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RECENT PROGRESS IN TIME DOMAIN BOUNDARY INTEGRAL EQUATIONS

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The last decade has witnessed very intense activity in the field of boundary integral equations applied to evolutionary processes. By Time Domain Boundary Integral Equations (TDBIE) we understand the use of layer potential theory based on the full dynamical problem (parabolic or hyperbolic), using either the time domain fundamental solution or the fundamental solution for the resolvent operator, as opposed to uses of static BIE combined with time stepping procedures for evolutionary processes.

The engineering and mathematical literature of TDBIEs goes back well into the 20th century, with the appearance of methods based on the heat kernel boundary representation of solutions of transient linear diffusive processes, or practical uses of Kirchhoff's formula for wave propagation problems. The mathematical literature on the topic typically credits the two-part article of Alain Bamberger and Tuong Ha-Duong **[2, 3]** with the eclosion of a rich literature on theory and applications of TDBIE for wave propagation problems. This happened very much at the same time that the hidden coercivity of the single layer potential for the heat equation **[1]** was uncovered, which sparked a wealth of discretization methods for space-time parabolic boundary integral equations. Discretization methods for the TDBIE for hyperbolic problems were mainly based on Galerkin-in-space discretization combined with weighted Galerkin-in-time time stepping, although it was clear from the beginning that the weight was an annoyance introduced by the theoretical analysis and its elimination led to practicable schemes. Christian Lubich's Convolution Quadrature (CQ) method was originally devised in another two-part paper **[9, 10]** as a discretization method for abstract causal convolution processes. Since TDBIEs are Volterra-Fredholm integral equations (convolutional Volterra in time, to be more precise), it was natural that CQ would soon be applied for TDBIE, first of parabolic **[12]** and then of hyperbolic type **[11]**.

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In 2003, two review papers by Tuong Ha-Duong [**7**] and Martin Costabel [**5**][1] set the state-of-the-art for Boundary Integral Equations for dynamical problems. For many years, they were the only mathematically oriented surveys of the area, and they remain dependable resources for the concepts of TDBIE and the 'early' literature. It can be said, though, that the mathematical literature of TDBIE was slightly dormant at the time of the arrival of the new millenium, while interest in applications in engineering and applied sciences persisted in the work of several computational groups in solid mechanics and electromagnetism. Like their frequency domain or steady-state counterparts, TDBIEs remain complicated-to-code methods that work excellently for a precise but very relevant family of problems. Even their stronger proponents seem to agree that they will never become the paradigm for easy numerical schemes, but their strength is well understood and appreciated.

Several papers restarted the interest of the mathematical community in theory and implementation of TDBIE, but it is possible that the work of Wolfgang Hackbusch, Wendy Kress, and Stefan Sauter [**6**] was one of the first to show the way forward. The Convolution Quadrature technique gained momentum, and the realm of applications as well as the depth of the mathematical literature grew rapidly. It was only a matter of time (not long) until Galerkin TDBIE came back on the publishing stage. CQ and Galerkin approaches now share the spotlight with alternative old and new time-stepping tools and novel space-and-time methods.

As the topic has matured and the community of practitioners and theorists has grown, more tutorial work is available. Two recent papers [**4, 8**] can be used as practical introductions to Convolution Quadrature based TDBIE for wave propagation problems, and the equally CQ-based wave-equation-focused monograph [**13**] collects part of the old and new theoretical approaches to the subject. (In the interest of full disclosure, one of the editors of this special issue is involved in the authorship of the monograph and of one of the tutorial papers.)

In this special issue, we collect six articles with surveys, comparisons, and novel results on the general topic of TDBIEs. Without going into the details of what each has to offer, let us give a short description of the topics that are covered in them.

- (Davies and Duncan) Volterrá convolutional equations of the first kind with singular or discontinuous kernels are a prototype of the difficulties for time-stepping of TDBIEs. Spline convolutional techniques are studied as low-dispersion high-order scheme for their numerical treatment.
- (Aimi, Diligenti, and Guardasoni) A simple model for wave propagation with damping in a one-dimensional domain is used as a testing field for comparison of different discretization techniques, posing the Galerkin TDBIE (also referred to as energetic BEM when no weight is used) as a competitor of Finite Element and Finite Difference discretizations.
- (Gimperlein, Maischak, and Stephan) Adaptive treatment of TDBIEs is an area of very recent development. Ideas on the time-and-space mesh adaptation for Galerkin schemes for an acoustic wave propagation problems are presented in the context of engineering applications.
- (Melenk and Rieder) A transient BEM-FEM scheme is presented for a linear Schrödinger equation, with the novelty that the Runge-Kutta CQ method is analyzed using time domain techniques instead of Laplace domain (resolvent) estimates.
- (Hassell, Qiu, Sánchez-Vizuet, and Sayas) A new approach to the analysis of semidiscrete TDBIEs is shown using theory of evolutionary equations, by rewriting the problem as a first order in space and time evolution equation in a Hilbert space.
- (Joly and Rodríguez) Finally, Galerkin BEM for acoustics are reviewed, with general arguments on the need for weights in the time stepping process and with full computation for the one-dimensional operators. This paper can be used as a very nicely presented tutorial on the topic.

We hope that the present volume will offer the reader a collection of papers offering the state-of-the-art of TDBIE as of the end of 2016.

ENDNOTES

1. A new edition of the Stein, De Boorst and Hughes, *Encyclopedia of computational mechanics* has been in the works for some time. This will contain a revised version of [**5**].

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