ON SOME REGULAR FRACTIONAL
STURM-LIOUVILLE PROBLEMS WITH
GENERALIZED DIRICHLET CONDITIONS

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ABSTRACT. The present work deals with some spectral properties of the problem

$$(\mathcal{P}) \begin{cases} D_{b,-}^\alpha (p(x) D_{a,+}^\alpha y)(x) + \lambda q(x)\, y(x) = 0, & a < x < b, \\ \lim_{\substack{x \to a \\ >}} (x-a)^{1-\alpha} y(x) = 0 = y(b), \end{cases}$$

where $p, q \in C([a,b])$, $p(x) > 0$, $q(x) > 0$, for all $x \in [a,b]$ and $1/2 < \alpha < 1$. $D_{b,-}^\alpha$ and $D_{a,+}^\alpha$ are the right- and left-sided Riemann-Liouville fractional derivatives of order $\alpha \in (0,1)$, respectively. λ is a scalar parameter.

First, we prove, using the spectral theory of linear compact operators, that this problem has an infinite sequence of real eigenvalues and the corresponding eigenfunctions form a complete orthonormal system in the Hilbert space $L_q^2[a,b]$. Then, we investigate some asymptotic properties of the spectrum as $\alpha \underset{<}{\to} 1$. We give, in particular, the asymptotic expansion of the first eigenvalue.

1. Introduction. Sturm-Liouville theory has an important role in mathematics, physics and engineering. During the last few years, fractional Sturm-Liouville problems have appeared in the literature. They were introduced by Klimek and Agrawal [**7, 8**]. Both Riemann-Liouville and Caputo derivatives were used. These authors proved that the eigenvalues are real and the eigenfunctions are orthogonal by using a Green-type formula as in the classical case.

Another important research area concerns the inverse problem in diffusion equations with fractional derivatives, namely, the problem of recovery of the diffusion potential from spectral data and the initial condition. Recently, much progress has been made in this direction by Tuan [18] and Bas [3], where one can find a wide number of references on the subject and many interesting comments. In [16], more problems have been presented. Also, similar problems to our $(\mathcal{P})$ have been considered by Bas et al. [2, 4] with singular coefficients $p(\cdot)$ as in Bessel-type equations.

The purpose of this paper is to investigate some basic spectral properties of the fractional Sturm-Liouville problem $(\mathcal{P})$ with generalized Dirichlet conditions. By *generalized Dirichlet conditions*, we mean

$$(1.1) \qquad \lim_{\substack{x \to a \\ >}} (x-a)^{1-\alpha} y(x) = 0 = y(b).$$

Note that this kind of condition involves continuous as well as non-continuous functions at the point a. For example,

$$\lim_{\substack{x \to a \\ >}} \ln(x-a) = -\infty \quad \text{but} \quad \lim_{\substack{x \to a \\ >}} (x-a)^{1-\alpha} \ln(x-a) = 0.$$

Obviously, Dirichlet conditions are a particular case of the generalized ones, and the limit in equation (1.1) implies that y is in L^1.

We start by considering the problem $(\mathcal{P})$ on the domain Dom where $y \in Dom$ if and only if y verifies the following conditions:

(C1) $y \in C(]a,b])$ and $\lim_{\substack{x \to a \\ >}} (x-a)^{1-\alpha} y(x) = 0$

(C2) $D_{a,+}^{\alpha} y \in C([a,b[)$ and $\lim_{\substack{x \to b \\ <}} (b-x)^{1-\alpha} D_{a,+}^{\alpha} y(x)$ exists,

with the aim of transform $(\mathcal{P})$ into an integral equation problem, which can be analyzed by classical tools of operator theory. Condition **(C1)** is nothing but the generalized Dirichlet condition at $x = a$. The second, **(C2)**, will appear naturally further on in the calculation of the integral operator.

The number λ is said to be an eigenvalue for problem $(\mathcal{P})$ if there exists a function in Dom, not identically zero, which satisfies the fractional regular Sturm-Liouville equation and the generalized Dirichlet conditions.

In what follows, we prove that the set of eigenvalues of $(\mathcal{P})$ is real and infinite. This result partially answers a question in [**16, page 9, Remark 1**], where the situation is quite similar. However, the simplicity of the eigenvalues is still an open problem.

The corresponding eigenfunctions form a complete orthonormal system in the Hilbert space $L_q^2[a, b]$ endowed with the usual scalar product and the associated norm:

$$\langle u, v \rangle_q = \int_a^b u(x)\,\overline{v(x)}\,q(x)\,dx, \quad \|u\|_q = \sqrt{\langle u, u \rangle_q}.$$

This result follows from the theory of linear compact operators and the fractional Green's function properties, following the approach in [**10**] adapted to our generalized boundary conditions. The second result investigates the asymptotic behavior of the first eigenvalue of $(\mathcal{P})$ when $\alpha \underset{<}{\to} 1$. This can be the starting point to put the classical and the fractional cases in the same theory.

2. Basic concepts. In this section, we recall some definitions and properties of Riemann-Liouville fractional integrals and fractional derivatives used below. For more details see [**11, 14, 15, 17**].

Definition 2.1 ([**11, 15**])**.** Let $f \in C([a, b])$. The right- and left-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined, respectively, by

$$(2.1) \qquad (I_{b,-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t)\,dt$$

and

$$(2.2) \qquad (I_{a,+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)\,dt$$

where Γ is the Euler gamma function.

Definition 2.2 ([**11, 15**])**.** The right- and left-sided Riemann-Liouville fractional derivatives of order $\alpha \in (0,1)$ are defined by:

$$(2.3) \qquad\qquad (D_{b,-}^{\alpha} f)(x) = -\frac{d}{dx}(I_{b,-}^{1-\alpha} f)(x)$$

and

$$(2.4) \qquad\qquad (D_{a,+}^{\alpha} f)(x) = \frac{d}{dx}(I_{a,+}^{1-\alpha} f)(x),$$

respectively.

Proposition 2.3 ([**11, 15, 17**])**.** *Let $0 < \alpha < 1$ and $f \in C((a,b))$.*

(i) $[(D_{b,-}^{\alpha} \; o \; I_{b,-}^{\alpha}) f](x) = f(x).$
(ii) *If $I_{b,-}^{1-\alpha} f \in AC([a,b])$, then we have*

$$[(I_{b,-}^{\alpha} o \, D_{b,-}^{\alpha}) f](x) = f(x) - \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} \lim_{\substack{x \to b \\ <}} (I_{b,-}^{1-\alpha} f)(x).$$

(iii) $[(D_{a,+}^{\alpha} \; o \; I_{a,+}^{\alpha}) f](x) = f(x).$
(iv) *If $I_{a,+}^{1-\alpha} f \in AC([a,b])$, then we have*

$$[(I_{a,+}^{\alpha} o \, D_{a,+}^{\alpha}) f](x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \lim_{\substack{x \to a \\ >}} (I_{a,+}^{1-\alpha} f)(x).$$

Lemma 2.4 ([**11,** page 151, Lemma 3.2])**.** *Let $0 < \alpha < 1$ and $y(x)$ be a Lebesgue measurable function on $[a,b]$.*

(i) *If there exists a limit*

$$\lim_{\substack{x \to a \\ >}} (x-a)^{1-\alpha} y(x) = c, \quad c \in \mathbb{C},$$

then the following limit also exists

$$\left(I_{a,+}^{1-\alpha} y\right)(a+) := \lim_{\substack{x \to a \\ >}} \left(I_{a,+}^{1-\alpha} y\right)(x) = c\,\Gamma(\alpha).$$

(ii) *Symmetrically, if there exists a limit*

$$\lim_{\substack{x \to b \\ <}} (b-x)^{1-\alpha} y(x) = d, \quad d \in \mathbb{C},$$

then the following limit also exists

$$\left(I_{b,-}^{1-\alpha}y\right)(b-) := \lim_{\substack{x \to b \\ <}} \left(I_{b,-}^{1-\alpha}y\right)(x) = d\,\Gamma(\alpha).$$

Finally, we recall the following fractional integration by parts formula, proved in [**17,** page 46, Corollary 2] (with simple sufficient conditions).

Proposition 2.5. *Let* $f, g, (D_{a,+}^{\alpha}f), (D_{b,-}^{\alpha}g) \in C([a,b])$. *Then*

$$(2.5) \qquad \int_a^b (D_{a,+}^{\alpha}f)(x)\, g(x)\, dx = \int_a^b f(x)\, (D_{b,-}^{\alpha}g)(x)\, dx.$$

3. Main results.

3.1. Spectral properties. In this subsection, we prove that problem $(\mathcal{P})$ admits an infinite sequence of eigenvalues, which are real and negative. Also, the normalized eigenfunctions form an orthonormal basis in $L_q^2[a,b]$.

Proposition 3.1. *Let* $\alpha \in ((1/2), 1]$. *Then*

(i) *for all* $y \in Dom$, *problem* $(\mathcal{P})$ *is equivalent to*

$$y(x) = \lambda(T_\alpha y)(x),$$

where T_α *is the linear operator defined by*

$$T_\alpha : Dom \longrightarrow Dom$$

$$y(\cdot) \longmapsto (T_\alpha\, y)(x) = \int_a^b G_\alpha(x,\tau)\, q(\tau)\, y(\tau)\, d\tau$$

$G_\alpha(x,\tau)$ *is the fractional Green's function defined by*

$$(3.1) \qquad G_\alpha(x,\tau) = \begin{cases} \dfrac{\phi(b,x)\,\phi(b,\tau)}{\phi(b,b)\,\Gamma(\alpha)} - \dfrac{\phi(x,\tau)}{\Gamma(\alpha)}, & \text{if } a \le \tau \le x \\[2mm] \dfrac{\phi(b,x)\,\phi(b,\tau)}{\phi(b,b)\,\Gamma(\alpha)} - \dfrac{\phi(\tau,x)}{\Gamma(\alpha)}, & \text{if } x \le \tau \le b, \end{cases}$$

and

(3.2) $$\phi(u, v) = I_{a,+}^{\alpha}\left(\frac{(u-v)^{\alpha-1}}{p(v)}\right)(v)$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^v (u-\tau)^{\alpha-1}(v-\tau)^{\alpha-1}\frac{d\tau}{p(\tau)}.$$

(ii) T_α *extends to a compact, self-adjoint operator on* $L_q^2[a, b]$ *and*

$$\mathrm{Ker}(T_\alpha) = \{0\}.$$

(iii) *All the eigenvalues of problem* $(\mathcal{P})$ *are negative.*

Proof.

(i) By applying $I_{b,-}^{\alpha}$ to the equation

$$D_{b,-}^{\alpha}(p(x)\,D_{a,+}^{\alpha}y)(x) + \lambda q(x)\,y(x) = 0,$$

we obtain

$$(D_{a,+}^{\alpha}\,y)(x) = \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)\,p(x)}\lim_{\substack{x\to b\\<}}(I_{b,-}^{1-\alpha}p(x)D_{a,+}^{\alpha}y)(x)$$

$$-\frac{\lambda}{p(x)}(I_{b,-}^{\alpha}q(x)\,y(x))(x).$$

Now, we apply the operator $I_{a,+}^{\alpha}$. Then

(3.3) $$y(x) = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\lim_{\substack{x\to a\\>}}(I_{a,+}^{1-\alpha}y)(x) + \frac{C\,\phi(b,x)}{\Gamma(\alpha)}$$

$$-\lambda I_{a,+}^{\alpha}\left(\frac{1}{p(x)}(I_{b,-}^{\alpha}q(x)\,y(x))\right)(x),$$

where, according to Proposition 2.3 (ii), we have

$$C = \lim_{\substack{x\to b\\<}}(I_{b,-}^{1-\alpha}p(x)D_{a,+}^{\alpha}y)(x)$$

and $\phi(b, x)$ is defined in equation (3.2).

- Note that the generalized Dirichlet condition

$$\lim_{\substack{x\to a\\>}}(x-a)^{1-\alpha}y(x) = 0$$

implies that

$$\lim_{\substack{x \to a \\ >}} (I_{a,+}^{1-\alpha} y)(x) = 0.$$

according to Lemma 2.4.

• The application of the second boundary value condition $y(b) = 0$ gives

$$C = \frac{\lambda \Gamma(\alpha)}{\phi(b,b)} \lim_{\substack{x \to b \\ <}} I_{a,+}^{\alpha} \left(\frac{1}{p(x)} (I_{b,-}^{\alpha} q(x) \, y(x)) \right)(x),$$

where $\phi(b,b) = \lim_{\substack{x \to b \\ <}} \phi(b,x)$ exists for $1/2 < \alpha \le 1$, since the integral

(3.4)
$$\int_a^b (b-x)^{2\alpha-2} \frac{dx}{p(x)}$$

converges if and only if $2\alpha - 1 > 0$.

Let us compute the constant C. By using Fubini's theorem, we obtain:

(3.5) $\quad I_{a,+}^{\alpha} \left(\frac{1}{p(x)} (I_{b,-}^{\alpha} q(x) \, y(x)) \right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \phi(x,\tau) \, q(\tau) \, y(\tau) \, d\tau$

$$+ \frac{1}{\Gamma(\alpha)} \int_x^b \phi(\tau,x) q(\tau) \, y(\tau) \, d\tau.$$

Then we have

$$C = \frac{\lambda \Gamma(\alpha)}{\phi(b,b)} \lim_{\substack{x \to b \\ <}} I_{a,+}^{\alpha} \left(\frac{1}{p(x)} (I_{b,-}^{\alpha} q(x) \, y(x)) \right)(x)$$

$$= \frac{\lambda}{\phi(b,b)} \int_a^b \phi(b,\tau) q(\tau) \, y(\tau) \, d\tau.$$

Thus, by substituting the first boundary value condition, the constant C and the integral (3.5) into equation (3.3) we obtain the first implication.

We can easily verify that, for all $y \in Dom$,

$$T_\alpha y \in Dom.$$

Then, the equivalence stated above follows immediately.

(ii) We can obtain the results from the properties of the fractional Green's function. The operator T_α defined on Dom can be continuously extended to $L_q^2[a, b]$ because Dom is dense, since it contains the space of infinitely differentiable functions with compact support. The compactness of T_α follows from the properties below. In fact,

(a) for all $(x, \tau) \in [a, b] \times [a, b]$, $G_\alpha(x, \tau)$ is a continuous function because $\phi(x, \tau)$ is a continuous function in the domain $a \leq \tau \leq x \leq b$ and $\lim\limits_{\substack{\tau \to x \\ <}} G_\alpha(x, \tau) = \lim\limits_{\substack{\tau \to x \\ >}} G_\alpha(x, \tau)$, the kernel $G_\alpha \in L_{q \otimes q}^2([a, b] \times [a, b])$ therefore T_α is a Hilbert-Schmidt operator, and thus compact (see [**5, 19**]).

(b) For all $(x, \tau) \in [a, b] \times [a, b]$, $G_\alpha(x, \tau) = G_\alpha(\tau, x)$, then G is a symmetric function on $[a, b] \times [a, b]$ so the operator T_α is self-adjoint in $L_q^2[a, b]$.

(c) To prove that the kernel of T_α is reduced to $\{0\}$, it suffices to solve the equation

$$(T_\alpha y)(x) = 0, \quad \text{for all } x \in [a, b].$$

By application of the left- and right-sided Riemann-Liouville fractional derivatives, respectively, we obtain the result after some simple calculations.

(iii) Let us consider the fractional Sturm-Liouville equation

$$D_{b,-}^\alpha (p(x) D_{a,+}^\alpha y)(x) + \lambda q(x)\, y(x) = 0.$$

By multiplying this equation by $y(x)$ and integrating it on $[a, b]$ we get

$$\int_a^b D_{b,-}^\alpha (p(x) D_{a,+}^\alpha y)(x)\, y(x)\, dx + \lambda \int_a^b q(x) y^2(x)\, dx = 0$$

Now, by applying Proposition 2.5 (2.5), which is also satisfied with the generalized Dirichlet conditions (1.1), we obtain

$$\int_a^b p(x) \left(D_{a,+}^\alpha y\right)^2 (x)\, dx + \lambda \int_a^b q(x) y^2(x)\, dx = 0;$$

hence, λ must be negative.

$\square$

Theorem 3.2. *Let us consider the fractional regular Sturm-Liouville problem* $(\mathcal{P})$ *for* $1/2 < \alpha < 1$. *Then*

(i) *The problem* $(\mathcal{P})$ *has an infinite countable set* $\{\lambda_n\}_{n\geq 1}$ *of real negative eigenvalues such that* $\lim_{n\to\infty} \lambda_n = -\infty$, *which can be ordered as*:

$$\cdots \leq \lambda_n \leq \cdots \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \leq 0.$$

(ii) *The normalized eigenfunctions* $\{\phi_n\}_n$ *corresponding to the eigenvalues* $\{\lambda_n\}_n$ *are orthogonal in* $L_q^2[a,b]$, *that is, for all* $i \neq j$,

$$\int_a^b \phi_i(x)\phi_j(x)q(x)\,dx = 0.$$

(iii) *Let* $\{\phi_{n,j}\}_{j=1}^{d_n}$ *be an orthonormal basis of* E_n, *the eigenspace corresponding to* λ_n $(d_n = \dim E_n)$. *The eigenfunctions* $\{\phi_{n,j}\}_{n,j}$ *form an orthonormal basis of the Hilbert space* $L_q^2[a,b]$, *that is, for all* $y \in L_q^2[a,b]$,

$$y = \lim_{N\to\infty} \sum_{n=1}^N \sum_{j=1}^{d_n} < y, \phi_{n,j} >_q \phi_{n,j}.$$

Proof. Let λ be a non-zero eigenvalue of $(\mathcal{P})$ and μ a non-zero eigenvalue of T_α. Then, according to Proposition 3.1, we have

$$(\mathcal{P}) \Longleftrightarrow \frac{1}{\lambda} y = T_\alpha y$$

and

$$\mu y = T_\alpha y \Longleftrightarrow \begin{cases} D_{b,-}^\alpha (p(x)D_{a,+}^\alpha y)(x) + \dfrac{1}{\mu}q(x)\,y(x) = 0 \\ \lim\limits_{\substack{x\to a \\ >}}(x-a)^{1-\alpha}\,y(x) = 0 = y(b). \end{cases}$$

The homogeneous problem does not have a non-zero solution; hence, zero is neither an eigenvalue of $(\mathcal{P})$ nor an eigenvalue of T_α.

Then, the non-zero eigenvalues of $(\mathcal{P})$ are exactly the inverses of those of T_α. These last inverses are real countable, negative, tend to

zero when $n \to \infty$ and they can be ordered as:

$$|\mu_1| \geq |\mu_2| \geq |\mu_3| \geq \cdots \geq |\mu_k| \geq \cdots \to 0$$

because T_α is self-adjoint and compact in $L_q^2[a, b]$. We can also compute the properties of the eigenfunctions.

Now, we prove the infinity of eigenvalues by contradiction. Let us assume that T_α has a finite sequence of non-zero eigenvalues. The spectral theorem of compact self-adjoint operators gives:

$$L_q^2[a, b] = (\operatorname{Ker} T_\alpha) \oplus \left(\bigoplus_{k=1}^{N} E_n \right).$$

Since the corresponding eigenspaces $E_1, \ldots, E_n$ are finite-dimensional and, according to Proposition 3.1 we have $\operatorname{Ker} T_\alpha = \{0\}$, then $L_q^2[a, b]$ must be finite-dimensional, which is a contradiction. $\square$

Remark 3.3. The integral (3.4) converges if and only if $\alpha > 1/2$. Thus, the operator T_α remains compact and self-adjoint even if $\alpha \geq 1$. But the corresponding fractional Sturm-Liouville problem must contain many more boundary conditions.

Recall the expression of the Hilbert-Schmidt norm of a compact integral operator on $L_q^2[a, b]$ [**19**]. If

$$(Sy)(x) = \int_a^b K(x, t)\, y(t) q(t) dt,$$

then

(3.6) $$\|S\|_{HS}^2 = \int_a^b \int_a^b |K(x, t)|^2 q(t) q(x)\, dt\, dx.$$

Proposition 3.4. *All of the eigenvalues of problem* $(\mathcal{P})$ *verify:*

$$|\lambda| \geq \frac{1}{\|T_\alpha\|_{HS}}.$$

Proof. If λ is an eigenvalue, then there exists a $y \neq 0$ such that

$$y = \lambda T_\alpha y$$

so

$$\|y\|_q = |\lambda| \cdot \|T_\alpha y\|_q \leq |\lambda| \cdot \|T_\alpha\| \cdot \|y\|_q \leq |\lambda| \cdot \|T_\alpha\|_{HS} \cdot \|y\|_q.$$

Then

$$|\lambda| \geq \frac{1}{\|T_\alpha\|_{HS}}. \qquad \square$$

Remark 3.5. The optimal disc free from eigenvalues has $1/\|T_\alpha\|$ as the radius. But the operator norm is more difficult to compute than the Hilbert-Schmidt. In our case,

$$\|T_\alpha\|_{HS}^2 = \int_a^b \int_a^b |G_\alpha(x,t)|^2 q(t) q(x) \, dt \, dx$$

can be computed for explicit functions p and q.

3.2. Asymptotic behavior. In this section, we prove that the first eigenvalue of T_α is close to the first eigenvalue of the following problem

$$(\mathcal{P}_1) \begin{cases} -(p(x)\,y')'(x) + \lambda q(x)\,y(x) = 0, & a < x < b \\ y(a) = 0 = y(b), \end{cases}$$

when $\alpha \underset{<}{\to} 1$.

To prove this result, we need some lemmas and propositions.

Lemma 3.6. *The nth derivative of $1/\Gamma(\alpha)$ and $\Gamma(\alpha)$ have the following properties. For all $n \geq 0$ there exists $M_n > 0$, for all $\alpha \in [1/2, 1]$,*

$$\left| \left(\frac{1}{\Gamma(\alpha)} \right)^{(n)} \right| \leq M_n$$

(3.7)
$$and$$

$$\left| (\Gamma(\alpha))^{(n)} \right| \leq M_n.$$

Proof. The functions $1/\Gamma(\cdot)$, $\Gamma(\cdot) \in C^\infty(]0, +\infty[)$, then they are bounded on compact sets, as well as all of their derivatives. $\qquad \square$

Lemma 3.7. *For $a \leq t \leq x \leq b$, we have the following property for all $\varepsilon > 0$ and for all $0 \leq j \leq k$, there exists $C_{k,j}(\varepsilon) > 0$ $/1/2 + \varepsilon < \alpha \leq 1$,*

$$(3.8) \quad \left| \int_a^t (x - \tau)^{\alpha-1} \ln^j (x - \tau)(t - \tau)^{\alpha-1} \ln^{k-j}(t - \tau)\, d\tau \right| \leq C_{k,j}(\varepsilon).$$

Proof. Let us consider the function $f_m(\xi, \beta) = \xi^\beta \ln^m(\xi)$ with $m \in \mathbb{N}$ and $0 < \beta < \alpha - (1/2)$. It is not hard to see that $f_m(\xi, \beta)$ is uniformly bounded in $[0, c] \times [0, \alpha - 1/2]$. Therefore,

$$\left| \int_a^t (x - \tau)^{\alpha-1} \ln^j (x - \tau)(t - \tau)^{\alpha-1} \ln^{k-j}(t - \tau)\, d\tau \right|$$

$$= \left| \int_a^t (x - \tau)^{\alpha-\beta-1} f_j((x - \tau), \beta)\, (t - \tau)^{\alpha-\beta-1}\, f_{k-j}((t - \tau), \beta)\, d\tau \right|$$

$$\leq A_{k,j}(\varepsilon) \int_a^t (x - \tau)^{\alpha-\beta-1}(t - \tau)^{\alpha-\beta-1}\, d\tau$$

$$\leq \frac{A_{k,j}(\varepsilon)}{(2\alpha - 2\beta - 1)} \sqrt{(x - a)^{2\alpha-2\beta-1} - (x - t)^{2\alpha-2\beta-1}} \cdot \sqrt{(t - a)^{2\alpha-2\beta-1}}$$

$$\leq \frac{A_{k,j}(\varepsilon)}{(2\alpha - 2\beta - 1)} (b - a)^{2\alpha-2\beta-1}$$

where we used the Cauchy-Schwartz inequality. By choosing $2\beta = \alpha - (1/2)$, we get

$$\left| \int_a^t (x - \tau)^{\alpha-1} \ln^j (x - \tau)(t - \tau)^{\alpha-1} \ln^{k-j}(t - \tau)\, d\tau \right| \leq \frac{A_{k,j}(\varepsilon)}{\varepsilon} d_\varepsilon \left(\frac{1}{2} \right)$$

$$= C_{k,j}(\varepsilon),$$

where

$$d_\varepsilon(\gamma) = \begin{cases} (b - a)^\gamma & \text{if } b \geq a + 1 \\ (b - a)^{2\gamma\varepsilon} & \text{if } b < a + 1. \end{cases} \qquad \square$$

Lemma 3.8. *Let g be the function defined by*

$$g(\alpha) = \int_a^b (b - \tau)^{2\alpha-2} \frac{d\tau}{p(\tau)}.$$

Then g has the following properties:

(i) *for all $\varepsilon > 0$, there exists*

$$K_{\min}, K_{\max} > 0/(1/2) + \varepsilon < \alpha \leq 1 \Longrightarrow K_{\min} \leq |g(\alpha)| \leq K_{\max}.$$

(ii) *For all $\varepsilon > 0$ and $n \geq 0$, there exists*

$$\eta_n > 0/(1/2) + \varepsilon < \alpha \leq 1 \Longrightarrow |g^{(n)}(\alpha)| \leq \eta_n.$$

(iii) *For all $\varepsilon > 0$ and $n \geq 0$, there exists*

$$\delta_n > 0/(1/2) + \varepsilon < \alpha \leq 1 \Longrightarrow |1/g(\alpha))^{(n)}| \leq \delta_n.$$

Proof. Let

$$g(\alpha) = \int_a^b (b - \tau)^{2\alpha - 2} \frac{d\tau}{p(\tau)}.$$

Since $p \in C([a, b])$ and $p(t) > 0$, then

$$\frac{1}{M} = \frac{1}{\max\limits_{\tau \in [a,b]} p(\tau)} \leq \frac{1}{p(\tau)} \leq \frac{1}{\min\limits_{\tau \in [a,b]} p(\tau)} = \frac{1}{m};$$

hence,

$$\frac{(b - a)^{2\alpha - 1}}{M(2\alpha - 1)} \leq |g(\alpha)| \leq \frac{(b - a)^{2\alpha - 1}}{m(2\alpha - 1)}.$$

Letting $\varepsilon > 0/(1/2) + \varepsilon < \alpha \leq 1$, we conclude that

$$\frac{c_\varepsilon}{M} \leq |g(\alpha)| \leq \frac{d_\varepsilon(1)}{m(2\varepsilon)}$$

where

$$c_\varepsilon = \begin{cases} (b - a)^{2\varepsilon} & \text{if } b \geq a + 1 \\ b - a & \text{if } b < a + 1, \end{cases}$$

which gives the first property.

For $g^{(n)}(\alpha)$, we have

$$g^{(n)}(\alpha) = \int_a^b 2^n \ln^n (b - \tau)(b - \tau)^{2\alpha - 2} \frac{d\tau}{p(\tau)}$$

$$= \int_a^b 2^n f_n(b - \tau, \beta)(b - \tau)^{2\alpha - \beta - 2} \frac{d\tau}{p(\tau)},$$

and then

$$\left|g^{(n)}(\alpha)\right| \leq \frac{2^n}{m} A_n\left(\varepsilon\right) \frac{(b-a)^{2\alpha-\beta-1}}{2\alpha-\beta-1}.$$

Now, we choose $\beta = \alpha - 1/2$ to obtain

$$(3.9) \qquad \left|g^{(n)}(\alpha)\right| \leq \frac{2^n}{m\,\varepsilon} A_n\left(\varepsilon\right) d_\varepsilon\left(\frac{1}{2}\right) = \eta_n$$

Finally, we can compute $(1/g(\alpha))^{(n)}$ by application of the nth derivative of composite functions (see [**13**]) to the following functions:

$$f(\alpha) = \frac{1}{\alpha} \quad \text{and} \quad g(\alpha) = \int_a^b (b-\tau)^{2\alpha-2} \frac{d\tau}{p(\tau)}.$$

Thus,

$$(3.10) \ \ (f \circ g)^{(n)}(\alpha) = \sum_{k=1}^{n} f^{(k)}(g(\alpha)) \sum_{l=0}^{k} \frac{(-1)^{k-l}}{l!(k-l)!} g^{k-l}(\alpha)\left(g^l(\alpha)\right)^{(n)},$$

where for all $k/1 \leq k \leq n$,

$$(3.11) \qquad f^{(k)}(g(\alpha)) = \frac{(-1)^k k!}{g^{k+1}(\alpha)}$$

and for all $l/0 \leq l \leq n$, $j_0 = 0$ and $j_{l+1} = n$, we have

$$(3.12)$$
$$\left(g^l(\alpha)\right)^{(n)} = \sum_{j_0=0}^{j_1} \sum_{j_1=0}^{j_2} \sum_{j_2=0}^{j_3} \cdots \sum_{j_l=0}^{j_{l+1}} \left[\prod_{i=1}^{l}\left(\binom{j_{i+1}}{j_i}(g(\alpha))^{(j_{i+1}-j_i)}\right)\right],$$

which can easily be proved by induction.

By substituting (3.11) and (3.12) into (3.10) we obtain:

$$(f \circ g)^{(n)}(\alpha) = \sum_{k=1}^{n} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^l}{g^{l+1}(\alpha)} \sum_{j_0=0}^{j_1} \cdots$$
$$\sum_{j_l=0}^{j_{l+1}} \left[\prod_{i=1}^{l}\left(\binom{j_{i+1}}{j_i}(g(\alpha))^{(j_{i+1}-j_i)}\right)\right],$$

where for all $l/0 \leq l \leq n$, $j_{l+1} = n$.

Using triangular inequality, the first property of $g(\alpha)$ and property (3.9), we conclude that, for all $\varepsilon > 0$ and $n \geq 0$, there exists:

$$(3.13) \qquad \delta_n > 0, \left/ \frac{1}{2} + \varepsilon < \alpha \leq 1 \Longrightarrow \left|(f \circ g)^{(n)}(\alpha)\right| \leq \delta_n. \right. \qquad \Box$$

Proposition 3.9. *For $a \leq t \leq x \leq b$ and*

$$\phi(x,t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1}(x-\tau)^{\alpha-1} \frac{d\tau}{p(\tau)},$$

we have for all $\varepsilon > 0$ and $n \geq 0$, there exists:

$$K_n > 0 \left/ \frac{1}{2} + \varepsilon < \alpha \leq 1 \Longrightarrow \left|\frac{\partial^n}{\partial \alpha^n} \phi(x,t)\right| \leq K_n. \right.$$

For all $\varepsilon > 0$ and $n \geq 0$, there exists

$$B_n > 0 \left/ \frac{1}{2} + \varepsilon < \alpha \leq 1 \Longrightarrow \left|\frac{\partial^n}{\partial \alpha^n}\left(\frac{1}{\phi(b,b)}\right)\right| \leq B_n. \right.$$

Proof.

- $\dfrac{\partial^n}{\partial\alpha^n}\phi(x,t) = \dfrac{\partial^n}{\partial\alpha^n}\left(\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^t (t-\tau)^{\alpha-1}(x-\tau)^{\alpha-1}\dfrac{d\tau}{p(\tau)}\right)$

by twice applying Leibniz's formula, we obtain

$$= \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{\Gamma(\alpha)}\right)^{(n-k)}\left(\int_a^t (t-\tau)^{\alpha-1}(x-\tau)^{\alpha-1}\frac{d\tau}{p(\tau)}\right)^{(k)}$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{k} \binom{n}{k}\binom{k}{j}\left(\frac{1}{\Gamma(\alpha)}\right)^{(n-k)}$$

$$\left(\int_a^t (x-\tau)^{\alpha-1}\ln^j(x-\tau)(t-\tau)^{\alpha-1}\ln^{k-j}(t-\tau)\frac{d\tau}{p(\tau)}\right).$$

Then, according to the triangular inequality, we obtain

$$\left|\frac{\partial^n}{\partial\alpha^n}\phi(x,t)\right| \leq \sum_{k=0}^{n}\sum_{j=0}^{k} \binom{n}{k}\binom{k}{j}\left|\left(\frac{1}{\Gamma(\alpha)}\right)^{(n-k)}\right|$$

$$\left|\int_a^t (x-\tau)^{\alpha-1}\ln^j(x-\tau)(t-\tau)^{\alpha-1}\ln^{k-j}(t-\tau)\frac{d\tau}{p(\tau)}\right|.$$

The facts that $p \in C([a,b])$ and $p(\tau) > 0$ give

$$\frac{1}{p(\tau)} \leq \frac{1}{\min\limits_{\tau \in [a,b]} p(\tau)} = \frac{1}{m}.$$

Then, by Lemmas 3.6 and 3.7, we obtain for all $\varepsilon > 0$ and $n \geq 0$ that there exists

$$K_n = \frac{1}{m} \sum_{k=0}^{n} \sum_{j=0}^{k} \left[\binom{n}{k} \binom{k}{j} M_{n-k} C_{k,j} \right] > 0/$$

$$\frac{1}{2} + \varepsilon < \alpha \leq 1 \Longrightarrow \left| \frac{\partial^n}{\partial \alpha^n} \phi(x,t) \right| \leq K_n.$$

- We have

$$\phi(b,b) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{2\alpha-2} \frac{d\tau}{p(\tau)} \iff \frac{1}{\phi(b,b)} = \frac{\Gamma(\alpha)}{g(\alpha)}.$$

Then, by application of Leibniz's formula, we obtain

$$(3.14) \qquad \frac{\partial^n}{\partial \alpha^n} \left(\frac{1}{\phi(b,b)} \right) = \sum_{k=0}^{n} \binom{n}{k} (\Gamma(\alpha))^{(n-k)} \left(\frac{1}{g(\alpha)} \right)^{(k)}.$$

Hence, from properties (3.7) and (3.13) we obtain the desired result. $\qquad \square$

Proposition 3.10. *Consider once more the operator T_α,*

$$(T_\alpha y)(x) = \int_a^b G_\alpha(x,\tau) q(\tau) \, y(\tau) \, d\tau,$$

where G_α is defined in equation (3.1). Then $T_\alpha^{(n)} = \partial^n/(\partial \alpha^n) T_\alpha$ is uniformly bounded for $(1/2) + \varepsilon \leq \alpha \leq 1$, that is, for all $\varepsilon > 0$ and $n \geq 0$, there exists:

$$(3.15) \qquad \theta_n > 0 / \frac{1}{2} + \varepsilon \leq \alpha \leq 1 \Longrightarrow \left\| T_\alpha^{(n)} \right\| \leq \theta_n.$$

Proof. We have

$$(T_\alpha^{(n)} y)(x) = \int_a^b G_\alpha^{(n)}(x,\tau) q(\tau) \, y(\tau) \, d\tau$$

and

- for $a \leq \tau \leq x$, then

$$G_\alpha^{(n)}(x,\tau) = \sum_{m=0}^{n} \binom{n}{m} \left(\frac{1}{\Gamma(\alpha)}\right)^{(n-m)}$$

$$\cdot \left[\sum_{l=0}^{m} \left(\binom{m}{l} \phi^{(m-l)}(b,x) \sum_{k=0}^{l} \binom{l}{k} \phi^{(l-k)}(b,\tau) \left(\frac{1}{\phi(b,b)}\right)^{(k)} \right) \right.$$

$$\left. - \phi^{(m)}(x,\tau) \right],$$

and

- for $x \leq \tau \leq b$, then

(3.16)

$$G_\alpha^{(n)}(x,\tau) = \sum_{m=0}^{n} \binom{n}{m} \left(\frac{1}{\Gamma(\alpha)}\right)^{(n-m)}$$

$$\cdot \left[\sum_{l=0}^{m} \left(\binom{m}{l} \phi^{(m-l)}(b,\tau) \sum_{k=0}^{l} \binom{l}{k} \phi^{(l-k)}(b,x) \left(\frac{1}{\phi(b,b)}\right)^{(k)} \right) \right.$$

$$\left. - \phi^{(m)}(\tau,x) \right],$$

obtained by Leibniz's formula.

Then, by the Cauchy-Schwartz inequality, the fact that $q \in C([a,b])$ and $q(\tau) > 0$, we can easily prove that

$$\|T_\alpha^{(n)}\| \leq \|T_\alpha^{(n)}\|_{H.S.} \leq \sup_{\tau \in [a,b]} q(\tau) \sqrt{\int_a^b \int_a^b \left[G_\alpha^{(n)}(x,\tau)\right]^2 \, d\tau \, dx}.$$

Thus, by application of Lemma 3.6, Proposition 3.9 and formula (3.16) of $G_\alpha^{(n)}$, we obtain result (3.15). $\qquad\square$

Now, we can state the second main result.

Theorem 3.11.

   (i) *Let $m \geq 0$ be a fixed integer. Then, T_α is an $(m+1)$th differentiable operator with respect to α at $\alpha = 1$, and we have the*

following expansion

$$(3.17) \qquad T_\alpha y = \sum_{k=0}^{m} \frac{\left(T_\alpha^{(k)} y\right)\big|_{\alpha=1}}{k!} (\alpha - 1)^k + R_m(\alpha)\, y,$$

where the Lagrange reminder is expressed by

$$(3.18) \qquad R_m(\alpha) y = \frac{(\alpha - 1)^{m+1}}{(m+1)!} \left(T_\alpha^{(m+1)} y\right)\Big|_{\alpha=\xi}, \quad \alpha < \xi < 1,$$

and

$$\|R_m(\alpha)\| = o(|\alpha - 1|^m), \quad \alpha \underset{<}{\to} 1.$$

(ii) *If $\mu_1^-(\alpha)$ is the first negative eigenvalue of the operator T_α, then*

$$(3.19) \qquad \lim_{\alpha \to 1 \atop <} \mu_1^-(\alpha) = \mu_1^-(1),$$

where $\mu_1^-(1)$ is the first negative eigenvalue of the operator T_1.

Proof.

(i) From expression (3.1) of the kernel $G_\alpha(x, \tau)$, one can see that $G_\alpha(x, \tau)$ is C^∞ in any small neighborhood of $\alpha = 1$. So we can write the Taylor formula with Lagrange reminder at this point as:

$$(T_\alpha y)(x) = \sum_{k=0}^{m} \frac{\left(T_\alpha^{(k)} y\right)(x)\big|_{\alpha=1}}{k!} (\alpha - 1)^k + R_m(\alpha),$$

where

$$\left(T_\alpha^{(k)} y\right)(x)\Big|_{\alpha=1} = \int_a^b G_\alpha^{(k)}(x, \tau)\Big|_{\alpha=1} q(\tau)\, y(\tau)\, d\tau,$$

and $R_m(\alpha)$ is the Lagrange reminder defined in equation (3.18).

Then,

$$\|R_m(\alpha)\| \le \|T_\xi^{(m+1)}\| \cdot \frac{|\alpha - 1|^{m+1}}{(m+1)!}, \quad \alpha < \xi < 1.$$

By applying property (3.15), we obtain that there exists:

$$\theta_{m+1} > 0 / \|R_m(\alpha)\| \leq \theta_{m+1} \cdot \frac{|\alpha - 1|^{m+1}}{(m+1)!},$$

which gives

$$\|R_m(\alpha)\| = o(|\alpha - 1|^m), \quad \alpha \underset{<}{\to} 1.$$

(ii) Let us take the previous expansion of T_α equation (3.17) with $m = 0$. Then

$$(T_\alpha y)(x) = (T_1 y)(x) + (\alpha - 1) \cdot \left(T'_\xi y\right)(x)$$
$$\Longleftrightarrow \left| \langle T_\alpha y, y \rangle_q - \langle T_1 y, y \rangle_q \right| = |\alpha - 1| \cdot \left| \langle T'_\xi y, y \rangle_q \right|.$$

However,

$$\left| \langle T'_\xi y, y \rangle_q \right| \leq \|T'_\xi\| \cdot \|y\|_q \leq \|T'_\xi\|, \quad \text{for all } y / \|y\|_q = 1.$$

Hence,

$$\left| \langle T_\alpha y, y \rangle_q - \langle T_1 y, y \rangle_q \right| \leq |\alpha - 1| \cdot \|T'_\xi\|$$
$$\Longrightarrow -|\alpha - 1| \cdot \|T'_\xi\| \leq \langle T_\alpha y, y \rangle_q - \langle T_1 y, y \rangle_q \leq |\alpha - 1| \cdot \|T'_\xi\|$$
$$\Longrightarrow -|\alpha - 1| \cdot \|T'_\xi\| + \langle T_1 y, y \rangle_q \leq \langle T_\alpha y, y \rangle_q \leq \langle T_1 y, y \rangle_q$$
$$+ |\alpha - 1| \cdot \|T'_\xi\|$$
$$\Longrightarrow -|\alpha - 1| \cdot \|T'_\xi\| + \inf_{\|y\|_q = 1} \langle T_1 y, y \rangle_q \leq \inf_{\|y\|_q = 1} \langle T_\alpha y, y \rangle_q$$

and

$$\inf_{\|y\|_q = 1} \langle T_\alpha y, y \rangle_q \leq \inf_{\|y\|_q = 1} \langle T_1 y, y \rangle_q + |\alpha - 1| \cdot \|T'_\xi\|$$
$$\Longrightarrow -|\alpha - 1| \cdot \|T'_\xi\| + \mu_1^-(1) \leq \mu_1^-(\alpha) \leq \mu_1^-(1) + |\alpha - 1| \cdot \|T'_\xi\|.$$

From this and the fact that T'_ξ verifies (3.15), we obtain

$$\lim_{\alpha \underset{<}{\to} 1} \mu_1^-(\alpha) = \mu_1^-(1),$$

and consequently,

$$\lim_{\alpha \underset{<}{\to} 1} \lambda_1^-(\alpha) = \lambda_1^-(1). \qquad \square$$

Remark 3.12. We think that the previous expansion of T_α can be of great importance in the expression of the next terms in asymptotic development of $\mu_1^-(\alpha)$.

4. Illustrative example. Let us show a simple example with $p \equiv 1$, $q \equiv 1$ and $[a, b] = [0, 1]$. Then, our regular fractional Sturm-Liouville problem reduces to

$$
\begin{cases}
\left(D_{1,-}^\alpha D_{0,+}^\alpha\right) y(x) + \lambda\, y(x) = 0, & \tfrac{1}{2} < \alpha < 1,\ 0 < x < 1 \\
\lim\limits_{\substack{x \to 0 \\ >}} x^{1-\alpha}\, y(x) = 0 = y(1),
\end{cases}
$$

and the "inverse" compact operator defined in $L^2([0, 1])$ is

$$
(T_\alpha y)(x) = \int\limits_0^1 G_\alpha(x, \tau)\, y(\tau)\, d\tau,
$$

where the fractional Green's function is

$$
G_\alpha(x, \tau) = \begin{cases}
(2\,\alpha - 1)\varphi(1, x)\varphi(1, \tau) - \varphi(x, \tau) & \text{if } 0 \le \tau \le x, \\
(2\,\alpha - 1)\varphi(1, x)\varphi(1, \tau) - \varphi(\tau, x) & \text{if } x \le \tau \le 1,
\end{cases}
$$

with

$$
\varphi(\tau, x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1}(\tau - t)^{\alpha-1}\, dt
$$

$$
= \frac{x^\alpha \tau^{\alpha-1}}{\Gamma(1 + \alpha)}\, {}_2F_1\left(\begin{matrix} 1 - \alpha & , & 1 \\ & & \\ & 1 + \alpha & \end{matrix}\ ;\ \frac{x}{\tau}\right),
$$

where ${}_2F_1$ is the Gauss-hypergeometric function which can be defined by its Euler representation, see [**12**],

$$
{}_2F_1\left(\begin{matrix} a & , & b \\ & & \\ & c & \end{matrix}\ ;\ z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 s^{b-1}(1 - s)^{c-b-1}(1 - z\,s)^{-a}\, ds.
$$

Thus, this problem has an infinite sequence of real negative eigenvalues. The Taylor expansion up to order 1 gives

$$
(T_\alpha y)(x) = (T_\alpha y)(x)|_{\alpha=1} + \left(T_\alpha^{(1)} y\right)(x)\Big|_{\alpha=1}(\alpha - 1) + R_1(\alpha)
$$

where the kernel $\widetilde{G}_\alpha(x,\tau)|_{\alpha=1} = x\tau - \min(x,\tau)$ is the classical Green's function of the $(\mathcal{P}_1)$ problem corresponding to our data.

$\widetilde{G}_\alpha^{(1)}(x,\tau)|_{\alpha=1}$ can be computed by integration by parts

$$\widetilde{G}_\alpha^{(1)}(x,\tau)\Big|_{\alpha=1}$$
$$= \begin{cases} 2\tau + 2(\gamma-1)x\tau - u(x) - u(\tau) + u(x-\tau) \\ \quad +u(x\tau) - xu(1-\tau) - \tau(1-x) & \text{if } 0 \le \tau \le x, \\ 2x + 2(\gamma-1)\tau x - u(\tau) - u(x) + u(\tau-x) \\ \quad +u(\tau x) - \tau u(1-x) - xu(1-\tau) & \text{if } x \le \tau \le 1, \end{cases}$$

where $u(x) = x\ln x$ and γ is the Euler constant.

We remark that the linear operator associated to this Green's function which contains logarithmic functions is not connected to any classical differential operator. Finally, we have

$$\lim_{\substack{\alpha \to 1 \\ <}} \lambda_1^-(\alpha) = -\pi^2.$$

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