)\|^2 + \vartheta\,\mathbb{E}\|u(t)\|^2 < +\infty. \tag{3.26}$$

Therefore, by (3.25) and the proof of Theorem 3.1, we know that the operator $\mathbb{F}$ defined by (3.1) maps $C_e(J, L^2(\Omega,\mathbb{H}))$ to $C_e(J, L^2(\Omega,\mathbb{H}))$, and it is continuous. For any $u, v \in C_e(J, L^2(\Omega,\mathbb{H}))$, by (2.4), (2.8), (3.1), assumption (P3) and Lemma 2.3, we have

$$\begin{aligned}
&e^{-t}\mathbb{E}\|(\mathbb{F}u)(t) - (\mathbb{F}v)(t)\|^2 \\
&\quad\le 2e^{-t}\mathbb{E}\bigg\|\sum_{k=1}^{\infty} c_k T(t) B \int_0^{t_k} T(t_k - s)[f(s,u(s)) - f(s,v(s))]\, d\mathbb{W}(s)\bigg\|^2 \\
&\qquad + 2e^{-t}\mathbb{E}\bigg\|\int_0^t T(t-s)[f(s,u(s)) - f(s,v(s))]\, d\mathbb{W}(s)\bigg\|^2 \\
&\quad\le \frac{2e^{-t}\left(\sum\limits_{k=1}^{\infty}|c_k|\right)^2 \vartheta}{\left(1 - e^{-\lambda_1 t_1}\sum\limits_{k=1}^{\infty}|c_k|\right)^2} \int_0^{t_k} e^{-2\lambda_1(t_k - s)}\mathbb{E}\|u(s) - v(s)\|^2 ds \\
&\qquad + 2\vartheta e^{-t}\int_0^t e^{-2\lambda_1(t-s)}\mathbb{E}\|u(s) - v(s)\|^2 ds \\
&\quad\le \Lambda\vartheta \sup_{t\in J} e^{-t}\mathbb{E}\|u(t) - v(t)\|^2,
\end{aligned} \tag{3.27}$$

where Λ is defined by (3.4). By (3.27) and condition (P3), we have

$$\|(\mathbb{F}u) - (\mathbb{F}v)\|_e \le \sqrt{\Lambda\vartheta}\|u - v\|_e < \|u - v\|_e. \tag{3.28}$$

Hence, $\mathbb{F} : C_e(J, L^2(\Omega, \mathbb{H})) \to C_e(J, L^2(\Omega, \mathbb{H}))$ is a contraction operator, and therefore, $\mathbb{F}$ has a unique fixed point $\widehat{u}$ in $C_e(J, L^2(\Omega, \mathbb{H}))$, which is, in turn, the unique mild solution of nonlocal problem (1.3)–(1.4) on $[0, +\infty)$.

Step 2. *Asymptotic stability.* By using a totally similar method as that used in Theorem 3.2, condition (P3) and the Banach contraction theorem, we know that, for any continuous function $f : J \times \mathbb{H} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ satisfying $\sup_{t\in J} \mathbb{E}\|f(t, \theta)\|^2 < +\infty$ and $u_0 \in \mathbb{H}$, the initial value problem of the stochastic evolution equation

$$\begin{cases} du(t) + Au(t)dt = f(t, u(t))\, d\mathbb{W}(t) & t \in J, \\ u(0) = u_0, \end{cases} \tag{3.29}$$

has a unique global mild solution $u \in C_e(J, L^2(\Omega, \mathbb{H}))$, and it satisfies

$$u(t) = T(t)u(0) + \int_0^t T(t - s)f(s, u(s))\, d\mathbb{W}(s), \quad t \in J. \tag{3.30}$$

By Step 1 and the semigroup representation of the solutions, the unique global mild solution $\widehat{u}$ of nonlocal problem (1.3)–(1.4) satisfies the integral equation (3.30). From (2.4), (3.30), Lemma 2.3 and condition (P3), we obtain that

$$\begin{aligned} \mathbb{E}\|\widehat{u}(t) - u(t)\|^2 &\le 2\mathbb{E}\|T(t)[\widehat{u}(0) - u(0)]\|^2 \\ &\quad + 2\mathbb{E}\left\| \int_0^t T(t - s)[f(s, \widehat{u}(s)) - f(s, u(s)]\, d\mathbb{W}(s) \right\|^2 \\ &\le 2e^{-2\lambda_1 t}\mathbb{E}\|\widehat{u}(0) - u(0)\|^2 \\ &\quad + 2\vartheta \int_0^t e^{-2\lambda_1(t-s)}\mathbb{E}\|\widehat{u}(s) - u(s)\|^2 ds \\ &= 2e^{-2\lambda_1 t}\mathbb{E}\|\widehat{u}(0) - u(0)\|^2 \\ &\quad + 2e^{-2\lambda_1 t}\vartheta \int_0^t e^{2\lambda_1 s}\mathbb{E}\|\widehat{u}(s) - u(s)\|^2 ds, \quad t \in J. \end{aligned} \tag{3.31}$$

Let $m(t) = e^{2\lambda_1 t}\mathbb{E}\|\widehat{u}(t) - u(t)\|^2$, $t \in J$. From (3.31), it follows that

$$m(t) \leq 2m(0) + 2\vartheta \int_0^t m(s)\,ds, \quad t \in J. \tag{3.32}$$

Hence, by Lemma 2.4 and (3.32), we know that

$$m(t) = e^{2\lambda_1 t}\mathbb{E}\|\widehat{u}(t) - u(t)\|^2 \leq 2m(0)e^{2\vartheta t}, \quad t \in J. \tag{3.33}$$

Set $\rho := 2(\lambda_1 - \vartheta)$. From condition (P3), we know that $\rho > 0$. Therefore, by (3.33), we have

$$\mathbb{E}\|\widehat{u}(t) - u(t)\|^2 \leq 2m(0)e^{-\rho t} \longrightarrow 0 \quad \text{as } t \to +\infty. \tag{3.34}$$

Hence, the global mild solution $\widehat{u}$ of nonlocal problem (1.3)–(1.4) is globally asymptotically stable. Furthermore, from the proof process, we can easily see that the global mild solution $\widehat{u}$ exponentially attracts every solution of the initial value problem (3.29). □

4. Application. In order to illustrate our main results, we consider the following one-dimensional semilinear parabolic stochastic evolution equation with nonlocal condition

$$\begin{cases} du(x,t) - \partial^2/\partial x^2 u(x,t)\,dt - \nu u(x,t)dt \\ \qquad\qquad = f(x,t,u(x,t))\,d\mathbb{W}(t), \quad x \in [0,1],\ t \in J, \\ u(0,t) = u(1,t) = 0, \quad t \in J, \\ u(x,0) = \sum_{k=1}^{\infty} 4/\pi \arctan 1/(2k^2)u(x,k), \quad x \in [0,1], \end{cases} \tag{4.1}$$

where $\nu < \pi^2$ is a constant, $\mathbb{W}(t)$ denotes a one-dimensional standard cylindrical Wiener process defined on a stochastic space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $J = [0, +\infty)$.

Let $\mathbb{H} = L^2(0,1)$ with the norm $\|\cdot\|_2$. Then $\mathbb{H}$ is a Hilbert space. Define an operator A in $\mathbb{H}$ by

$$Au = -\frac{\partial^2}{\partial x^2}u - \nu u; \tag{4.2}$$

its domain $D(A)$ is defined by

$$D(A) = H^2(0,1) \cap H_0^1(0,1). \tag{4.3}$$

From **[15, 20]**, we know that A is a positive definite self-adjoint operator on $\mathbb{H}$ and $-A$ is the infinitesimal generator of an analytic, compact semigroup $T(t)$, $t \geq 0$. Moreover, A has a discrete spectrum with eigenvalues $\lambda_n = n^2\pi^2 - \nu$, $n \in \mathbb{N}$ and associated normalized eigenvectors $v_n(x) = \sqrt{2}\sin(n\pi x)$. The set $\{v_n : n \in \mathbb{N}\}$ is an orthonormal basis of $\mathbb{H}$ and

$$T(t)u = \sum_{n=1}^{\infty} e^{(\nu - n^2\pi^2)t}(u, v_n)v_n. \tag{4.4}$$

Therefore, for every $t \geq 0$, we have

$$\|T(t)\| \leq e^{(\nu - \pi^2)t}. \tag{4.5}$$

Let $u(t) = u(\cdot, t)$, $f(t, u(t)) = f(\cdot, t, u(\cdot, t))$, $c_k = 4/\pi \arctan 1/(2k^2)$, $t_k = k$, $k = 1, 2, \ldots$. Then the nonlocal problem of semilinear parabolic stochastic evolution equation (4.1) can be rewritten into the abstract form of nonlocal problem (1.3)–(1.4).

Conclusion 4.1. If $\nu \leq 5$ and $f(x, t, u(x, t)) = e^{-|u(x,t)|}/(1 + |u(x, t)|)$, then the nonlocal problem of semilinear parabolic stochastic evolution equation (4.1) has at least one global mild solution $u \in C_e([0, 1] \times [0, +\infty))$.

Proof. By the fact that

$$\sum_{k=1}^{\infty} \arctan \frac{1}{2k^2} = \pi/4,$$

we know that

$$\sum_{k=1}^{\infty} |c_k| = 1 < e^{\pi^2 - \nu},$$

and therefore, condition (P1) holds. From the definition of nonlinear term f, we can easily verify that condition (P2) is satisfied with $\alpha = \beta = 2$. Therefore, our conclusion follows from Theorem 3.1 □

Conclusion 4.2. If $\nu \leq 7$ and $f(x, t, u(x, t)) = \cos(\pi t)/\sqrt{2}(1 + |u(x, t)|)$. Then the nonlocal problem of semilinear parabolic stochastic evolution equation (4.1) has a unique global mild solution $\widehat{u} \in C_e([0, 1] \times [0, +\infty))$, and it is globally asymptotically stable.

Proof. By the fact that

$$\sum_{k=1}^{\infty} \arctan \frac{1}{2k^2} = \pi/4,$$

we know that

$$\sum_{k=1}^{\infty} |c_k| = 1 < e^{\pi^2 - \nu},$$

and therefore, condition (P1) holds. From the definition of nonlinear term f, we can easily verify that condition (P3) is satisfied with $\vartheta = 1$. Therefore, our conclusion follows from Theorem 3.2. $\square$

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