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WELL-POSEDNESS OF FRACTIONAL DEGENERATE DIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN VECTOR-VALUED FUNCTIONAL SPACES

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ABSTRACT. We study the well-posedness of degenerate fractional differential equations with infinite delay (P_α) : $D^\alpha(Mu)(t) = Au(t)+\int_{-\infty}^{t} a(t-s)Au(s)\,ds+f(t)$, $0 \le t \le 2\pi$, in Lebesgue-Bochner spaces $L^p(\mathbb{T};X)$ and Besov spaces $B^s_{p,q}(\mathbb{T};X)$, where A and M are closed linear operators on a Banach space X satisfying $D(A) \subset D(M)$, $\alpha > 0$ and $a \in L^1(\mathbb{R}_+)$ are fixed. Using well known operator-valued Fourier multiplier theorems, we completely characterize the well-posedness of (P_α) in the above vector-valued function spaces on $\mathbb{T}$.

1. Introduction. In a series of publications, operator-valued Fourier multipliers on vector-valued function spaces were studied, see e.g., **[2, 3, 14]**. They are needed to study the existence and uniqueness of differential equations on Banach spaces **[7, 8, 9, 10, 12, 13, 14]**. Recently, problems of the characterization of well-posedness for degenerate differential equations with periodic initial conditions have been extensively studied. For instance, first order degenerate differential equations:

$$(Mu)'(t) = Au(t) + f(t), \quad 0 \le t \le 2\pi, \tag{1.1}$$

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with periodic boundary condition $(Mu)(0) = (Mu)(2\pi)$, have recently been studied by Lizama and Ponce **[10]**, where A and M are closed linear operators on a Banach space X. Under suitable assumptions on the modified resolvent operator determined by (1.1), they gave necessary and sufficient conditions to ensure the well-posedness of (1.1) in Lebesgue-Bochner spaces $L^p(\mathbb{T}; X)$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$ and periodic Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$. In **[4]**, Bu studied the second order degenerate differential equations:

$$(1.2) \qquad (Mu')'(t) = Au(t) + f(t), \quad 0 \le t \le 2\pi,$$

with periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$, where A and M are closed linear operators on a Banach space X. He also obtained necessary or sufficient conditions for the well-posedness of (1.2) in Lebesgue-Bochner spaces $L^p(\mathbb{T}; X)$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$ and periodic Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$ under some suitable conditions on the modified resolvent operator determined by (1.2).

Poblete and Pozo studied fractional order neutral differential equations with finite delay:

$$(1.3) \quad D^\alpha(u(t) - Bu(t-r)) = Au(t) + Fu_t + GD^\beta u_t + f(t), \quad 0 \le t \le 2\pi,$$

where $r > 0$ is fixed, A and B are closed linear operators on a Banach space X satisfying $D(A) \subset D(B)$, $u_t(\theta) = u(t+\theta)$, and F and G are bounded linear operators on an appropriate space. Under suitable assumptions on delay operators F and G, the authors were able to give a sufficient condition for (1.3) to be well-posed in Besov spaces $B^s_{p,q}(\mathbb{T}; X)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$ **[13]**.

On the other hand, Bu considered the well-posedness in different function spaces of the following equations with fractional derivative with infinite delay:

$$(1.4) \quad D^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t), \quad 0 \le t \le 2\pi,$$

with symmetric boundary conditions, where A is a closed linear operator on a Banach space X, $\alpha > 0$ and $D^\alpha u$ is the fractional derivative of u in the sense of Weyl, $a \in L^1(\mathbb{R}_+)$. Under suitable assumptions on the Laplace transform of a, the author completely characterized the well-posedness of (1.4) on Lebesgue-Bochner spaces $L^p(\mathbb{T}; X)$ and Besov spaces $B^s_{p,q}(\mathbb{T}; X)$ [**5**].

In this paper, we study the following degenerate fractional differential equations with infinite delay:

$$(P_\alpha) \quad D^\alpha(Mu)(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t),\ 0 \le t \le 2\pi,$$

where A and M are closed linear operators on a Banach space X satisfying $D(A) \subset D(M)$, $a \in L^1(\mathbb{R}_+)$, $\alpha > 0$. It is clear that (1.4) is a special case of (P_α) when $M = I_X$. When Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, m is a non-negative bounded measurable function defined on Ω and X is the Hilbert space $H^{-1}(\Omega)$, we can consider M as the multiplication operator on X by m. One may also consider M as a differential operator on $H^{-1}(\Omega)$ or $L^2(\Omega)$ with different boundary conditions on $\partial\Omega$.

The purpose of this paper is to characterize the well-posedness of (P_α) in Lebesgue-Bochner spaces $L^p(\mathbb{T}; X)$ and Besov spaces $B^s_{p,q}(\mathbb{T}; X)$. Our characterizations of the well-posedness of (P_α) involve the Rademacher boundedness (or norm boundedness) of the M-resolvent set of A. Our main tools in the study of the well-posedness of (P_α) are the operator-valued Fourier multiplier theorems obtained by Arendt and Bu [**2, 3**] on $L^p(\mathbb{T}; X)$ and $B^s_{p,q}(\mathbb{T}; X)$. Indeed, we will transform the well-posedness of (P_α) to an operator-valued Fourier multiplier problem in the corresponding vector-valued function space. In this paper, we are able to characterize the well-posedness of (P_α) by the boundedness of the M-resolvent set of A. For instance, we show that, under suitable assumptions on a, when the underlying Banach space X is a UMD Banach space and $1 < p < \infty$, then (P_α) is L^p well-posed if and only if

$$\left\{ \frac{r_k^{(\alpha)}}{1 + c_k} : k \in \mathbb{Z} \right\} \subset \rho_M(A)$$

and the set

$$\left\{ r_k^{(\alpha)} M [r_k^{(\alpha)} M - (1 + c_k)A]^{-1} : k \in \mathbb{Z} \right\}$$

is R-bounded, where $\rho_M(A)$ is the M-resolvent set of A (see the precise definition in Section 2), $r_k^{(\alpha)} = |k|^\alpha e^{(1/2)\,\mathrm{sgn}(k)\pi i\alpha}$ when $k \neq 0$, and $r_0^{(\alpha)} = 0$, $c_k = \int_0^{+\infty} e^{-ikt} a(t)\,dt$ is the Fourier transform of a.

The results obtained in this paper recover the known results presented in **[5]** in the simpler case when $M = I_X$. Our results also recover the results obtained in **[10]** in the special case when $\alpha = 1$, $a = 0$. Thus, one may also consider our results as generalizations of the previous results obtained in **[2, 3]**.

This work is organized as follows. In Section 2, we study the well-posedness of (P_α) in vector-valued Lebesgue spaces $L^p(\mathbb{T}; X)$. In Section 3, we consider the well-posedness of (P_α) in Besov spaces $B^s_{p,q}(\mathbb{T}; X)$. In the last section, we give some examples that our abstract results may be applied.

2. Well-posedness of (P_α) in Lebesgue-Bochner spaces. Let X and Y be complex Banach spaces, and let $\mathbb{T} := [0, 2\pi]$. We denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y. If $X = Y$, we will simply denote it by $\mathcal{L}(X)$. For $1 \leq p < \infty$, we denote by $L^p(\mathbb{T}; X)$ the space of all equivalent classes of X-valued measurable functions f defined on $\mathbb{T}$ satisfying

$$\|f\|_{L^p} := \left(\int_0^{2\pi} \|f(t)\|^p \frac{dt}{2\pi} \right)^{1/p} < \infty.$$

For $f \in L^1(\mathbb{T}; X)$, we denote by

$$\widehat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} e_{-k}(t) f(t)\,dt$$

the kth Fourier coefficient of f, where $k \in \mathbb{Z}$ and $e_k(t) = e^{ikt}$ when $t \in \mathbb{T}$. We denote by $e_k \otimes x$ the X-valued function defined on $\mathbb{T}$ by $(e_k \otimes x)(t) = e_k(t)x$.

The main tool in our study of L^p well-posedness of (P_α) is the next L^p-Fourier multiplier theorem **[2]**.

Definition 2.1. Letting X and Y be complex Banach spaces and $1 \leq p < \infty$, we say that $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an L^p-Fourier multiplier if, for each $f \in L^p(\mathbb{T}; X)$, there exists a $u \in L^p(\mathbb{T}; Y)$ such that $\widehat{u}(k) = M_k \widehat{f}(k)$ for all $k \in \mathbb{Z}$.

It easily follows from the Closed Graph theorem that, when $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ is an L^p-Fourier multiplier, then there exists a bounded linear operator $T \in \mathcal{L}(L^p(\mathbb{T};X), L^p(\mathbb{T};Y))$ satisfying $(Tf)^\wedge(k) = M_k\widehat{f}(k)$ when $f \in L^p(\mathbb{T};X)$ and $k \in \mathbb{Z}$. The operator-valued Fourier multiplier theorem on $L^p(\mathbb{T};X)$ obtained in **[2]** involves the Rademacher boundedness for sets of bounded linear operators. Let γ_j be the jth Rademacher function on $[0,1]$ given by $\gamma_j(t) = \operatorname{sgn}(\sin(2^j\pi t))$ when $j \geq 1$. For $x \in X$, we denote by $\gamma_j \otimes x$ the vector-valued function $t \to r_j(t)x$ on $[0,1]$.

Definition 2.2. Let X and Y be Banach spaces. A set $\mathbf{T} \subset \mathcal{L}(X,Y)$ is said to be Rademacher bounded (R-bounded, in short), if a $C > 0$ exists such that

$$\Big\|\sum_{j=1}^{n} \gamma_j \otimes T_j x_j\Big\|_{L^1([0,1];Y)} \leq C\Big\|\sum_{j=1}^{n} \gamma_j \otimes x_j\Big\|_{L^1([0,1];X)}$$

for all $T_1,\ldots,T_n \in \mathbf{T}$, $x_1,\ldots,x_n \in X$ and $n \in \mathbb{N}$.

Remark 2.3.

(i) Let $\mathbf{S},\mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then it can be easily seen from the definition that

$$\mathbf{ST} := \{ST : S \in \mathbf{S}, T \in \mathbf{T}\}$$

and

$$\mathbf{S}+\mathbf{T} := \{S+T : S \in \mathbf{S}, T \in \mathbf{T}\}$$

are still R-bounded.

(ii) Let X be a UMD Banach space, and let $M_k = m_k I_X$ with $m_k \in \mathbb{C}$, where I_X is the identity operator on X, if $\sup_{k\in\mathbb{Z}} |m_k| < \infty$ and $\sup_{k\in\mathbb{Z}} |k(m_{k+1}-m_k)| < \infty$. Then $(M_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier whenever $1 < p < \infty$ **[2]**.

The next results will be fundamental in the proof of our main result of this section. For the notion of UMD Banach spaces, we refer the reader to **[2]** and the references therein.

Proposition 2.4 ([**2,** Proposition 1.11]). *Let X and Y be Banach spaces, $1 \leq p < \infty$, and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ be an L^p-Fourier multiplier. Then the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

Theorem 2.5 ([**2,** Theorem 1.3]). *Let X and Y be* UMD *Banach spaces and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. If the sets $\{M_k : k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier whenever $1 < p < \infty$.*

The derivative operator (of order 1), denoted by D in $L^p(\mathbb{T};X)$, was defined in [**2**] as

$$Du := \sum_{k\in\mathbb{Z}} ike_k \otimes \widehat{u}(k)$$

with domain $W^{1,p}(\mathbb{T};X)$, where

$$W^{1,p}(\mathbb{T};X) := \big\{u \in L^p(\mathbb{T};X) : \text{ there exists } v \in L^p(\mathbb{T};X) \text{ such that } \widehat{v}(k) = ik\widehat{u}(k) \text{ for } k \in \mathbb{Z}\big\} \tag{2.1}$$

is the first periodic Sobolev space. Let $u \in L^p(\mathbb{T};X)$. Then $u \in W^{1,p}(\mathbb{T};X)$ if and only if u is differentiable almost everywhere on $\mathbb{T}$ and $u' \in L^p(\mathbb{T};X)$. In this case, u is actually continuous and $u(0) = u(2\pi)$ [**2,** Lemma 2.1].

The unbounded operator D is non negative in $L^p(\mathbb{T};X)$ [**11**]; thus, its fractional power makes sense. Let $\alpha > 0$. The fractional power D^α of D is given by

$$D^\alpha u := \sum_{k\in\mathbb{Z}} r_k^{(\alpha)} e_k \otimes \widehat{u}(k)$$

with domain $W^{\alpha,p}(\mathbb{T};X)$, where $W^{\alpha,p}(\mathbb{T};X)$ is the fractional Sobolev space of order α defined by

$$W^{\alpha,p}(\mathbb{T};X) := \big\{u \in L^p(\mathbb{T};X) : \text{ there exists } v \in L^p(\mathbb{T};X) \text{ such that } \widehat{v}(k) = r_k^{(\alpha)}\widehat{u}(k) \text{ for } k \in \mathbb{Z}\big\}. \tag{2.2}$$

Here,

$$r_k^{(\alpha)} := \begin{cases} |k|^\alpha e^{(1/2)\operatorname{sgn}(k)\pi i\alpha} & k \neq 0, \\ 0 & k = 0. \end{cases} \tag{2.3}$$

This notation $r_k^{(\alpha)}$ will be fixed throughout this paper. D^α is called the *fractional derivative* (in the sense of Weyl) of u of order α [**11**]. It is clear that definition (2.2) coincides with (2.1) when $\alpha = 1$ and $D = D^1$. See [**9**] for an equivalent definition of the fractional derivative D^α on $L^p(\mathbb{T}; X)$. $W^{\alpha,p}(\mathbb{T}; X)$ is a Banach space with the norm

$$\|u\|_{W^{\alpha,p}} := \|u\|_{L^p} + \|D^\alpha u\|_{L^p}.$$

For $\beta > 0$, we let $a_k = 1/r_k^{(\beta)}$ for $k \neq 0$ and $a_0 = 0$, and

$$F_\beta := \sum_{k\in\mathbb{Z}} e_k \otimes a_k.$$

Then $F_\beta \in L^1(\mathbb{T})$ [**15,** Chapter V, (1.5), (1.14)]. This implies that, when $\alpha_1 \leq \alpha_2$, then $W^{\alpha_2,p}(\mathbb{T}; X) \subset W^{\alpha_1,p}(\mathbb{T}; X)$ by Young's inequality. It is clear from the definition and [**2,** Lemma 2.1] that, when $\alpha > 1$, then $u \in W^{\alpha,p}(\mathbb{T}; X)$ if and only if u is differentiable almost everywhere and $u' \in W^{\alpha-1,p}(\mathbb{T}; X)$.

It was shown in [**15,** Chapter XII, (9.1)] that, when $1/p < \alpha < 1 + 1/p$, then $W^{\alpha,p}(\mathbb{T}; X) \subset C_{\text{per}}^{\alpha-1/p}(\mathbb{T}; X)$, where $C_{\text{per}}^{\alpha-1/p}(\mathbb{T}; X)$ is the space of all X valued $(\alpha - 1/p)$-Hölder continuous functions u on $\mathbb{T}$ satisfying $u(0) = u(2\pi)$. This implies that, if $\alpha > 0$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ are such that

$$n + \frac{1}{p} < \alpha < n + 1 + \frac{1}{p},$$

and, if $u \in W^{\alpha,p}(\mathbb{T}; X)$, then u is n-times continuously differentiable on $\mathbb{T}$, and $u^{(k)}(0) = u^{(k)}(2\pi)$ when $0 \leq k \leq n$. This means that (P_α) is in fact a problem with symmetric boundary conditions when $1/p < \alpha$.

A scalar sequence $(b)_{k\in\mathbb{Z}} \subset \mathbb{C}\backslash\{0\}$ is called 1-regular if the sequence $(k(b_{k+1} - b_k)/b_k)_{k\in\mathbb{Z}}$ is bounded; it is called 2-regular if it is 1-regular and the sequence $(k^2(b_{k+2} - 2b_{k+1} + b_k)/b_k)_{k\in\mathbb{Z}}$ is bounded.

Remark 2.6. An easy computation shows that $(r_k^{(\alpha)})_{k\in\mathbb{Z}}$ is 2-regular whenever $\alpha > 0$.

For $a \in L^1(\mathbb{R}_+)$ and $u \in L^p(\mathbb{T}; D(A))$, we define

$$(a * Au)(t) := \int_{-\infty}^{t} a(t-s)Au(s)\,ds, \quad t \in \mathbb{T}. \tag{2.4}$$

Here we consider $D(A)$ as a Banach space equipped with its graph norm. It is clear that $a * Au \in L^p(\mathbb{T};X)$ by Young's inequality and $\|a * Au\|_{L^p} \leq \|a\|_{L^1}\|Au\|_{L^p}$. Let $\widetilde{a}(\lambda) := \int_0^{+\infty} e^{-\lambda t}a(t)\,dt$ be the Laplace transform of a for $\mathrm{Re}\lambda \geq 0$. An easy computation shows that:

$$\widehat{a * Au}(k) = \widetilde{a}(ik)A\widehat{u}(k) \tag{2.5}$$

when $k \in \mathbb{Z}$. We note that $\widetilde{a}(ik)$ exists for all $k \in \mathbb{Z}$ as $a \in L^1(\mathbb{R}_+)$. In what follows, we always use the notation:

$$c_k := \widetilde{a}(ik), \tag{2.6}$$

for all $k \in \mathbb{Z}$.

Remark 2.7. Under the above assumptions on a, if $\widetilde{a}(ik) \neq -1$ for all $k \in \mathbb{Z}$, then the sequences $(\widetilde{a}(ik))_{k\in\mathbb{Z}}$ and $(1/(1+\widetilde{a}(ik)))_{k\in\mathbb{Z}}$ are bounded by the Riemann-Lebesgue lemma.

Let $(b_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ be a scalar sequence. We will use the following hypotheses:

(**A**1) $b_k \neq -1$ for all $k \in \mathbb{Z}$, $(k(b_{k+1} - b_k))_{k\in\mathbb{Z}}$ is bounded.

(**A**2) $b_k \neq -1$ for all $k \in \mathbb{Z}$, $(k(b_{k+1} - b_k))_{k\in\mathbb{Z}}$ and $(k^2(b_{k+2} - 2b_{k+1} + b_k)_{k\in\mathbb{Z}}$ are bounded.

Note that the sequences $(b_{k+1} - b_k)_{k\in\mathbb{Z}}$, $(b_{k+2} - 2b_{k+1} + b_k))_{k\in\mathbb{Z}}$ and $(b_{k+3} - 3b_{k+2} + 3b_{k+1} - b_k))_{k\in\mathbb{Z}}$ may be considered as the first derivative, the second derivative and the third derivative of $(b_k)_{k\in\mathbb{Z}}$, respectively.

Let $1 \leq p < \infty$, $a \in L^1(\mathbb{R}_+)$. We define the solution space of (P_α) in the L^p well-posedness case by

$$S_p(A, M) := \{u \in L^p(\mathbb{T}; D(A)) : Mu \in W^{\alpha,p}(\mathbb{T}; X)\}.$$

Here, again, we consider $D(A)$ to be a Banach space equipped with its graph norm. If $u \in S_p(A, M)$, then $a * Au \in L^p(\mathbb{T};X)$ by Young's inequality. $S_p(A, M)$ is a Banach space with the norm

$$\|u\|_{S_p(A,M)} := \|u\|_{L^p} + \|Au\|_{L^p} + \|Mu\|_{W^{\alpha,p}}.$$

Now we are ready to introduce the well-posedness of (P_α).

Definition 2.8. Let $1 \le p < \infty$ and $f \in L^p(\mathbb{T}; X)$; $u \in S_p(A, M)$ is called a *strong* L^p*-solution* of (P_α), if (P_α) is satisfied almost everywhere on $\mathbb{T}$. We say that (P_α) is L^p well-posed if, for each $f \in L^p(\mathbb{T}; X)$, there exists a unique strong L^p-solution of (P_α).

If (P_α) is L^p well-posed, there exists a constant $C > 0$ such that, for each $f \in L^p(\mathbb{T}; X)$, if $u \in S_p(A, M)$ is the unique strong L^p-solution of (P_α), then

$$\|u\|_{S_p(A,M)} \le C\|f\|_{L^p}. \tag{2.7}$$

This is an easy consequence of the Closed Graph theorem.

Now we introduce the M-resolvent set of A. We recall that, under the assumption that $D(A) \subset D(M)$, for any $\lambda \in \mathbb{C}$, the sum operator $\lambda M - A$ is a linear operator $D(A)$ into X. We define

$$\rho_M(A) := \{\lambda \in \mathbb{C} : \lambda M - A : D(A) \to X \text{ is bijective and } (\lambda M - A)^{-1} \in \mathcal{L}(X)\}$$

as the M-resolvent set of A. If $\lambda \in \rho_M(A)$, then the operator $M(\lambda M - A)^{-1}$ is well defined by the assumption $D(A) \subset D(M)$, and $M(\lambda M - A)^{-1} \in \mathcal{L}(X)$ by the closedness of M and the boundedness of $(\lambda M - A)^{-1}$.

In the proof of our main result of this section, we will use the next result.

Proposition 2.9. *Let A and M be closed linear operators defined on a* UMD *Banach space X such that $D(A) \subset D(M)$, $a \in L^1(\mathbb{R}_+)$. Assume that $(c_k)_{k\in\mathbb{Z}}$ defined by* (2.6) *satisfies* (**A**1)*. We assume that $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ is* 1*-regular and satisfies*

$$\left\{\frac{a_k}{1 + c_k} : k \in \mathbb{Z}\right\} \subset \rho_M(A).$$

Then the following assertions are equivalent.

(i) $(a_k M[a_k M - (1 + c_k)A]^{-1})_{k\in\mathbb{Z}}$ *is an L^p-Fourier multiplier for* $1 < p < \infty$;
(ii) *the set* $\{a_k M[a_k M - (1 + c_k)A]^{-1} : k \in \mathbb{Z}\}$ *is R-bounded.*

Proof. Let $N_k = [a_k M - (1+c_k)A]^{-1}$ and $M_k = a_k M N_k$. The implication (i) $\Rightarrow$ (ii) is clearly true by Proposition 2.4. Now assume that the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded. To show that (i) is true, it will suffice to show that the set $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ is R-bounded by Theorem 2.5. We have

(2.8)

$$\begin{aligned} N_{k+1} - N_k &= N_{k+1}[N_k^{-1} - N_{k+1}^{-1}]N_k \\ &= N_{k+1}[a_k M - (1+c_k)A - a_{k+1}M + (1+c_{k+1})A]N_k \\ &= N_{k+1}(a_k - a_{k+1})MN_k + N_{k+1}(c_{k+1} - c_k)AN_k \\ &= N_{k+1}\frac{a_k - a_{k+1}}{a_k}M_k + N_{k+1}(c_{k+1} - c_k)AN_k, \end{aligned}$$

when $k \neq 0$. It follows that

$$\begin{aligned} &k(M_{k+1} - M_k) \\ &\quad = k[a_{k+1}MN_{k+1} - a_k MN_k] \\ &\quad = k[a_{k+1}M(N_{k+1} - N_k) + (a_{k+1} - a_k)MN_k] \\ &\quad = ka_{k+1}MN_{k+1}\frac{a_k - a_{k+1}}{a_k}M_k \\ &\qquad + ka_{k+1}MN_{k+1}(c_{k+1} - c_k)AN_k + k(a_{k+1} - a_k)MN_k \\ &\quad = M_{k+1}\frac{k(a_k - a_{k+1})}{a_k}M_k \\ &\qquad + M_{k+1}k(c_{k+1} - c_k)\frac{1}{1+c_k}[M_k - I_X] + \frac{k(a_{k+1} - a_k)}{a_k}M_k, \end{aligned}$$

when $k \neq 0$. Hence, the set $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ is R-bounded as $(a_k)_{k\in\mathbb{Z}}$ is 1-regular and $(c_k)_{k\in\mathbb{Z}}$ satisfies (**A**1). This completes the proof. □

The next result gives a necessary and sufficient condition for (P_α) to be L^p-well-posed.

Theorem 2.10. *Let X be a* UMD *Banach space, $1 < p < \infty$, $\alpha > 0$, and let A, M be closed linear operators on X satisfying $D(A) \subset D(M)$, $a \in L^1(\mathbb{R}_+)$. We assume that $(c_k)_{k\in\mathbb{Z}}$ defined by* (2.6) *satisfies* (**A**1). *Then the following assertions are equivalent*:

(i) (P_α) *is L^p-well-posed*;

(ii) $\left\{r_k^{(\alpha)}1 + c_k : k \in \mathbb{Z}\right\} \subset \rho_M(A)$ *and the set*

$$\left\{r_k^{(\alpha)} M[r_k^{(\alpha)} M - (1 + c_k)A]^{-1} : k \in \mathbb{Z}\right\}$$

is R-bounded, where $r_k^{(\alpha)}$ *is defined by* (2.3).

Proof.

(ii) $\Rightarrow$ (i). We assume that

$$\left\{\frac{r_k^{(\alpha)}}{1 + c_k} : k \in \mathbb{Z}\right\} \subset \rho_M(A)$$

and the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded, where $N_k = [r_k^{(\alpha)} M - (1 + c_k)A]^{-1}$ and $M_k = r_k^{(\alpha)} M N_k$. It follows from Proposition 2.9 that $(M_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier as the sequence $(r_k^{(\alpha)})_{k\in\mathbb{Z}}$ is clearly 1-regular. Then, for all $f \in L^p(\mathbb{T}; X)$, there exists $u \in L^p(\mathbb{T}; X)$ satisfying

$$\widehat{u}(k) = M_k \widehat{f}(k) \tag{2.9}$$

for all $k \in \mathbb{Z}$. We note that

$$AN_k = \frac{1}{1 + c_k}[M_k - I_X]. \tag{2.10}$$

$(I_X/1 + c_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier by Theorem 2.5 as we have assumed that $(c_k)_{k\in\mathbb{Z}}$ satisfies (**A**1). We deduce that $(AN_k)_{k\in\mathbb{Z}}$ is an L^p Fourier multiplier as the product of L^p Fourier multipliers is still an L^p Fourier multiplier. Thus, $v \in L^p(\mathbb{T}; X)$ exists and satisfies $\widehat{v}(k) = AN_k \widehat{f}(k)$ for all $k \in \mathbb{Z}$. We note that A^{-1} is an isomorphism from X onto $D(A)$ as $0 \in \rho_M(A)$ by assumption; here, we consider $D(A)$ as a Banach space equipped with its graph norm. Hence, $A^{-1}\widehat{v}(k) = N_k \widehat{f}(k)$. Setting $w = A^{-1}v$, then $w \in L^p(\mathbb{T}; D(A))$ and

$$\widehat{w}(k) = N_k \widehat{f}(k) \tag{2.11}$$

for all $k \in \mathbb{Z}$. This implies, in particular, that $(N_k)_{k\in\mathbb{Z}}$ is an L^p Fourier multiplier. It is clear that the sequence $(I_X/r_k^{(\alpha)})_{k\in\mathbb{Z}}$ satisfies the first order Marcinkiewicz condition in Theorem 2.1; thus, it is an L^p Fourier multiplier. We deduce that $(MN_k)_{k\in\mathbb{Z}}$ is an L^p Fourier multiplier. This implies that $w \in L^p(\mathbb{T}; D(M))$. Here, $D(M)$ is equipped with its graph norm so that it becomes a Banach space.

Following from (2.9) and (2.11), we obtain that

$$\widehat{u}(k) = r_k^{(\alpha)} M \widehat{w}(k) = r_k^{(\alpha)} (Mw)^\wedge(k),$$

which implies that $Mw \in W^{\alpha,p}(\mathbb{T}; X)$. We have shown that $w \in S_p(A, M)$. By (2.11), we have

$$r_k^{(\alpha)} (Mw)^\wedge(k) = A\widehat{w}(k) + c_k A\widehat{w}(k) + \widehat{f}(k)$$

for all $k \in \mathbb{Z}$. Thus, $D^\alpha(Mw)(t) = Aw(t) + (a * Aw)(t) + f(t)$ for $t \in \mathbb{T}$ by the uniqueness theorem [**2,** page 314]. This shows the existence.

To show the uniqueness, we let $u \in S_p(A, M)$ be another solution of $D^\alpha(Mw)(t) = Aw(t) + (a * Aw)(t) + f(t)$. Then $D^\alpha M(u - w)(t) = A(u - w)(t) + (a * A(u - w))(t)$ almost everywhere on $\mathbb{T}$. Taking the Fourier transform on both sides, we obtain $r_k^{(\alpha)} M(\widehat{u}(k) - \widehat{v}(k)) = (1 + c_k)A(\widehat{u}(k) - \widehat{w}(k))$ when $k \in \mathbb{Z}$. This implies that $[r_k^{(\alpha)} M - (1 + c_k)A](\widehat{u}(k) - \widehat{w}(k)) = 0$ when $k \in \mathbb{Z}$. Thus, $\widehat{u}(k) - \widehat{w}(k) = 0$ as $r_k^{(\alpha)} M - (1 + c_k)A$ is invertible, and so $u = w$ by the uniqueness theorem [**2,** page 314]. We have shown that the implication (ii) $\Rightarrow$ (i) is true.

(i) $\Rightarrow$ (ii). Assume that (P_α) is L^p well-posed. We shall show that $r_k^{(\alpha)}/1 + c_k \in \rho_M(A)$ for all $k \in \mathbb{Z}$. Let $k \in \mathbb{Z}$ and $y \in X$ be fixed. We define $f(t) = e^{ikt} y$ $(t \in \mathbb{T})$. Then, $f \in L^p(\mathbb{T}; X)$, $\widehat{f}(k) = y$ and $\widehat{f}(n) = 0$ when $n \neq k$. There exists a unique $u \in S_p(A, M)$ such that

$$D^\alpha(Mu)(t) = Au(t) + (a * Au)(t) + f(t) \tag{2.12}$$

almost everywhere on $\mathbb{T}$. We have $\widehat{u}(n) \in D(A)$ when $n \in \mathbb{Z}$ by [**2,** Lemma 3.1] as $u \in L^p(\mathbb{T}; D(A))$. Taking Fourier transforms on both sides of (2.12), we obtain

$$(r_k^{(\alpha)} M - (1 + c_k)A)\widehat{u}(k) = y \tag{2.13}$$

and $(r_n^{(\alpha)} M - (1 + c_n)A)\widehat{u}(n) = 0$ when $n \neq k$. Thus, $r_k^{(\alpha)} M - (1 + c_k)A$ is surjective. To show that $r_k^{(\alpha)} M - (1 + c_k)A$ is also injective, we take $x \in D(A)$ such that

$$(r_k^{(\alpha)} M - (1 + c_k)A)x = 0.$$

Let $u(t) = e^{ikt}x$ when $t \in \mathbb{T}$. Then $u \in S_p(A, M)$ and (P_α) hold almost everywhere on $\mathbb{T}$ when taking $f = 0$. Thus, u is a strong L^p-solution of (P_α) when $f = 0$. We obtain $x = 0$ by the uniqueness assumption. We have shown that $r_k^{(\alpha)}M - (1 + c_k)A$ is injective. Therefore, $r_k^{(\alpha)}M - (1 + c_k)A$ is bijective from $D(A)$ onto X.

Next, we show that $[r_k^{(\alpha)}M - (1+c_k)A]^{-1} \in \mathcal{L}(X)$. For $f(t) = e^{ikt}y$, we let u be the unique strong L^p-solution of (P_α). Then

$$\widehat{u}(n) = \begin{cases} 0 & n \neq k, \\ [r_k^{(\alpha)}M - (1 + c_k)A]^{-1}y & n = k, \end{cases}$$

by (2.13). This means that $u(t) = e^{ikt}[r_k^{(\alpha)}M - (1 + c_k)A]^{-1}y$. By (2.7), there exists a constant $C > 0$ independent from $f \in L^p(\mathbb{T}; X)$ such that

$$\|u\|_{L^p} + \|Au\|_{L^p} + \|Mu\|_{W^{\alpha,p}} \leq C\|f\|_{L^p}.$$

In particular $\|u\|_{L^p} \leq C\|f\|_{L^p}$. Hence,

$$||[r_k^{(\alpha)}M - (1 + c_k)A]^{-1}y|| \leq C\|y\|,$$

which implies

$$||[r_k^{(\alpha)}M - (1 + c_k)A]^{-1}|| \leq C.$$

We have shown that $r_k^{(\alpha)}/1 + c_k \in \rho_M(A)$ for all $k \in \mathbb{Z}$. Thus,

$$\left\{ \frac{r_k^{(\alpha)}}{1 + c_k} : k \in \mathbb{Z} \right\} \subset \rho_M(A).$$

Finally, we prove that, if $M_k = r_k^{(\alpha)}M[r_k^{(\alpha)}M - (1 + c_k)A]^{-1}$ when $k \in \mathbb{Z}$, then $(M_k)_{k\in\mathbb{Z}}$ defines an L^p Fourier multiplier. Let $f \in L^p(\mathbb{T}; X)$. Then there exists $u \in S_p(A, M)$, a strong L^p-solution of (P_α) by assumption. Taking Fourier transforms on both sides of (P_α), we obtain that $\widehat{u}(k) \in D(A)$ by **[2,** Lemma 3.1] and

$$[r_k^{(\alpha)}M - (1 + c_k)A]\widehat{u}(k) = \widehat{f}(k), (k \in \mathbb{Z})$$

for all $k \in \mathbb{Z}$. Since $r_k^{(\alpha)}M - (1 + c_k)A$ is invertible, we have

$$\widehat{u}(k) = [r_k^{(\alpha)}M - (1 + c_k)A]^{-1}\widehat{f}(k)$$

for all $k \in \mathbb{Z}$. It follows from $Mu \in W^{\alpha,p}(\mathbb{T}; X)$ that $[D^\alpha(Mu)]^\wedge(k) = r_k^{(\alpha)} M\widehat{u}(k)$, which implies that

$$[D^\alpha(Mu)]^\wedge(k) = r_k^{(\alpha)} M\widehat{u}(k) = M_k \widehat{f}(k)$$

for all $k \in \mathbb{Z}$. We conclude that $(M_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier as $Mu \in W^{\alpha,p}(\mathbb{T}; X) \subset L^p(\mathbb{T}; X)$. We deduce that $(M_k)_{k\in\mathbb{Z}}$ is R-bounded by Proposition 2.4. Therefore, the implication (i) $\Rightarrow$ (ii) is also true. This finishes the proof. □

Since the second statement in Theorem 2.10 does not depend on the space parameter p, we immediately have the next corollary.

Corollary 2.11. *Let X be a* UMD *Banach space, and let A, M be closed linear operators on X satisfying $D(A) \subset D(M)$, $\alpha > 0$, $a \in L^1(\mathbb{R}_+)$. We assume that $(c_k)_{k\in\mathbb{Z}}$ defined by* (2.6) *satisfies* (**A**1). *Then, if (P_α) is L^p well-posed for some $1 < p < \infty$, then it is L^p well-posed for all $1 < p < \infty$.*

When the underlying Banach space is isomorphic to a Hilbert space, then each norm bounded subset of $\mathcal{L}(X)$ is actually R-bounded [**2,** Proposition 1.13]. This fact, together with Theorem 2.5, immediately gives the following result.

Corollary 2.12. *Let H be a Hilbert space, $1 < p < \infty$, $\alpha > 0$, and let A, M be closed linear operators on H satisfying $D(A) \subset D(M)$, $a \in L^1(\mathbb{R}_+)$. We assume that $(c_k)_{k\in\mathbb{Z}}$ defined by* (2.6) *satisfies* (**A**1). *Then the following assertions are equivalent*:

(i) *(P_α) is L^p well-posed;*
(ii) *$\{r_k^{(\alpha)}/(1 + c_k) : k \in \mathbb{Z}\} \subset \rho_M(A)$, and the set*

$$\left\{ r_k^{(\alpha)} M[r_k^{(\alpha)} M - (1 + c_k)A]^{-1} : k \in \mathbb{Z} \right\}$$

is norm bounded,

where $r_k^{(\alpha)}$ is defined by (2.3).

3. Well-posedness of (P_α) in Besov and Triebel-Lizorkin spaces. In this section, we study the well-posedness of (P_α) in Besov spaces $B^s_{p,q}(\mathbb{T};X)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T};X)$. Firstly, we briefly recall the definition of Besov spaces in the vector-valued case introduced in [**3**]. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms $\|f\|_\alpha = \sup_{x\in\mathbb{T}} |f^{(\alpha)}(x)|$ for $\alpha \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Let $\mathcal{D}'(\mathbb{T};X) := \mathcal{L}(\mathcal{D}(\mathbb{T}),X)$ be the space of all continuous linear operators from $\mathcal{D}(\mathbb{T})$ to X. In order to define Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : |t| \le 2\}, \qquad I_k = \{t \in \mathbb{R} : 2^{k-1} < |t| \le 2^{k+1}\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ satisfying $\mathrm{supp}(\phi_k) \subset \overline{I}_k$ for each $k \in \mathbb{N}_0$, $\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1$ for $x \in \mathbb{R}$, and, for each $\alpha \in \mathbb{N}_0$, $\sup_{x\in\mathbb{R},k\in\mathbb{N}_0} 2^{k\alpha}|\phi_k^{(\alpha)}(x)| < \infty$. Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \phi(\mathbb{R})$ be fixed. For $1 \le p,\, q \le \infty$, $s \in \mathbb{R}$, the X-valued Besov space is defined by

$$B^s_{p,q}(\mathbb{T};X) = \Bigg\{ f \in \mathcal{D}'(\mathbb{T};X) : \|f\|_{B^s_{p,q}} := \bigg(\sum_{j\ge 0} 2^{sjq} \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\widehat{f}(k) \Big\|_p^q \bigg)^{1/q} < \infty \Bigg\}$$

with the usual modification if $q = \infty$. The space $B^s_{p,q}(\mathbb{T};X)$ is independent from the choice of ϕ, and different choices of ϕ lead to equivalent norms on $B^s_{p,q}(\mathbb{T};X)$. $B^s_{p,q}(\mathbb{T};X)$ equipped with the norm $\|\cdot\|_{B^s_{p,q}}$ is a Banach space. It is known that, if $s_1 \le s_2$, then $B^{s_1}_{p,q}(\mathbb{T};X) \subset B^{s_2}_{p,q}(\mathbb{T};X)$, and the embedding is continuous [**3,** Theorem 2.3]. It was shown [**3,** Theorem 2.3] that, when $s > 0$, then $f \in B^{s+1}_{p,q}(\mathbb{T};X)$ if and only if f is differentiable almost everywhere on $\mathbb{T}$ and $f' \in B^s_{p,q}(\mathbb{T};X)$ (this is equivalent to saying that $Df \in B^s_{p,q}(\mathbb{T};X)$). More generally, for $\alpha > 0$ and $s > 0$, $f \in B^{\alpha+s}_{p,q}(\mathbb{T};X)$ if and only if $D^\alpha f \in B^s_{p,q}(\mathbb{T};X)$. See [**3,** Section 2] for more information about the space $B^s_{p,q}(\mathbb{T};X)$.

Next, we give the definition of operator-valued Fourier multipliers in the context of Besov spaces, which is fundamental in the proof of our main result of this section.

Definition 3.1. Let X and Y be Banach spaces, $1 \le p,\ q \le \infty$, $s \in \mathbb{R}$, and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. We say that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$ Fourier multiplier if, for each $f \in B^s_{p,q}(\mathbb{T};X)$, there exists $u \in B^s_{p,q}(\mathbb{T};Y)$, such that $\widehat{u}(k) = M_k \widehat{f}(k)$ for all $k \in \mathbb{Z}$.

The next result was obtained in [**3,** Theorem 4.5], which gives a sufficient condition for an operator-valued sequence to be a $B^s_{p,q}$ Fourier multiplier.

Theorem 3.2. *Let X, Y be Banach spaces, and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. We assume that*

$$\sup_{k\in\mathbb{Z}} \big(\|M_k\| + \|k(M_{k+1} - M_k)\|\big) < \infty, \tag{3.1}$$

$$\sup_{k\in\mathbb{Z}} \|k^2\big(M_{k+2} - 2M_{k+1} + M_k\big)\| < \infty. \tag{3.2}$$

Then $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier whenever $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. If X is B-convex, then the first order condition (3.1) *is already sufficient for $(M_k)_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-multiplier.*

Recall that a Banach space X is B-convex if it does not contain l_1^n uniformly. This is equivalent to saying that X has a Fourier type $1 < p \le 2$, i.e., the Fourier transform is a bounded linear operator from $L^p(\mathbb{R};X)$ to $L^q(\mathbb{R};X)$, where $1/p + 1/q = 1$. It is well known that, when $1 < p < \infty$, then $L^p(\mu)$ has Fourier type $\min\{p, p/(p-1)\}$ [**3**].

Remark 3.3.

(i) If $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$ Fourier multiplier, then there exists a bounded linear operator T from $B^s_{p,q}(\mathbb{T};X)$ to $B^s_{p,q}(\mathbb{T};Y)$ satisfying $\widehat{Tf}(k) = M_k \widehat{f}(k)$ when $k \in \mathbb{Z}$. This implies in particular that $(M_k)_{k\in\mathbb{Z}}$ must be bounded.

(ii) If $(M_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$ Fourier multipliers, it can be easily seen that the product sequence $(M_k N_k)_{k\in\mathbb{Z}}$ and the sum sequence $(M_k + N_k)_{k\in\mathbb{Z}}$ are still $B^s_{p,q}$ Fourier multipliers.

(iii) It is easy to see that the sequence $((1/k)I_X)_{k\in\mathbb{Z}}$ satisfies conditions (3.1) and (3.2). Thus, the sequence $((1/k)I_X)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$ Fourier multiplier by Theorem 3.2.

Letting $1 \le p,\ q \le \infty$, $s > 0$ and $a \in L^1(\mathbb{R}_+)$, we define the solution space of (P_α) in the $B^s_{p,q}$ well-posedness case by

$$S_{p,q,s}(A,M) := \{u \in B^s_{p,q}(\mathbb{T}; D(A)) : Mu \in B^{\alpha+s}_{p,q}(\mathbb{T}; X)\}.$$

Here again we consider $D(A)$ as a Banach space equipped with its graph norm. When $u \in S_{p,q,s}(A,M)$, then $a * Au \in B^s_{p,q}(\mathbb{T}; X)$, by Young's inequality. $S_{p,q,s}(A,M)$ is a Banach space with the norm

$$\|u\|_{S_{p,q,s}(A,M)} := \|u\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} + \|Mu\|_{B^{s+\alpha}_{p,q}}.$$

Now, we give the definition of the $B^s_{p,q}$ well-posedness of (P_α).

Definition 3.4. Let $1 \le p,\ q \le \infty$, $s > 0$ and $f \in B^s_{p,q}(\mathbb{T}; X)$; $u \in S_{p,q,s}(A,M)$ is called a *strong* $B^s_{p,q}$*-solution* of (P_α), if (P_α) is satisfied almost everywhere on $\mathbb{T}$. We say that (P_α) is $B^s_{p,q}$ well-posed if, for each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists a unique strong $B^s_{p,q}$-solution of (P_α).

If (P_α) is $B^s_{p,q}$ well-posed, a constant $C > 0$ exists such that, for each $f \in B^s_{p,q}(\mathbb{T}; X)$, if $u \in S_{p,q,s}(A,M)$ is the unique strong $B^s_{p,q}$-solution of (P_α), then

$$\|u\|_{S_{p,q,s}(A,M)} \le C\|f\|_{B^s_{p,q}}. \tag{3.3}$$

This can easily be obtained by the closedness of the operators A and M and the closed graph theorem.

We need the following preparation in the proof of our main result of this section.

Proposition 3.5. *Let $1 \le p,\ q \le \infty$, $s > 0$, and let A and M be closed linear operators defined on a Banach space X such that $D(A) \subset D(M)$, $a \in L^1(\mathbb{R}_+)$. We assume that $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ is 2-regular, and $(c_k)_{k\in\mathbb{Z}} \subset \mathbb{C}\backslash\{0\}$ defined by (2.6) satisfies (**A**2), such that*

$$\left\{\frac{a_k}{1+c_k} : k \in \mathbb{Z}\right\} \subset \rho_M(A).$$

Then the following assertions are equivalent:

(i) $(a_kM[a_kM-(1+c_k)A]^{-1})_{k\in\mathbb{Z}}$ *is a* $B^s_{p,q}$*-Fourier multiplier.*
(ii) $\sup_{k\in\mathbb{Z}}\|a_kM[a_kM-(1+c_k)A]^{-1}\|<\infty$.

Proof. Let $M_k=a_kMN_k$, where $N_k=[a_kM-(1+c_k)A]^{-1}$ when $k\in\mathbb{Z}$. The implication (i) $\Rightarrow$ (ii) is clearly true by Remark 3.3.

We need only show that the implication (ii) $\Rightarrow$ (i) is true. Assume that $\sup_{k\in\mathbb{Z}}\|M_k\|<\infty$. It follows from the proof of Proposition 2.9 that

$$\sup_{k\in\mathbb{Z}}\|k(M_{k+1}-M_k)\|<\infty. \tag{3.4}$$

On the other hand, we observe that

$$\begin{aligned}
&k^2(M_{k+2}-2M_{k+1}+M_k)\\
&\quad=k^2[a_{k+2}MN_{k+2}-2a_{k+1}MN_{k+1}+a_kMN_k]\\
&\quad=k^2MN_{k+2}[a_{k+2}N_k^{-1}-2a_{k+1}N_{k+2}^{-1}N_{k+1}N_k^{-1}+a_kN_{k+2}^{-1}]N_k\\
&\quad=k^2MN_{k+2}\{a_{k+2}N_k^{-1}-2a_{k+1}[a_{k+2}M-(1+c_{k+2})A]N_{k+1}N_k^{-1}\\
&\qquad\qquad+a_k[a_{k+2}M-(1+c_{k+2})A]\}N_k\\
&\quad=k^2MN_{k+2}\{a_{k+2}N_k^{-1}-2a_{k+1}[N_{k+1}^{-1}+(a_{k+2}-a_{k+1})M\\
&\qquad\qquad+(c_{k+1}-c_{k+2})A]N_{k+1}N_k^{-1}\\
&\qquad\qquad+a_k[N_k^{-1}+(a_{k+2}-a_k)M+(c_k-c_{k+2})A]\}N_k\\
&\quad=k^2MN_{k+2}\{(a_{k+2}-2a_{k+1}+a_k)I_X\\
&\qquad\qquad-2(a_{k+2}-a_{k+1})M_{k+1}+(a_{k+2}-a_k)M_k\\
&\qquad\qquad\quad+2a_{k+1}(c_{k+2}-c_{k+1})AN_{k+1}-a_k(c_{k+2}-c_k)AN_k\}\\
&\quad=k^2MN_{k+2}\{(a_{k+2}-2a_{k+1}+a_k)(I_X-M_{k+1})\\
&\qquad\qquad-(a_{k+2}-a_k)(M_{k+1}-M_k)\\
&\qquad\qquad+2(a_{k+1}-a_k)(c_{k+2}-c_{k+1})AN_{k+1}\\
&\qquad\qquad+a_k(c_{k+2}-2c_{k+1}+c_k)AN_{k+1}\\
&\qquad\qquad\qquad+a_k(c_{k+2}-c_k)A(N_{k+1}-N_k)\}\\
&\quad=M_{k+2}\Big\{\frac{k^2(a_{k+2}-2a_{k+1}+a_k)}{a_{k+2}}(I_X-M_{k+1})
\end{aligned}\tag{3.5}$$

$$
\begin{aligned}
&- \frac{k(a_{k+2} - a_k)}{a_{k+2}} k(M_{k+1} - M_k) \\
&+ \frac{2k(a_{k+1} - a_k)}{a_{k+2}} k(c_{k+2} - c_{k+1}) \frac{(M_{k+1} - I_X)}{1 + c_{k+1}} \\
&+ \frac{a_k}{a_{k+2}} k^2 (c_{k+2} - 2c_{k+1} + c_k) \frac{(M_{k+1} - I_X)}{1 + c_{k+1}} \\
&\qquad\qquad + \frac{a_k}{a_{k+2}} k(c_{k+2} - c_k) kA(N_{k+1} - N_k) \Big\},
\end{aligned}
$$

when $k \neq -2$. We note that, by (2.8),

$$
\begin{aligned}
&kA(N_{k+1} - N_k) \\
&\quad = AN_{k+1} \frac{k(a_k - a_{k+1})}{a_k} M_k + AN_{k+1} k(c_{k+1} - c_k) AN_k \\
&\quad = \frac{(M_{k+1} - I_X)}{1 + c_{k+1}} \frac{k(a_k - a_{k+1})}{a_k} M_k \\
&\qquad + \frac{(M_{k+1} - I_X)}{1 + c_{k+1}} k(c_{k+1} - c_k) \frac{(M_k - I_X)}{1 + c_k},
\end{aligned}
\tag{3.6}
$$

when $k \neq 0$. Noticing the assumption that $(a_k)_{k\in\mathbb{Z}}$ satisfies (**A**2) and $(c_k)_{k\in\mathbb{Z}}$ is 2-regular, we deduce from (3.5) and (3.6) that

$$
\sup_{k\in\mathbb{Z}} \|k^2(M_{k+2} - 2M_{k+1} + M_k)\| < \infty. \tag{3.7}
$$

This implies that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier by Theorem 3.2. Therefore, the implication (ii) $\Rightarrow$ (i) is also true. This completes the proof. □

Lemma 3.6. *Let X be a Banach space and $1 \le p$, $q \le \infty$, $s > 0$, $a \in L^1(\mathbb{R}_+)$. Suppose that $(c_k)_{k\in\mathbb{Z}}$ defined by* (2.6) *satisfies* (**A**2). *Then $(1/(1 + c_k)I_X)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier.*

Proof. It is clear that $(c_k)_{k\in\mathbb{Z}}$ and $(1/(1 + c_k))_{k\in\mathbb{Z}}$ are bounded by Remark 2.3. We observe that

$$
k\left(\frac{1}{1 + c_{k+1}} - \frac{1}{1 + c_k}\right) = \frac{-k(c_{k+1} - c_k)}{(1 + c_k)(1 + c_{k+1})} \tag{3.8}
$$

and

(3.9)

$$k^2\left(\frac{1}{1+c_{k+2}}-\frac{2}{1+c_{k+1}}+\frac{1}{1+c_k}\right)$$

$$= \frac{k^2}{(1+c_k)(1+c_{k+1})(1+c_{k+2})}[(1+c_k)(1+c_{k+1})-2(1+c_k)(1+c_{k+2}) + (1+c_{k+1})(1+c_{k+2})]$$

$$= \frac{k^2}{(1+c_k)(1+c_{k+1})(1+c_{k+2})}[-(1+c_k)(c_{k+2}-2c_{k+1}+c_k) + (c_{k+2}-c_k)(c_{k+1}-c_k)]$$

$$= \frac{1}{(1+c_k)(1+c_{k+1})(1+c_{k+2})}[-(1+c_k)k^2(c_{k+2}-2c_{k+1}+c_k) + k(c_{k+2}-c_k)k(c_{k+1}-c_k)].$$

Noting that assumption $(c_k)_{k\in\mathbb{Z}}$ satisfies (**A**2), it follows from (3.8) and (3.9) that

$$\sup_{k\in\mathbb{Z}}\left|k\left(\frac{1}{1+c_{k+1}}-\frac{1}{1+c_k}\right)\right| < \infty,$$

and

$$\sup_{k\in\mathbb{Z}}\left|k^2\left(\frac{1}{1+c_{k+2}}-\frac{2}{1+c_{k+1}}+\frac{1}{1+c_k}\right)\right| < \infty.$$

By Theorem 3.2, $(1/(1+c_k)I_X)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. This finishes the proof. □

The next theorem is the main result of this section which gives a necessary and sufficient condition for (P_α) to be $B^s_{p,q}$ well-posed.

Theorem 3.7. *Let X be a Banach space, $1 \le p,\ q \le \infty$, $s > 0$, and let A and M be closed linear operators on X satisfying $D(A) \subset D(M)$, $\alpha > 0$ and $a \in L^1(\mathbb{R}_+)$. We assume that $(c_k)_{k\in\mathbb{Z}}$ defined by* (2.6) *satisfies* (**A**2). *Then the following assertions are equivalent*:

(i) *(P_α) is $B^s_{p,q}$-well-posed*;

(ii) *$\{r_k^{(\alpha)}/(1+c_k) : k \in \mathbb{Z}\} \subset \rho_M(A)$ and $\sup_{k\in\mathbb{Z}} \|r_k^{(\alpha)}M[r_k^{(\alpha)}M - (1+c_k)A]^{-1}\| < \infty$.*

Proof.

(ii) $\Rightarrow$ (i). We assume that

$$\left\{\frac{r_k^{(\alpha)}}{1+c_k} : k \in \mathbb{Z}\right\} \subset \rho_M(A)$$

and $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$, where $N_k = [r_k^{(\alpha)} M - (1+c_k)A]^{-1}$ and $M_k = r_k^{(\alpha)} M N_k$ when $k \in \mathbb{Z}$. It follows from Proposition 3.5 that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. Then, for all $f \in B^s_{p,q}(\mathbb{T};X)$, there exists $u \in B^s_{p,q}(\mathbb{T};X)$ satisfying

$$\widehat{u}(k) = M_k \widehat{f}(k) \tag{3.10}$$

for all $k \in \mathbb{Z}$. By Lemma 3.6, $(I_X/(1+c_k))_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. We note that

$$AN_k = \frac{1}{1+c_k}[M_k - I_X]. \tag{3.11}$$

Thus, $(AN_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier by Remark 3.3. Thus, $v \in B^s_{p,q}(\mathbb{T};X)$ exists and satisfies $\widehat{v}(k) = AN_k\widehat{f}(k)$ for all $k \in \mathbb{Z}$. We note that A^{-1} is an isomorphism from X onto $D(A)$ as $0 \in \rho_M(A)$. By assumption, here we consider $D(A)$ as a Banach space equipped with its graph norm. Hence,, $A^{-1}\widehat{v}(k) = N_k\widehat{f}(k)$. Putting $w = A^{-1}v$, then $w \in B^s_{p,q}(\mathbb{T};D(A))$ and

$$\widehat{w}(k) = N_k \widehat{f}(k) \tag{3.12}$$

for all $k \in \mathbb{Z}$. This implies, in particular, that $(N_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. It is clear that the sequence $(I_X/r_k^{(\alpha)})_{k\in\mathbb{Z}}$ satisfies the second order Marcinkiewicz condition in Theorem 3.7; thus, it is a $B^s_{p,q}$-Fourier multiplier. We deduce that $(MN_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. This implies that $w \in B^s_{p,q}(\mathbb{T};D(M))$. Here, $D(M)$ is equipped with its graph norm so that it becomes a Banach space.

Combining (3.10) and (3.12), we obtain that

$$\widehat{u}(k) = r_k^{(\alpha)} M\widehat{w}(k) = r_k^{(\alpha)}\widehat{Mw}(k)$$

for all $k \in \mathbb{Z}$, which implies that $Mw \in B^{\alpha+s}_{p,q}(\mathbb{T};X)$. We have shown that $w \in S_{p,q,s}(A, M)$. By (3.12), we have

$$r_k^{(\alpha)} \widehat{Mw}(k) = A\widehat{w}(k) + c_k A\widehat{w}(k) + \widehat{f}(k)$$

for $k \in \mathbb{Z}$. Thus, $D^\alpha(Mw)(t) = Aw(t) + (a * Aw)(t) + f(t)$ for $t \in \mathbb{T}$ by the uniqueness theorem [**7,** page 314]. This shows the existence.

To show the uniqueness, we let $u \in S_{p,q,s}(A, M)$ be another solution of $D^\alpha(Mw)(t) = Aw(t) + (a * Aw)(t) + f(t)$. Then $D^\alpha M(u - w)(t) = A(u-w)(t) + (a * A(u-w))(t)$. Taking the Fourier transform, we have

$$r_k^{(\alpha)} M(u-w)^\wedge(k) = (1 + c_k)A(u-w)^\wedge(k)).$$

This implies that $[r_k^{(\alpha)} M - (1 + c_k)A](u - w)^\wedge(k)) = 0$. Thus, $(u - w)^\wedge(k) = 0$ as $r_k^{(\alpha)} M - (1 + c_k)A$ is invertible, and so $u = w$ by the uniqueness theorem [**7,** page 314]. Therefore, the implication (ii) $\Rightarrow$ (i) is true.

(i) $\Rightarrow$ (ii). Assume that (P_α) is $B^s_{p,q}$ well-posed. We shall show that $r_k^{(\alpha)}/(1 + c_k) \in \rho_M(A)$ for all $k \in \mathbb{Z}$. Let $k \in \mathbb{Z}$ and $y \in X$ be fixed. We define $f(t) = e^{ikt}y$, $t \in \mathbb{T}$. Then, $f \in B^s_{p,q}(\mathbb{T}; X)$, $\widehat{f}(k) = y$ and $\widehat{f}(n) = 0$ for $n \neq k$. There exists a unique $u \in S_{p,q,s}(A, M)$ satisfying

$$D^\alpha(Mu)(t) = Au(t) + (a * Au)(t) + f(t)$$

almost everywhere on $\mathbb{T}$. We have $\widehat{u}(n) \in D(A)$ when $n \in \mathbb{Z}$ by [**7,** Lemma 3.1]. Taking Fourier transforms on both sides, we obtain

$$(r_k^{(\alpha)} M - (1 + c_k)A)\widehat{u}(k) = y \tag{3.13}$$

and $(r_n^{(\alpha)} M - (1 + c_n)A)\widehat{u}(n) = 0$ when $n \neq k$. Thus, $r_k^{(\alpha)} M - (1 + c_k)A$ is surjective. To show that $r_k^{(\alpha)} M - (1 + c_k)A$ is also injective, we let $x \in D(A)$ be such that

$$(r_k^{(\alpha)} M - (1 + c_k)A)x = 0.$$

Let $u(t) = e^{ikt}x$ when $t \in \mathbb{T}$. Then, clearly, we have $u \in S_{p,q,s}(A, M)$ and (P_α) holds almost everywhere on $\mathbb{T}$ when $f = 0$. Thus, u is a strong $B^s_{p,q}$-solution of (P_α) when $f = 0$. We obtain $x = 0$ by the uniqueness assumption. We have shown that $r_k^{(\alpha)} M - (1 + c_k)A$ is injective. Therefore, $r_k^{(\alpha)} M - (1 + c_k)A$ is bijective from $D(A)$ onto X.

Next, we show that $[r_k^{(\alpha)}M-(1+c_k)A]^{-1}\in\mathcal{L}(X)$. For $f(t)=e^{ikt}y$, we let u be the unique strong $B^s_{p,q}$-solution of (P_α). Then

$$\widehat{u}(n)=\begin{cases}0 & n\neq k,\\ [r_k^{(\alpha)}M-(1+c_k)A]^{-1}y & n=k,\end{cases}$$

by (3.13). This implies that $u(t)=e^{ikt}[r_k^{(\alpha)}M-(1+c_k)A]^{-1}y$. By (3.3), a constant $C>0$ exists independent from $f\in B^s_{p,q}(\mathbb{T};X)$ such that

$$\|u\|_{B^s_{p,q}}+\|Au\|_{B^s_{p,q}}+\|Mu\|_{B^{s+\alpha}_{p,q}}\le C\|f\|_{B^s_{p,q}}.$$

Hence,

$$||[r_k^{(\alpha)}M-(1+c_k)A]^{-1}y||\le C\|y\|,$$

which implies that $||[r_k^{(\alpha)}M-(1+c_k)A]^{-1}||\le C$. We have shown $r_k^{(\alpha)}/(1+c_k)\in\rho_M(A)$ for all $k\in\mathbb{Z}$. Thus, $\{r_k^{(\alpha)}/(1+c_k):k\in\mathbb{Z}\}\subset\rho_M(A)$.

Finally, we prove that, if $M_k=r_k^{(\alpha)}M[r_k^{(\alpha)}M-(1+c_k)A]^{-1}$ when $k\in\mathbb{Z}$, then $(M_k)_{k\in\mathbb{Z}}$ defines a $B^s_{p,q}$-Fourier multiplier. Let $f\in B^s_{p,q}(\mathbb{T};X)$. Then there exists $u\in S_{p,q,s}(A,M)$, a strong $B^s_{p,q}$-solution of (P_α) by assumption. Taking Fourier transforms on both sides of (P_α) we have that $\widehat{u}(k)\in D(A)$ by Lemma 3.6 and

$$[r_k^{(\alpha)}M-(1+c_k)A]\widehat{u}(k)=\widehat{f}(k)$$

for all $k\in\mathbb{Z}$. Since $r_k^{(\alpha)}M-(1+c_k)A$ is invertible, we have

$$\widehat{u}(k)=[r_k^{(\alpha)}M-(1+c_k)A]^{-1}\widehat{f}(k)$$

for all $k\in\mathbb{Z}$. It follows from $Mu\in B^{\alpha+s}_{p,q}(\mathbb{T};X)$ that $[D^\alpha(Mu)]^\wedge(k)=r_k^{(\alpha)}M\widehat{u}(k)$. We obtain

$$[D^\alpha(Mu)]^\wedge(k)=r_k^{(\alpha)}M\widehat{u}(k)=M_k\widehat{f}(k)$$

when $k\in\mathbb{Z}$. We conclude that $(M_k)_{k\in\mathbb{Z}}$ defines a $B^s_{p,q}$-Fourier multiplier as $D^\alpha(Mu)\in B^s_{p,q}(\mathbb{T};X)$. Thus, we have $\sup_{k\in\mathbb{Z}}\|M_k\|<\infty$ by Remark 3.3. Therefore, the implication (i) $\Rightarrow$ (ii) is also true. The proof is completed. □

Since Theorem 3.7 (ii) does not depend on the parameters p,q and s, we immediately have the next corollary.

Corollary 3.8. *Let X be a Banach space, and let A and M be closed linear operators on X satisfying $D(A) \subset D(M)$, $\alpha > 0$, $a \in L^1(\mathbb{R}_+)$. We assume that $(c_k)_{k\in\mathbb{Z}}$ defined by* (2.6) *satisfies* (**A**2)*. Then, if (P_α) is $B^s_{p,q}$ well-posed for some $1 \le p, q \le \infty$ and $s > 0$, then it is $B^s_{p,q}$ well-posed for all $1 \le p$, $q \le \infty$ and $s > 0$.*

When the underlying Banach space X is B-convex and $1 \le p$, $q \le \infty$, $s \in \mathbb{R}$, the first order condition (3.1) is already sufficient for an operator-valued sequence to be a $B^s_{p,q}$-Fourier multiplier by Theorem 3.7. From this fact and the proof of Theorem 2.10, we easily deduce the following result on the $B^s_{p,q}$ well-posedness of the problem (P_α) under a weaker assumption on the sequence $(c_k)_{k\in\mathbb{Z}}$ when X is B-convex.

Corollary 3.9. *Let X be a B-convex Banach space, $1 \le p$, $q \le \infty$, $s > 0$, and let A and M be closed linear operators on X satisfying $D(A) \subset D(M)$, $\alpha > 0$, $a \in L^1(\mathbb{R}_+)$. We assume that $(c_k)_{k\in\mathbb{Z}}$ defined by* (2.6) *satisfies* (**A**1)*. Then the following assertions are equivalent*:

(i) (P_α) *is $B^s_{p,q}$-well-posed*;

(ii) $\{r_k^{(\alpha)}/1 + c_k : k \in \mathbb{Z}\} \subset \rho_M(A)$ *and*

$$\sup_{k\in\mathbb{Z}} ||r_k^{(\alpha)} M[r_k^{(\alpha)} M - (1 + c_k)A]^{-1}|| < \infty.$$

A Hölder continuous function space is a particular case of Besov space $B^s_{p,q}(\mathbb{T}; X)$. From [**8,** Theorem 3.1], we have $B^\beta_{\infty,\infty}(\mathbb{T}; X) = C^\beta_{\text{per}}(\mathbb{T}; X)$ whenever $0 < \beta < 1$, where $C^\beta_{\text{per}}(\mathbb{T}; X)$ is the space of all X-valued functions f defined on $\mathbb{T}$ satisfying $f(0) = f(2\pi)$ and

$$\sup_{s\neq t} \frac{\|f(s) - f(t)\|}{|s - t|^\beta} < \infty.$$

Moreover, the norm

$$\|f\|_{C^\beta_{\text{per}}} := \max_{t\in\mathbb{T}} \|f(t)\| + \sup_{s\neq t} \frac{\|f(s) - f(t)\|}{|s - t|^\beta}$$

on $C^\beta_{\text{per}}(\mathbb{T}; X)$ is an equivalent norm of the Besov space $B^\alpha_{\infty,\infty}(\mathbb{T}; X)$. We can similarly give the definition of C^β well-posedness of (P_α) when $0 < \beta < 1$ as well as a characterization of the C^β-well-posedness of (P_α)

as a special case of Theorem 3.7 when $p = q = +\infty$ and $0 < s < 1$. We omit the details.

We may also introduce the well-posedness of (P_α) in Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$. Using known operator-valued Fourier multiplier results on $F^s_{p,q}(\mathbb{T}; X)$, we may give a similar characterization of the $F^s_{p,q}$ well-posedness under a stronger condition than $(\mathbf{A}2)$ on the sequence $(c_k)_{k\in\mathbb{Z}}$.

4. Applications. In this section, we give some examples where our abstract results (Theorems 2.10 and 3.7) may be applied. The degenerate fractional differential equations we consider depend on the value of $\alpha > 0$.

Example 4.1. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and m a non-negative bounded measurable function defined on Ω. Let X be the Hilbert space $H^{-1}(\Omega)$. We consider the following degenerate fractional differential equations with infinite delay:

$$\begin{cases} D^\alpha(m(x)u(t,x)) = \Delta u(t,x) + \int_{-\infty}^t a(t-s)(\Delta u)(s,x)\,ds + f(t,x), \\ \qquad\qquad\qquad\qquad\qquad (t,x) \in [0,2\pi]\times\Omega, \\ u(t,x) = 0 \qquad\qquad\qquad\quad (t,x) \in [0,2\pi]\times\partial\Omega, \end{cases}$$

where $a \in L^1(\mathbb{R}_+)$, the fractional derivative D^α in the sense of Weyl, acts on the first variable $t \in [0, 2\pi]$ and the Laplacian operator Δ acts on the second variable $x \in \Omega$.

Let M be the multiplication operator by m on $H^{-1}(\Omega)$ with domain of definition $D(M)$. We assume that $D(\Delta) \subset D(M)$, where Δ is the Laplacian operator on $H^{-1}(\Omega)$ with Dirichlet boundary condition. Then, it follows from [**6,** Section 3.7] that a constant $C \geq 0$ exists such that

$$\|M(zM-\Delta)^{-1}\| \leq \frac{C}{1+|z|} \tag{4.1}$$

when $\mathrm{Re}\,(z) \geq -\beta(1 + |\mathrm{Im}\,(z)|)$ for some positive constant β depending only on m. We assume that $\{r_k^{(\alpha)}/1+c_k : k \in \mathbb{Z}\} \subset \rho_M(\Delta)$ and

$$\sup_{k\in\mathbb{Z}} ||r_k^{(\alpha)} M[r_k^{(\alpha)} M - (1+c_k)\Delta]^{-1}|| < \infty,$$

where c_k is defined by (2.6).

We note that, if $\alpha > 0$, then $\arg(r_k^{(\alpha)}) = \alpha\pi/2$ when $k \geq 1$, and $\arg(r_k^{(\alpha)}) = -\alpha\pi/2$ when $k \leq -1$. This, together with fact that $\lim_{|k|\to+\infty} c_k = 0$, implies that

$$\lim_{|k|\to+\infty} \arg\left(\frac{r_k^{(\alpha)}}{1+c_k}\right) = \operatorname{sgn}(k)\frac{\alpha\pi}{2} \tag{4.2}$$

when $k \neq 0$. If $4n \leq \alpha \leq 4n+1/2$ for some non negative integer n, then the estimates (4.1) and (4.2) imply that the above problem is L^p well-posed for all $1 < p < \infty$ by Theorem 2.10 whenever $(c_k)_{k\in\mathbb{Z}}$ satisfies (**A**1). Here, we have used the fact that, in a Hilbert space H, every norm bounded subset $\mathbf{T} \subset \mathcal{L}(H)$ is actually R-bounded [**7,** Proposition 1.13].

When the sequence $(c_k)_{k\in\mathbb{Z}}$ defined by (2.6) satisfies (**A**2), the estimates (4.1) and (4.2) imply that the above problem is $B_{p,q}^s$ well-posed for all $1 \leq p, q \leq \infty$ and $s > 0$ by Theorem 3.7.

Under the same assumptions on Ω, m and a, one may also consider the degenerate fractional differential equations:

$$\begin{cases} D^\alpha(m(x)u(t,x)) + \Delta u(t,x) = -\int_{-\infty}^t a(t-s)(\Delta u)(s,x)\,ds + f(t,x), \\ \qquad\qquad\qquad\qquad\qquad (t,x) \in [0,2\pi] \times \Omega, \\ u(t,x) = 0 \qquad\qquad\qquad (t,x) \in [0,2\pi] \times \partial\Omega. \end{cases}$$

The same argument used above shows that, when $\{r_k^{(\alpha)}/(1+c_k) : k \in \mathbb{Z}\} \subset \rho_M(-\Delta)$ and

$$\sup_{k\in\mathbb{Z}} ||r_k^{(\alpha)} M[r_k^{(\alpha)} M + (1+c_k)\Delta]^{-1}|| < \infty,$$

if $4n+1 \leq \alpha \leq 4n+2$ for some non negative integer n and $(c_k)_{k\in\mathbb{Z}}$ defined by (2.6) satisfies (**A**1), then the above problem is L^p well-posed for all $1 < p < \infty$ by Theorem 2.10. If, furthermore, $(c_k)_{k\in\mathbb{Z}}$ defined by (2.6) satisfies (**A**2), then the above problem is $B_{p,q}^s$ well-posed for all $1 \leq p, q \leq \infty$ and $s > 0$ by Theorem 3.7.

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