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L^p-SOLUTIONS FOR A CLASS OF FRACTIONAL INTEGRAL EQUATIONS

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ABSTRACT. This paper considers the existence of L^p-solutions for a class of fractional integral equations involving abstract Volterra operators in a separable Banach space. Some applications for the existence of L^p-solutions for different classes of fractional differential equations are given.

1. Introduction. Consider the fractional integral equation in a Banach space E:

(1.1)
$$u(t) = (Pu)(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,(Qu)(s)\,ds \text{ almost everywhere } t > 0,$$

where $P, Q : L^p_{\mathrm{loc}}([0,a), E) \to L^p_{\mathrm{loc}}([0,a), E)$, $1 < p < \infty$, are continuous abstract Volterra operators, $(Pu)(0) = u_0$ for all $u \in L^p_{\mathrm{loc}}([0,a), E)$ for a given $u_0 \in E$, $\alpha > 0$ and $0 < a \leq \infty$. We recall that an operator $Q : L^p_{\mathrm{loc}}([0,a), E) \to L^p_{\mathrm{loc}}([0,a), E)$ is a *causal operator* or an *abstract Volterra operator* if, for each $\tau \in [0,a)$ and for all $u(\cdot), v(\cdot) \in L^p_{\mathrm{loc}}([0,a), E)$ with $u(t) = v(t)$ for every $t \in [0,\tau]$, we have $Qu(t) = Qv(t)$ for $t \in [0,\tau]$ almost everywhere. As we will show, taking different particular classes of operators, this type of integral equation covers a large variety of integral equations of Volterra type as well as fractional differential equations. The existence of L^p-solutions for different kinds of integral equations has been intensively studied by many authors, such as **[1]**–**[39]**. The existence of L^p-solutions for fractional

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differential equations and the integral fractional equation has been studied in only a few papers; see, for example, [**4, 5, 10, 11, 17, 25**].

The paper is organized as follows. In Section 2, we recall some definitions and results required in this paper. In Section 3, under some suitable conditions involving compactness type conditions, we establish a result of the existence of solutions in the space $L^p([0,a],E)$ for the integral equation (1.1). Also, by using some Lipschitz type conditions, we obtain the existence and uniqueness of solution in the space $L^p([0,a],E)$ for the integral equation (1.1). In Section 4, we obtain a result on the existence of L^p-solutions on an infinite interval. More precisely, we shall prove the existence of solutions in the space $L^p_{\text{loc}}([0,\infty),E)$ for the integral equation (1.1). In Section 5, we will give some examples which highlight the wide range of applicability of our results.

2. Preliminaries. Let E be a real Banach space endowed with the norm $\|\cdot\|$. If A nonempty subset in E, then $\overline{A}$, $\text{conv}(A)$ and $\overline{\text{conv}}(A)$ denote the closure of A, the convex hull of A and the closure of the convex hull of A, respectively. We shall denote by $C([0,a],E)$ the Banach space of continuous bounded functions from $[0,a]$ into E endowed with the norm $\|u(\cdot)\| = \sup_{0\le t\le a}\|u(t)\|$. The space of all (equivalence classes of) strongly measurable and Bochner integrable functions $u(\cdot):[0,a]\to E$ such that

$$\|u(\cdot)\|_p := \left(\int_0^a \|u(t)\|^p\,dt\right)^{1/p} < \infty$$

for $1\le p<\infty$, will be denoted by $L^p([0,a],E)$. Then $L^p([0,a],E)$ is a Banach space with respect to the norm $\|u(\cdot)\|_p$. Also, we shall denote by $L^\infty([0,a],E)$ the space of all (equivalence classes of) strongly measurable functions $u(\cdot):[0,a]\to E$ which are essentially bounded on $[0,a]$. Then $L^\infty([0,a],E)$ is a Banach space with respect to the norm

$$\|u(\cdot)\|_\infty := \operatorname*{ess\,sup}_{t\in[0,a]}\|u(t)\| = \inf\{M\ge 0; \|u(t)\| \le M \text{ for almost every } t\in[0,a]\}.$$

We recall that, if $1\le p<q\le\infty$, then

$$L^q([0,a],E)\subset L^p([0,a],E),$$

and

$$\|u\|_p \leq a^{1/p-1/q}\,\|u\|_q \quad \text{for every } u \in L^q([0,a],E).$$

The Kuratowski measure of non-compactness of a nonempty bounded set $A \subset E$ is defined by ([**27**]):

> $\beta(A) = \inf\{\delta > 0;\ A$ can be expressed as the union of a finite number of sets such that the diameter of each set does not exceed $\delta\}$.

We recall some properties of β, see [**23**]. If A, B are bounded subsets of E, then:

(1) $\beta(A) = 0$ if and only if $\overline{A}$ is compact;
(2) $\beta(A) = \beta(\overline{A}) = \beta(\overline{\mathrm{conv}}(A))$;
(3) $\beta(\lambda A) = |\lambda|\beta(A)$ for every $\lambda \in \mathbb{R}$;
(4) $\beta(A) \leq \beta(B)$ if $A \subset B$;
(5) $\beta(A+B) = \beta(A) + \beta(B)$.

In the following, we let β_p denote the Kuratowski measures of noncompactness of sets in space $L^p([0,a],E)$.

We recall the next lemma due to Heinz [**21**].

Lemma 2.1. *Let A be a countable set of strongly measurable functions $u : [0,a] \to E$ such that there exists an $m(\cdot) \in L^1([0,a],\mathbb{R}_+)$ such that $\|u(t)\| \leq m(t)$ for each $u(\cdot) \in A$ and for almost every $t \in [0,a]$. Then the function $t \mapsto v(t) := \beta(\{u(t); u(\cdot) \in A\})$ is integrable on $[0,a]$ and, for each $t \in [0,a]$, we have*

$$\beta\left(\left\{\int_0^t u(s)\,ds;\, u\,(\cdot) \in A\right\}\right) \leq 2\int_0^t v(s)\,ds.$$

Lemma 2.2. *Let $A \subset L^p([0,a],E)(1 < p < \infty)$ be countable such that there exists an $m(\cdot) \in L^1([0,a],\mathbb{R}_+)$such that $\|u(t)\| \leq m(t)$ for each $u(\cdot) \in A$ and for almost every $t \in [0,a]$.*

(i) ([**20,** Lemma 1.2.2]). *If*

$$\lim_{h\to 0}\sup_{u(\cdot)\in A}\int_0^a \|u(t+h) - u(t)\|^p dt = 0, \tag{2.1}$$

then

$$\beta_p(A) \le 2\bigg(\int_0^a [\beta(A(t))]^p\, dt\bigg)^{1/p}.$$

(ii) ([**20,** Theorem 1.2.8]). *A is relatively compact if and only if* (2.1) *is satisfied and $A(t)$ is relatively compact* (*in E*) *for almost every $t \in [0,a]$.*

Lemma 2.3. (*Mönch fixed point theorem* [**33,** Theorem 2.1]). *Let E be a Banach space, $K \subset E$ closed and convex, and $F : K \to K$ continuous with the further property that, for some $x \in K$, we have*

$$C \subset K \text{ countable, } \overline{C} \subset \overline{\mathrm{co}}\,(\{x\} \cup F(C)) \Longrightarrow C \text{ is relatively compact.}$$

Then F has a fixed point in K.

3. Existence of L^p-solutions on finite intervals. In this section, we obtain a result of the existence of solutions in the space for the following integral fractional equation

$$u(t) = (Pu)(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,(Qu)(s)\, ds \tag{3.1}$$

almost everywhere $t \in [0,a]$, where $P, Q : L^p([0,a],E) \to L^p([0,a],E)$ are abstract Volterra operators, $a \in (0,\infty)$ and $1 < p < \infty$ is a real number such that $p > 1/\alpha$ with $\alpha \in (0,1)$. We also assume that:

(**H1**) $P : L^p([0,a],E) \to L^p([0,a],E)$ is a compact abstract Volterra operator such that $b(\cdot) \in L^p([0,a],\mathbb{R}_+)$ exists with

$$\|(Pu)(t)\| \le b(t) \text{ for almost every } t \in [0,a].$$

(**H2**) $Q : L^p([0,a],E) \to L^p([0,a],E)$ is a continuous abstract Volterra operator such that $c(\cdot) \in L^p([0,a],\mathbb{R}_+)$ and $d > 0$ exist with

$$\|(Qu)(t)\| \le c(t) + d\,\|u(t)\| \text{ for almost every } t \in [0,a],$$

and for every $u(\cdot) \in L^p([0,a],E)$.

(**H3**) There exists a $k_1 > 0$ such that

$$\beta((QH)(t)) \le k_1 \beta(H(t)) \tag{3.2}$$

for $t \in [0,a]$ and for each bounded subset $H \subset L^p([0,a],E)$.

Remark 3.1. Unfortunately, the measure of non-compactness cannot be eliminated from the assumptions because of the criteria for compactness in the spaces $C([0,a],E)$ and $L^p([0,a],E)$, when E is an infinite-dimensional Banach space. It is known that the fractional Riemann-Liouville integral of order $\alpha > 0$, given by

$$(I^\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds,$$

defines a bounded linear operator I^α from $L^p([0,\alpha],E)$ into itself, and also from $C([0,\alpha],E)$ into itself. However, in the case of infinite-dimensional Banach spaces, this operator does not map bounded sets of continuous functions into compact sets on bounded intervals. For example, if we consider the ball $B(0,\Gamma(\alpha+1)/a^\alpha)$ in $C([0,a],E)$, then $I^\alpha(B(0,\Gamma(\alpha+1)/a^\alpha)) = B(0,1) \subset C([0,a],E)$. However, the unit ball is not a compact set in $C([0,a],E)$.

Theorem 3.2. *Assume that conditions* (H1)–(H3) *are satisfied. Then the fractional integral equation* (3.1) *has at least one solution in* $L^p([0,a],E)$*, provided*

$$\gamma := \frac{a^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\alpha p}\right)^{1/p} \left(\frac{p-1}{\alpha p - 1}\right)^{1-1/p} \le \frac{1}{d}. \tag{3.3}$$

Proof. First, we will show that each solution of (3.1) is a priori bounded in $L^p([0,a],E)$. Indeed, since

$$\|u(t)\| \le \|(Pu)(t)\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|(Qu)(s)\|\, ds$$

then, using Holder's inequality, we have

$$\begin{aligned} \|u(t)\| &\le \|(Pu)(t)\| + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{q(\alpha-1)} ds\right)^{1/q} \|(Qu)(\cdot)\|_p \\ &\le b(t) + \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p-1}\right)^{1-1/p} t^{\alpha-1/p} \left[\|c(\cdot)\|_p + d\|u(\cdot)\|_p\right], \end{aligned}$$

so that

$$\|u(\cdot)\|_p \le \|b(\cdot)\|_p + \gamma \left[\|c(\cdot)\|_p + d\|u(\cdot)\|_p\right].$$

Then, using (3.3), we have $\|u(\cdot)\|_p \leq r$, where

$$r \geq \frac{\|b\,(\cdot)\,\|_p + \gamma\|c\,(\cdot)\,\|_p}{1-d\gamma}.$$

Next, we consider the operator A defined by

$$(Au)(t) = (Pu)(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(Qu)(s)\,ds \tag{3.4}$$

almost everywhere $t \in [0,a]$. If we put $B := \{u(\cdot) \in L^p([0,a],E); \|u(\cdot)\|_p \leq r\}$, then it is easy to check that $A(B) \subset B$, that is, A is an operator from B into itself.

Now, we show that A is a continuous operator. For this, let $\{u_n(\cdot)\}_{n\geq 1}$ be a convergent sequence in $L^p([0,a],E)$ such that $u_n(\cdot) \to u(\cdot)$ as $n \to \infty$. Since

$$\begin{aligned}\|(Au_n)\,(t) - (Au)\,(t)\| &\leq \|(Pu_n)\,(t) - (Pu)\,(t)\| \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|(Qu_n)\,(s) - (Qu)\,(s)\|\,ds \\ &\leq \|(Pu_n)\,(t) - (Pu)\,(t)\| + \frac{1}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-1/p} \\ &\quad \cdot t^{\alpha-(1/p)}\,\|Qu_n\,(\cdot) - Qu\,(\cdot)\|_p\,,\end{aligned}$$

it follows that

$$\begin{aligned}\|(Au_n)\,(\cdot) - (Au)\,(\cdot)\|_p &\leq \|(Pu_n)\,(\cdot) - (Pu)\,(\cdot)\|_p \\ &\quad + \gamma\,\|(Qu_n)\,(\cdot) - (Qu)\,(\cdot)\|_p\,.\end{aligned}$$

Since P and Q are continuous operators, $\|Au_n(\cdot) - Au(\cdot)\|_p \to 0$ as $n \to \infty$, so that A is a continuous operator. In the next step, we show that

$$\lim_{h\to 0}\sup_{u(\cdot)\in B_0}\int_0^a \|(Au)(t+h) - (Au)(t)\|^p dt = 0$$

for every countable subset B_0 of B. If $t \in [0,a]$ and $h > 0$ are such that $t+h \in [0,a]$, then, for every $u(\cdot) \in B_0$, we have

$$\begin{aligned}\|(Au)\,(t+h) - (Au)\,(t)\| &\leq \|(Pu)\,(t+h) - (Pu)\,(t)\| \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \left|(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right|\end{aligned}$$

$$\begin{aligned}
&\cdot \| (Qu)(s)\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_t^{t+h} (t+h-s)^{\alpha-1} \|(Qu)(s)\| \, ds \\
&= \|(Pu)(t+h) - (Pu)(t)\| \\
&+ \frac{1}{\Gamma(\alpha)} \left[\eta_1(t,h) + \eta_2(t,h)\right].
\end{aligned}$$

Now, using Holder's inequality, we have

$$\begin{aligned}
\eta_1(t,h) &\le \int_0^t \left|(t-s)^{\alpha-1} - (t+h-s)^{\alpha-1}\right| \| (Qu)(s)\| \, ds \\
&\le \left(\frac{p-1}{\alpha p-1}\right)^{1-1/p} \\
&\quad \cdot \left(h^{(p-1)/(\alpha p-1)} + t^{(p-1)/(\alpha p-1)} - (t+h)^{(p-1)/(\alpha p-1)}\right)^{1-1/p} \\
&\quad \cdot \| (Qu)(\cdot) \|_p \\
&\le \eta \left(\frac{p-1}{\alpha p-1}\right)^{1-1/p} h^{\alpha-1/p},
\end{aligned}$$

and

$$\eta_2(t,h) = \int_t^{t+h} (t+h-s)^{\alpha-1} \|Qu(s)\| \, ds \le \eta \left(\frac{p-1}{\alpha p-1}\right)^{1-1/p} h^{\alpha-1/p},$$

where $\eta := \|c(\cdot)\|_p + dr$. Hence,

$$\int_0^a \eta_i^p(t,h) \, dt \le a\eta^p \left(\frac{p-1}{\alpha p-1}\right)^{p-1} h^{\alpha p-1} \longrightarrow 0 \quad \text{as } h \to 0, \quad i = 1,2. \tag{3.5}$$

Since P is a compact operator, using Lemma 2.2, we have

$$\lim_{h\to 0} \sup_{u(\cdot)\in B_0} \int_0^a \|(Pu)(t+h) - (Pu)(t)\|^p dt = 0. \tag{3.6}$$

Consequently, since

$$\begin{aligned}
&\int_0^a \|(Au)(t+h) - (Au)(t)\|^p dt \\
&\quad \le 2^{p-1} \int_0^a \|(Pu)(t+h) - (Pu)(t)\|^p dt
\end{aligned}$$

$$+\frac{2^{p-1}}{[\Gamma(\alpha)]^p}\int_0^a [\eta_1(t,h)+\eta_2(t,h)]^p\,dt,$$

and using (3.5) and (3.6), we obtain

$$\lim_{h\to 0}\sup_{u(\cdot)\in B_0}\int_0^a \|(Au)(t+h)-(Au)(t)\|^p dt = 0. \tag{3.7}$$

Next, let H be a countable subset of B such that $H\subset\overline{\text{co}}((AH)\cup\{0\})$. We will use the compactness criteria from Lemma 2.2 to show that H is a relatively compact set in $L^p([0,a],E)$. First, from (3.7), we have

$$\lim_{h\to 0}\sup_{u(\cdot)\in H}\int_0^a \|u(t+h)-u(t)\|^p dt = 0. \tag{3.8}$$

Since H is a bounded set in $L^p([0,a],E)$, from (3.8) and Lemma 2.2, we have

$$\beta_p(H)\le 2\bigg(\int_0^a [\alpha(H(t))]^p\,dt\bigg)^{1/p}. \tag{3.9}$$

On the other hand, using the properties of the Kuratowski measures of noncompactness and (3.2), we have

$$\beta(H(t))\le\beta(\overline{\text{co}}(((AH)(t))\cup\{0\}))=\beta((AH)(t)).$$

Using compactness of the operator P, we have

$$\begin{aligned}\beta(H(t))&\le\beta\bigg((PH)(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(QH)(s)\,ds\bigg)\\&\le\beta((PH)(t))+\beta\bigg(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(QH)(s)\,ds\bigg)\\&\le\frac{k_1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\beta(H(s))\,ds,\end{aligned}$$

and thus, by [**22,** Lemma 7.1.2], it follows that $\beta(H(t))=0$ for all $t\in[0,a]$. Now, by (3.9), we obtain $\beta_p(H)=0$, that is, H is a relatively compact set in $L^p([0,a],E)$. Summarizing, we have shown that $A:B\to B$ is a continuous operator with the property that, for a countable subset H of B such that $H\subset\overline{\text{co}}((AH)\cup\{0\})$, H is relatively compact. Since B is a closed and convex set in $L^p([0,a],E)$ then, by the Mönch fixed point theorem, it follows that there exists $u(\cdot)\in B$

such that $u(\cdot) = (Au)(\cdot)$, that is, the integral equation (3.1) has at least one solution $u(\cdot) \in B$. $\square$

Theorem 3.3. *Let conditions* (H1) *and* (H2) *be satisfied. Also, suppose that there exist* $0 < L_1 < 1$ *and* $L_2 > 0$ *such that the following Lipschitz type conditions are satisfied*:

$$\|(Pu)(t) - (Pv)(t)\| \leq L_1 \|u(t) - v(t)\| \tag{3.10}$$

and

$$\|(Qu)(t) - (Qv)(t)\| \leq L_2 \|u(t) - v(t)\|, \tag{3.11}$$

for almost every $t \in [0, a]$ *and* $u(\cdot)$, $v(\cdot) \in L^p([0, a], E)$. *Then,* (3.1) *has a unique solution, provided* (3.3) *holds and*

$$L_1 + \gamma L_2 < 1. \tag{3.12}$$

Proof. Let $A : L^p([0, a], E) \to L^p([0, a], E)$ be the operator defined by (3.4). Then, for any $u(\cdot)$, $v(\cdot) \in L^p([0, a], E)$, and for almost every $t \in [0, a]$, we have

$$\begin{aligned}\|(Au)(t) - (Av)(t)\| &\leq \|(Pu)(t) - (Pv)(t)\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|(Qu)(s) - (Qv)(s)\|\, ds \\ &\leq L_1 \|u(t) - v(t)\| \\ &\quad + \frac{L_2}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p - 1}\right)^{1-1/p} t^{\alpha - 1/p} \|u(\cdot) - v(\cdot)\|_p,\end{aligned}$$

so that

$$\|Au(\cdot) - Av(\cdot)\|_p \leq (L_1 + \gamma L_2) \|u(\cdot) - v(\cdot)\|_p .$$

Since $L_1 + \gamma L_2 < 1$, it follows that A is a contraction in $L^p([0, a], E)$. Consequently, A has a unique fixed point, and thus, (3.1) has a unique solution $u(\cdot) \in L^p([0, a], E)$. $\square$

3.1. Fractional differential equations. Let $\alpha > 0$, and let $AC([0, a], E)$ be the space of all absolutely continuous functions from $[0, a]$ to E. It is well known, see, for example, **[43]**, that, if $u(\cdot) \in L^p([0, a], E)$, then the *fractional Riemann-Liouville integral of order* $\alpha > 0$, defined by

$$(I^\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \tag{3.13}$$

exists for almost every $t \in [0,a]$. Moreover, I^α is a bounded linear abstract Volterra operator from $L^p([0,a],E)$ into itself, see **[43]**. If $\alpha \in (0,1)$, then the *fractional Riemann-Liouville derivative of order* $\alpha \in (0,1)$ is defined by

$$(D^\alpha u)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s)\, ds. \tag{3.14}$$

It is easy to see that $(D^\alpha u)(t)$ exists for almost every $t \in [0,a]$ if and only if $u(\cdot) \in L^p([0,a],E)$ is such that $(I^{1-\alpha}u)(\cdot) \in AC([0,a],E)$ and $(d/dt)(I^{1-\alpha}u)(t)$ exists for almost every $t \in [0,a]$. The *Caputo fractional derivative of order* $\alpha \in (0,1)$ is defined by

$$\left({}^C D^\alpha u\right)(t) = \left(D^\alpha (u(\cdot) - u(0))\right)(t).$$

Then $({}^C D^\alpha u)(t)$ exists for almost every $t \in [0,a]$ if and only if $u(\cdot) \in L^p([0,a],E)$ is such that $(I^{1-\alpha}u)(\cdot) \in AC([0,a],E)$ and $(d/dt)(I^{1-\alpha}u)(t)$ exists for almost every $t \in [0,a]$. For $\alpha \in (0,1)$ and $p \geq 1$, we denote by $\Omega^{p,\alpha}([0,a],E)$ the space of all functions $u(\cdot) \in L^p([0,a],E)$ is such that $(I^{1-\alpha}u)(\cdot) \in AC([0,a],E)$ and $(d/dt)(I^{1-\alpha}u)(t)$ exists for almost every $t \in [0,a]$. It is easy to see that $u(\cdot) \in \Omega^{p,\alpha}([0,a],E)$ if and only if $u(\cdot) \in L^p([0,a],E)$ and $(D^\alpha u)(\cdot) \in L^1([0,a],E)$.

Remark 3.4. Let $g(\cdot) \in L^p([0,a],E)$ be a given function. The constant operator $P : L^p([0,a],E) \to L^p([0,a],E)$, defined by $(Pu)(\cdot) = g(\cdot)$ for all $u(\cdot) \in L^p([0,a],E)$, is evidently a compact abstract Volterra operator. Also, it is well known that the following initial value problem involving the fractional Caputo derivative

$$\begin{cases} ({}^C D^\alpha u)(t) = (Qu)(t) \quad \text{almost everywhere } t \in [0,a], \\ u(0) = u_0 \end{cases} \tag{3.15}$$

is equivalent to the following integral equation

$$u(t) = u_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Qu)(s)\, ds \text{ almost everywhere } t \in [0,a].$$

Therefore, the next results hold.

Corollary 3.5. *If $g(\cdot) \in L^p([0,a],E)$ and*

$$Q : L^p([0,a],E) \longrightarrow L^p([0,a],E)$$

are continuous abstract Volterra operators satisfying the conditions from Theorem 3.2*, then the fractional integral equation*

$$u(t) = g(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Qu)(s)\, ds$$

has at least one solution in $L^p([0,a],E)$, provided $1 - d\gamma > 0$.

Corollary 3.6. *If $Q : L^p([0,a],E) \to L^p([0,a],E)$ is a continuous, abstract Volterra operator satisfying the conditions from Theorem* 3.2*, then the initial value problem* (3.15) *has at least one solution in $\Omega^{p,\alpha}([0,a],E)$, provided $1 - d\gamma > 0$.*

4. Existence of L^p-solutions on infinite intervals. Let us denote by $L^p_{\mathrm{loc}}([0,\infty),E)$, $1 < p < \infty$, the space of locally integrable functions on $[0,\infty)$. It is well known that $L^p_{\mathrm{loc}}([0,\infty),E)$ is a Fréchet space with respect to the seminorms

$$\|u\,(\cdot)\|_{p,k} := \left(\int_0^{t_k} \|u(t)\|^p\, dt \right)^{1/p} \quad \text{for } 1 < p < \infty,$$

with $t_k \in (0,\infty)$, $k = 1,2,\ldots$.

In the next theorem, we obtain a result of the existence of global solutions to the fractional integral equation in the space $L^p_{\mathrm{loc}}([0,\infty),E)$ for the following fractional integral equation

(4.1)
$$u(t) = (Pu)(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Qu)(s)\, ds \text{ almost everywhere } t > 0,$$

where $P, Q : L^p_{\mathrm{loc}}([0,\infty),E) \to L^p_{\mathrm{loc}}([0,\infty),E)$ are abstract Volterra operators.

Theorem 4.1. *Let $p \geq 1$ be such that $p > 1/\alpha$ with $\alpha \in (0,1)$. Suppose that, for every $a \in (0,\infty)$, the operators $P, Q : L^p([0,a],E) \to$*

$L^p([0,a],E)$ *satisfy* (H1)–(H3). *Then, the fractional integral equation* (4.1) *has at least one solution in* $L^p_{\text{loc}}([0,\infty),E)$ *provided* $1-d\gamma>0$.

Proof. Let $0<t_1<t_2<\cdots<t_n<\cdots$ be such that $t_n\to\infty$ as $n\to\infty$. Then, from Theorem 3.2, for each $n=1,2,\ldots$, there exists $u_n(\cdot)\in L^p([0,t_n],E)$ which solves

$$u_n(t)=(Pu_n)(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(Qu_n)(s)\,ds \tag{4.2}$$

almost everywhere $t\in[0,t_n]$. In addition, there exist constants $r_k\in[0,\infty)$, $k=1,2,\ldots$, such that $n\geq k$ implies

$$\|u\,(\cdot)\|_{p,k}=\left(\int_0^{t_k}\|u_n(t)\|^p dt\right)^{1/p}\leq r_k.$$

As in the proof of Theorem 3.2 it is easy to show that $\{u_n(\cdot)\}_{n\geq k}$ is relatively compact in $L^p([0,t_k],E)$ for $k=1,2,\ldots.$ In particular, $\{u_n(\cdot)\}_{n\geq 1}$ is relatively compact in $L^p([0,t_1],E)$. Therefore, there exist an infinite set $N_1\subset\{1,2,\ldots\}$ and a function $v_1(\cdot)\in L^p[0,t_1]$ such that

$$\int_0^{t_1}\|u_n(t)-v_1(t)\|^p dt\longrightarrow 0\quad\text{as } n\longrightarrow\infty \text{ in } N_1.$$

Since $\{u_n(\cdot)\}_{n\in N_1}$ is a Cauchy sequence in $L^p([0,t_1],E)$, and it converges to $v_1(\cdot)$, there exists an infinite set $N_1^1\subset N_1$ such that $u_n(t)\to v_1(t)$ almost everywhere on $[0,t_1]$ as $n\to\infty$ in N_1^1. Let $N_1^{12}=N_1^1\setminus\{1\}$. As before, $\{u_n(\cdot)\}_{n\in N_1^{12}}$ is relatively compact in $L^p([0,t_2],E)$. Therefore, there exist an infinite set $N_2\subset N_1^{12}$ and a function $v_2(\cdot)\in L^p[0,t_2]$ such that

$$\int_0^{t_2}\|u_n(t)-v_2(t)\|^p dt\to 0\quad\text{as } n\to\infty \text{ in } N_2.$$

Since $\{u_n(\cdot)\}_{n\in N_2}$ is a Cauchy sequence in $L^p([0,t_2],E)$, there exists an infinite set $N_2^1\subset N_2$ such that $u_n(t)\to v_2(t)$ almost everywhere on $[0,t_2]$ as $n\to\infty$ in N_2^1.

Next, take $N_2^{12}=N_2^1\setminus\{2\}$, and continue the argument inductively. Since $N_2^1\subset N_1^1$, it follows that $v_1(\cdot)=v_2(\cdot)$ almost everywhere on $[0,t_1]$. Now, we define the function $u(\cdot):[0,\infty)\to E$, by $u(t)=v_k(t)$

for almost every $t \in [0, t_k]$ and $k = 1, 2, \ldots$. Since

$$\int_0^{t_n} \|u(t)\|^p dt = \int_0^{t_n} \|v_n(t)\|^p dt \le r_n \quad \text{for } n = 1, 2 \ldots,$$

it follows that $u(\cdot) \in L^p_{\text{loc}}([0, \infty), E)$.

Next, we show that the function $u(\cdot)$ is a solution for the integral equation. Since $Q : L^p([0, t_k], E) \to L^p([0, t_k], E)$ is continuous and $u_n(\cdot) \to y_k(\cdot)$ in $L^p([0, t_k], E)$ as $n \to \infty$ in N_k, there exists an infinite set $M_k \subset N_k$ such that $(Qu_n)(\cdot) \to (Qy_k)(\cdot)$ almost everywhere on $[0, t_k]$ as $n \to \infty$ in M_k. On the other hand, if $t \in (0, t_k]$ is given, then it is easy to see that

$$s \longmapsto m_k(s) := (t-s)^{\alpha-1}(b(t) + \gamma \left\|c(\cdot)\right\|_p + dr_k)$$

belong to $L^p([0, t_k], E)$, and $(t-s)^{\alpha-1}\|(Qu_n)(s)\| \le m_k(s)$ for almost every $s \in [0, t_k]$. Hence, taking $n \to \infty$ in M_k in (4.2), by Lebesgue's dominated convergence theorem, we obtain

$$v_k(t) = (Pv_k)(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Qv_k)(s)\, ds.$$

Since $v_k(t) = u(t)$ for almost every $t \in [0, t_k]$, we find

$$u(t) = (Pu)(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Qu)(s)\, ds$$

almost everywhere $t \in [0, t_k]$, and thus, $u(\cdot)$ is a solution for (4.1) in $L^p_{\text{loc}}([0, \infty), E)$. $\square$

5. Examples.

Example 5.1. Riemann-Liouville fractional differential equations. Consider the fractional differential equation involving the Riemann-Liouville derivative in Banach space E:

$$\begin{cases} D^\alpha u(t) = f(t, u(t)) & t > 0, \\ t^{1-\alpha} u(t)|_{t=0} = u_0, \end{cases} \tag{5.1}$$

where $f(\cdot, \cdot) : [0, \infty) \times E \to E$ satisfies the following conditions:

(F1) (a) $f(\cdot,\cdot)$ is a Carathéodory function, that is, $t \mapsto f(t,u)$ is strongly measurable for every $u \in E$, $u \to f(t,u)$ is continuous for almost every $t \in [0,\infty)$;

(b) there exist $c(\cdot) \in L^p_{\text{loc}}([0,\infty),\mathbb{R}_+)$ and $d > 0$ with

$$\|f(t,u(t))\| \le c(t) + d\|u(t)\|, \text{ for almost every } t \in [0,\infty); \tag{5.2}$$

(F2) there exists a $k_1 > 0$ such that

$$\beta(f(t,B)) \le k_1\beta(B) \text{ almost everywhere on } [0,\infty), \tag{5.3}$$

for every bounded set $B \in E$.

Let $a \in (0,\infty)$ be given. It is well known that, under condition (F1), the Nemytskii operator, defined by $(Qu)(t) := f(t,u(t))$, is a continuous abstract Volterra operator for $L^p([0,a],E)$ into itself, see **[19,** Theorem 3.4.4]. Also, by (F2), we have that

$$\beta\left((QB)(t)\right) \le \beta(f(t,B)) \le k_1\beta(B) \text{ almost everywhere on } [0,a], \tag{5.4}$$

for every bounded set $B \in E$. Consequently, (H1)–(H3) are satisfied.

Next, since

$$\left(\int_0^a t^{p(\alpha-1)}\,dt\right)^{1/p} = \left(\frac{1}{p(\alpha-1)+1}\right)^{1/p} a^{[p(\alpha-1)+1]/p} < \infty$$
$$\text{for } p < \frac{1}{1-\alpha},$$

it follows that the function $g(t) := t^{\alpha-1}u_0$ belongs to $L^p([0,a],E)$ if and only if $p < (1/1-\alpha)$. Also, it is well known, see **[43]**, that a function $u(\cdot) \in \Omega^{p,\alpha}([0,a],E)$ is a solution of (5.1) if and only if $u(\cdot)$ is a solution of the integral equation

$$u(t) = t^{\alpha-1}u_0 + \int_0^t \frac{t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\,ds,$$

which can be written as

$$u(t) = g(t) + \int_0^t \frac{t-s)^{\alpha-1}}{\Gamma(\alpha)}(Qu)(s))\,ds \text{ almost everywhere } t \in [0,a]. \tag{5.5}$$

Then, from

$$\begin{aligned}\|u(t)\| &\le \|g(t)\| + \int_0^t \frac{t-s)^{\alpha-1}}{\Gamma(\alpha)} \|(Qu)(s))\| \, ds \\ &\le \int_0^t \frac{t-s)^{\alpha-1}}{\Gamma(\alpha)} [c(t) + d\|u(t)\|] \, ds \\ &\le \|g(t)\| + \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p - 1}\right)^{1-1/p} \\ &\quad \cdot t^{\alpha-(1/p)} \left[\|c(\cdot)\|_p + d\|u(\cdot)\|_p\right],\end{aligned}$$

it follows that

$$\|u(\cdot)\|_p \le \|g(\cdot)\|_p + \gamma \left[\|c(\cdot)\|_p + d\|u(\cdot)\|_p\right].$$

This leads to

$$\|u(\cdot)\|_p \le r := \frac{\|g(\cdot)\|_p + \gamma\|c(\cdot)\|_p}{1-d\gamma}$$

if and only if $1/\alpha < p$ and $1 - d\gamma > 0$. Hence, a solution of the integral equation (5.5) is a priori bounded if and only if $1/\alpha < p$ and $1-d\gamma > 0$.

Summarizing, we have $1/\alpha < p < (1/(1-\alpha))$ and $\alpha \in (1/2, 1)$. Then, from Corollary 3.5, it follows that the integral equation (5.5) has at least one solution in $L^p([0,a],E)$, for every $a \in (0,\infty)$.

Consequently, by Theorem 4.1, we obtain the next result.

Theorem 5.2. *Let $p > 1$ be such that $(1/\alpha) < p < (1/(1-\alpha))$ with $\alpha \in (1/2, 1)$. If conditions* (F1) *and* (F2) *hold, then the fractional differential equation* (5.6) *has at least one solution in $L^p_{\mathrm{loc}}([0,\infty),E)$, provided $1 - d\gamma > 0$.*

Example 5.3. Neutral fractional differential equations. Consider the following neutral fractional differential equation

(5.6)

$$\begin{cases} {}^C D^\alpha [u(t) - \int_0^t K(t,s) g(s,u(s)) ds] = f(t,u(t)) & \text{for almost every } t > 0, \\ u(0) = u_0, \end{cases}$$

involving the Caputo derivative where $0 < \alpha < 1$, $f(\cdot,\cdot), g(\cdot,\cdot) : [0,\infty) \times E \to E$ and $K : \triangle = \{(s,t) : 0 \le s,t \le \infty\} \to \mathcal{L}(E)$. Assume that the following conditions hold.

(HF) $f(\cdot,\cdot)$ satisfies condition (F1),

(HG) (a) $g(\cdot,\cdot)$ is a Carathéodory function, and there exist $b_1(\cdot) \in L^p_{\mathrm{loc}}([0,\infty),\mathbb{R}_+)$ and $d_1 > 0$ such that

$$\|g(t,u(t))\| \le b_1(t) + d_1 \|u(t)\| \text{ for almost every } t \ge 0;$$

(b) there exists a $k_2 > 0$ such that

$$\beta(g(t,B)) \le k_2 \beta(B) \text{ almost everywhere on } [0,\infty), \tag{5.7}$$

for every bounded set $B \in E$;

(HK) Also, for each $a > 0$, there exists an $M_a > 0$ such that

$$\operatorname*{ess\,sup}_{t\in[0,a]} \left(\int_0^a \|K(t,s)\|^q ds \right)^{1/q} := M_a < \infty.$$

for $p, q > 1$ with $p(1-1/q) > 1$.

We recall that a function $u(\cdot) : [0,a] \to E$ is said to be a solution of (5.6) on $[0,a]$ if $u(0) = u_0$, the function

$$t \longmapsto u(t) - \int_0^t K(t,s) g(s,u(s))\, ds$$

is absolutely continuous and differentiable almost everywhere on $[0,a]$, and satisfies (5.6) for $t \in [0,a]$. Note that $u(\cdot)$ itself may not be absolutely continuous and differentiable almost everywhere on $[0,a]$.

For a given $a > 0$, it is easy to see that, if $u(\cdot) : [0,a] \to E$ is a solution of (5.6), then it satisfies the integral equation:

$$\begin{aligned} u(t) = u_0 &+ \int_0^t K(t,s) g(s,u(s))\, ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds, \quad t \in [0,a], \end{aligned} \tag{5.8}$$

Conversely, if $u(\cdot)$ satisfies (5.8) then $u(\cdot)$ is a solution of (5.6) on $[0,a]$. Also, it is easy to see that (5.8) can be written as

$$u(t) = (Pu)(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (Qu)(s)\, ds, \quad t \in [0,a], \tag{5.9}$$

where the operators P and Q are defined as follows. The Nemytskii operator, defined by $(Qu)(t) := f(t,u(t))$, is a continuous abstract

Volterra operator for $L^p([0,a],E)$ into itself, and it satisfies (5.4). Consequently, Q satisfies conditions (H2) and (H3).Also, under conditions (HG) (a) and (HK), the Volterra operator

$$(Pu)(t) = u_0 + \int_0^t K(t,s)g(s,u(s))\,ds$$

is a continuous, abstract Volterra operator from $L^p([0,a],E)$ into itself, see [**20,** Lemma 2.3.1] and [**16,** Theorem 9.5.6]. Moreover, condition (HG) (b) assures the compactness of the operator P, see [**20,** Theorem 2.3.1]. Consequently, operator P satisfies condition (H1). Then from Theorem 3.2, it follows that the integral equation (5.9) has at least one solution in $L^p([0,a],E)$ for every $a \in (0,\infty)$. Hence, by Theorem 4.1, we obtain the following result.

Theorem 5.4. *If conditions* (HF), (HG) *and* (HK) *hold, then the fractional differential equation* (5.9) *has at least one solution in* $L^p_{\mathrm{loc}}([0,\infty), E)$*, provided* $1 - d\gamma > 0$.

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