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ON THE APPLICATION OF SEQUENTIAL AND FIXED-POINT METHODS TO FRACTIONAL DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

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ABSTRACT. This article analyzes the existence and approximation of solutions to initial value problems for nonlinear fractional differential equations of arbitrary order. Several new approaches are furnished in the environment of fractional differential equations, such as the sequential technique of Cauchy-Peano and the Leray-Schauder topological degree. In addition, some well-known ideas are optimized in the context of Banach's fixed-point theorem. A general version of Gronwall's inequality is also established. A recurring theme throughout the work is the incorporation of desirable qualities of the classical Mittag-Leffler function. A YouTube video presentation by the author designed to complement this work is available at http://tinyurl.com/Tisdell-JIEA.

1. Introduction. This article explores the existence and approximation of solutions to the following initial value problem (IVP) of arbitrary order $q > 0$

$$D^q \left(x - T_{\lceil q \rceil - 1}[x]\right)(t) = f(t, x(t)); \tag{1.1}$$

$$x(0) = A_0, \quad x'(0) = A_1, \dots, \; x^{(\lceil q \rceil - 1)}(0) = A_{\lceil q \rceil - 1}; \tag{1.2}$$

where $\lceil q \rceil$ is the integer such that $q - 1 < \lceil q \rceil \leq q$; D^q represents the Riemann-Liouville fractional differentiation operator of arbitrary order $q > 0$ (a full definition is given in (2.2) a little later); $f : [0, a] \times D \subset \mathbf{R}^2 \to \mathbf{R}$; $T_{\lceil q \rceil - 1}[x]$ is the Maclaurin polynomial of order $\lceil q \rceil - 1$ of $x = x(t)$; $a > 0$ and the A_i are constants.

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The left-hand side of (1.1) is known as the Caputo derivative of x of order $q > 0$ with the notation ${}^{C}D^{q}(x) := D^{q}(x - T_{\lceil q \rceil - 1}[x])$ sometimes used. Note that the classical derivatives of the function x (from order zero to order $\lceil q \rceil - 1$ each at $t = 0$) appear in (1.1) and (1.2). This particular form was suggested by Caputo [**3**] in response to a need for improved accuracy in modeling the initial conditions of phenomena. It seems that equations whose initial conditions feature derivatives of integer order are more useful for practical purposes than initial conditions featuring derivatives of fractional order, which may be unavailable or whose physical meaning may be rather vague.

Early contributions to the qualitative analysis of the solutions to the nonlinear IVP (1.1), (1.2) appear in [**6**], where some foundational results on the existence, uniqueness and approximation of solutions can be found. In the spirit of [**6**], several papers such as [**20–22**] and the monographs [**5**] and [**18**, subsection 3.5, pages 198–212] have presented additional results for solutions to (1.1) and (1.2). The methods employed in the above works may be summarized as: the sequential technique of successive approximations also known as Picard iterations, and the classical fixed-point approaches of Banach and Schauder.

In contrast to the works [**6, 18, 20–22**], the approach herein involves: the sequential techniques of Cauchy-Peano type and the Leray-Schauder topological degree. Several results herein address remarks in the recent monograph [**5**]. In this way, the methods complement those already in the literature. Consequently, the results in this article contribute additional foundational knowledge to the field of nonlinear IVPs for fractional differential equations.

In accord with [**6, 18, 20–22**] (and partially motivated by them) Banach's fixed-point theorem and successive approximations are invoked in some places herein, but in novel ways. For example, a new metric is defined in the fractional differential equation environment which greatly simplifies the application of Banach's theorem for existence and uniqueness proofs and also gives a nice evaluation for the convergence of the iterations. In addition, a new example in the fractional differential equation setting is furnished which illustrates that the continuity of the right-hand side of (1.1) is insufficient to ensure the convergence of the successive approximations. Furthermore, the idea of "enveloping" of solutions via successive approximations and monotone iterations is discussed.

A guiding principle in the writing of this paper has been to incorporate desirable elements of the Mittag-Leffler function into the working and results, where possible. For example, the Mittag-Leffler function appears in some Gronwall-Bellman type inequalities and also in the definition of a new metric. Although the Mittag-Leffler function has been studied in great detail [**11**, Chapter 16], [**15, 25, 26**], [**29**, Chapter 1] it appears that its rich potential has yet to be fully realized in the qualitative analysis of solutions to the IVP (1.1), (1.2).

Although fractional differential equations are centuries old, it is surprising to discover that much of the basic qualitative and quantitative foundational theory is yet to be fully developed. Such ideas of existence and approximation of solutions would form the bedrock to underpin advanced studies in the area, especially with respect to applications, and thus appear to be of significant interest. Indeed, this is one of the aims of this paper.

2. Preliminaries. To understand the notation used throughout and to keep the paper somewhat self-contained, this section contains some preliminary definitions and associated notation.

A solution to the IVP (1.1), (1.2) on an interval I is defined to be a q-th (fractionally)-differentiable function $x : I \subseteq [0, a] \to \mathbf{R}$ such that the points $(t, x(t))$ lie in $I \times D$ for all $t \in I$ and $x(t)$ satisfying (1.1) for all $t \in I$, and (1.2).

Instead of directly dealing with problem (1.1), (1.2) the analysis will often involve an equivalent integral equation, as these equations are of a more tractable nature. The following lemma is fundamental to the ideas in this work.

Lemma 2.1. *If $f : [0, a] \times D \subset \mathbf{R}^2 \to \mathbf{R}$ is continuous, then the initial value problem* (1.1)*,* (1.2) *is equivalent to the integral equation*

$$x(t) = \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t^i}{i!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \quad t \in [0, a]. \tag{2.1}$$

Proof. This result is well known, but a proof is included for completeness and so that some basic notation may also be introduced in an optimal manner. Define the Riemann-Liouville fractional derivative and integral of order $q > 0$ of a function y, respectively, by:

$$
\begin{aligned}
D^q y(t) &:= \frac{d^{\lceil q \rceil}}{dt^{\lceil q \rceil}} \frac{1}{\Gamma(\lceil q \rceil - q)} \int_0^t (t-s)^{\lceil q \rceil - 1 - q} y(s)\, ds; \\
I^q y(t) &:= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds
\end{aligned} \tag{2.2}
$$

with the Caputo derivative defined via

$$
{}^C D^q y(t) := D^q \left(y - T_{\lceil q \rceil - 1}[y] \right)(t).
$$

If x is a solution to (2.1) on $[0, a]$, then from direct differentiation and substitution into (2.1) we have

$$
x(0) = A_0, \ldots, x^{(\lceil q \rceil - 1)}(0) = A_{\lceil q \rceil - 1}.
$$

For all $t \in [0, a]$, we then have

$$
\begin{aligned}
x(t) &= \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t^i}{i!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \\
&= \sum_{i=0}^{\lceil q \rceil - 1} \frac{x_i(0) t^i}{i!} + I^q \left(f(\cdot, x(\cdot)) \right)(t).
\end{aligned}
$$

Thus, a rearrangement and application of ${}^C D^q$ to both sides yields for all $t \in [0, a]$ we have

$$
\begin{aligned}
{}^C D^q(x)(t) &:= D^q \left(x - \sum_{i=0}^{\lceil q \rceil - 1} \frac{x_i(0) t^i}{i!} \right)(t) \\
&= D^q \left(I^q (f(\cdot, x(\cdot))) \right)(t) \\
&= f(t, x(t)),
\end{aligned}
$$

where we have used the identity $D^q[I^q(y(t))] = y(t)$ from [**17**, (1.7), page 153] or, equivalently, ${}^C D^q[I^q(y(t))] = y(t)$ from [**18**, (2.4.38), page

96]. Thus, every solution to (2.1) on $[0, a]$ is also a solution to (1.1) and (1.2) on $[0, a]$.

Now let x be a solution to (1.1), (1.2) on $[0, a]$. By applying I^q to both sides of (1.1) and using the identity from [**18**, (2.4.42), page 96], namely, $I^q[{}^C Dy(t)] = y(t) - \sum_{i=0}^{\lceil q \rceil - 1} y^{(i)}(0)t^i/i!$, we obtain (2.1). Thus, every solution to (1.1), (1.2) on $[0, a]$ is also a solution to (2.1) on $[0, a]$. $\square$

An important theme of this work is to utilize the rich qualities of the so called Mittag-Leffler function in a way that simplifies methods and optimizes results. In the fractional calculus the Mittag-Leffler function plays a similar role to that of the exponential function in classical calculus. The Mittag-Leffler function of order $q > 0$ is defined and denoted by

$$E_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}, \quad z \in \mathbf{C},$$

and we shall be interested in the particular function

$$E_q(\beta t^q) := \sum_{k=0}^{\infty} \frac{(\beta t^q)^k}{\Gamma(qk+1)}, \quad t \in [0, a] \subset \mathbf{R}$$

where $q > 0$, and $\beta > 0$ is a constant.

A very important property in the context of this work is that $E_q(\beta t^q)$ is the unique solution to the initial value problem

$$\begin{gathered} {}^C D^q x(t) := D^q \left(x - T_{\lceil q \rceil - 1}[x]\right)(t) = \beta x(t), \\ x(0) = 1,\ x'(0) = 0, \ldots, x^{\lceil q \rceil - 1}(0) = 0 \end{gathered}$$

for $t \geq 0$.

3. Some inequalities and estimates. In this section a basic fractional integral inequality is formulated that ensures a bound on the function involved. The bound is in terms of the Mittag-Leffler function, and such an idea might be considered as a fractional integral analogue of the famous inequality of Gronwall, Bellman and Reid [**1, 12**], [**31**, page 296] involving ordinary derivatives and integrals. The following lemma is an extension of [**7**, Lemma 4.3] (also see [**8**]) and will be frequently applied in subsequent sections.

Lemma 3.1. *Let A, B and C be non-negative constants, and let $\rho : [0, a] \to [0, \infty)$ be continuous. If*

$$\rho(t) \leq A + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [B\rho(s) + C]\, ds, \quad \text{for all } t \in [0, a], \tag{3.1}$$

then for all $t \in [0, a]$, we have

$$\rho(t) \leq \begin{cases} AE_q(Bt^q) + (C/B)\,[E_q(Bt^q) - 1] & \text{for } B > 0; \\ A + (Ct^q)/(\Gamma(q+1)) & \text{for } B = 0. \end{cases} \tag{3.2}$$

Proof. The proof follows that of [**7**, Lemma 4.3]. *Case $B > 0$*: Let $\varepsilon > 0$, and define the function

$$\phi(t) := \varepsilon + A + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [B\phi(s) + C]\, ds, \quad \text{for all } t \in [0, a]. \tag{3.3}$$

Note that (3.3) has the solution

$$\phi(t) = \left(\varepsilon + A + \frac{C}{B}\right) E_q(Bt^q) - \frac{C}{B}, \quad t \in [0, a].$$

From (3.1) we have $\rho(0) \leq A$, and from (3.3) we have $\phi(0) = A + \varepsilon$ and thus $0 \leq \rho(0) < \phi(0)$. We claim that $\rho < \phi$ on $[0, a]$. Argue by contradiction and assume that there is a $t_0 \in [0, a]$ such that

$$\rho(t) < \phi(t), \quad \text{for all } t \in [0, t_0), \text{ and } \rho(t_0) = \phi(t_0)$$

(such a t_0 exists by the continuity of the functions involved and the intermediate value theorem). Thus, $\rho \leq \phi$ on $[0, t_0]$ and

$$\begin{aligned} \rho(t_0) &\leq A + \frac{1}{\Gamma(q)} \int_0^{t_0} (t_0 - s)^{q-1} [B\rho(s) + C]\, ds \\ &\leq A + \frac{1}{\Gamma(q)} \int_0^{t_0} (t_0 - s)^{q-1} [B\phi(s) + C]\, ds \\ &< \varepsilon + A + \frac{1}{\Gamma(q)} \int_0^{t_0} (t_0 - s)^{q-1} [B\phi(s) + C]\, ds = \phi(t_0), \end{aligned}$$

and we have a contradiction to $\rho(t_0) = \phi(t_0)$.

Thus, for all $t \in [0, a]$, we must have

$$\rho(t) < \phi(t) = \left(\varepsilon + A + \frac{C}{B}\right) E_q(Bt^q) - \frac{C}{B}. \tag{3.4}$$

Now since the inequality in (3.4) holds for all $\varepsilon > 0$, we obtain

$$\rho(t) \le AE_q(Bt^q) + \frac{C}{B}\left[E_q(Bt^q) - 1\right], \quad \text{for all } t \in [0, a]. \tag{3.5}$$

Case $B = 0$: Inequality (3.1) may be integrated directly to obtain (3.2). $\square$

The following theorem gives some sufficient conditions under which all possible solutions to the IVP (1.1), (1.2) are bounded uniformly on $[0, a]$. The idea is known as an "*a priori* bound" as the solutions do not need to be explicitly known in order to formulate this bound. This idea will be used repeatedly in the sections that follow.

Theorem 3.2. *Let* K *and* K_1 *be non-negative constants. If* $x : [0, a] \to \mathbf{R}$ *has a continuous derivative of order* $q > 0$ *and*

$$\left|D^q\left(x - T_{\lceil q\rceil - 1}[x]\right)(t)\right| \le K|x(t)| + K_1, \quad \textit{for all } t \in [0, a], \tag{3.6}$$

then

$$|x(t)| \le \begin{cases} \sum_{i=0}^{\lceil q\rceil - 1} \frac{|x^{(i)}(0)|a^i}{i!} E_q(Kt^q) + \frac{K_1}{K}\left[E_q(Kt^q) - 1\right] & \textit{for } K > 0; \\ \sum_{i=0}^{\lceil q\rceil - 1} \frac{|x^{(i)}(0)|a^i}{i!} + \frac{K_1 t^q}{\Gamma(q+1)} & \textit{for } K = 0. \end{cases} \tag{3.7}$$

Proof. The basic idea of the proof is to apply Lemma 3.1.

For all $t \in [0, a]$ we have

$$\begin{aligned} |x(t)| - \left|T_{\lceil q\rceil-1}[x](t)\right| &\le |x(t) - T_{\lceil q\rceil-1}[x](t)| = \left|I^q\left[D^q\left(x - T_{\lceil q\rceil-1}[x]\right)(t)\right]\right| \\ &= \left|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} D^q\left(x - T_{\lceil q\rceil-1}[x]\right)(s)\, ds\right| \\ &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[K|x(s)| + K_1\right] ds, \end{aligned}$$

where we have used (3.6). Rearranging the above, we see that

$$|x(t)| \leq \sum_{i=0}^{\lceil q\rceil-1} \frac{|x^{(i)}(0)|a^i}{i!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[K|x(s)| + K_1\right] ds, \quad \text{for all } t \in [0,a],$$

and so (3.1) holds with: $\rho(t) = |x(t)|$; $A = \sum_{i=0}^{\lceil q\rceil-1} (|x^{(i)}(0)|a^i)/i!$; $B = K$; and $C = K_1$, which, by Lemma 3.1, yields (3.7). $\square$

4. Cauchy-Peano approximate solution approach. The notion of ε-approximate solution techniques for ordinary differential equations seems to be attributed to Cauchy and Peano, dating back to the 19th century. Roughly speaking, the approach first involves illustrating that a sequence of approximate solutions to the equation under consideration does exist; and secondly showing that there is a subsequence of these approximate solutions that converges to an actual solution of the problem.

In this section, the concept of an ε-approximate solution to (1.1), (1.2) is introduced. The ideas are combined with sequential arguments to form new existence and approximation results for solutions to (1.1), (1.2).

A significant advantage of the ε-approximate solution approach over fixed-point methods is that the arguments do not require a knowledge of functional analysis. Rather, a milder prerequisite of uniform convergence is all that is necessary.

Definition 4.1. Define a function $x : [0,a] \to \mathbf{R}$ as an ε-approximate integral-type solution to (1.1), (1.2) on $[0,a]$ if:

x is continuous on $[0,a]$;

$f(s,x(s))$ is continuous with respect to $s \in [0,a]$;

$$\left| x(t) - \sum_{i=0}^{\lceil q\rceil-1} \frac{A_i t^i}{i!} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))\, ds \right| \leq \varepsilon, \quad \text{for all } t \in [0,a]. \tag{4.1}$$

In what follows, define and denote the infinite strip $S \subset \mathbf{R}$ by

$$S := \{(t,p) : t \in [0,a],\ p \in \mathbf{R}\}.$$

The following result forms a theoretical basis for the results in this section. It provides some (rather abstract) conditions under which (1.1), (1.2) will admit at least one solution on $[0,a]$.

Theorem 4.2. *Let $f : S \to \mathbf{R}$ be continuous, and let $\{\varepsilon_m\}$ be a sequence of positive constants such that $\varepsilon_m \to 0$ as $m \to \infty$. If $\{x_m\} = \{x_m(t)\}$ is a corresponding sequence of ε_m-approximate integral-type solutions to* (1.1), (1.2) *on $[0,a]$ for which there exist non-negative constants K and K_1 such that*

$$|f(t,x_m(t))| \le K|x_m(t)| + K_1, \text{ for all } t \in [0,a] \text{ and } m = 1,2,\dots, \tag{4.2}$$

then there exists a subsequence $\{x_{m_k}\} = \{x_{m_k}(t)\}$ of $\{x_m\}$ ($m_1 < m_2 < \cdots$) that converges uniformly to a solution $x = x(t)$ of (1.1), (1.2) *on $[0,a]$.*

Proof. Let $\{x_m\}$ be a corresponding sequence of ε_m-approximate integral-type solutions to (1.1), (1.2) on $[0,a]$, and define the sequence of functions $\{r_m\} = \{r_m(t)\}$ for all $t \in [0,a]$ via

$$x_m(t) = \sum_{i=0}^{\lceil q\rceil - 1} \frac{A_i t^i}{i!} + r_m(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_m(s))\, ds, \tag{4.3}$$
$$\text{for all } m = 1,2,\dots.$$

Now, as f and each x_m are continuous functions, it follows that each r_m is continuous on $[0,a]$. Furthermore, since each x_m is an ε_m-approximate integral-type solution, combining (4.1) and (4.3) we have $|r_m(t)| \le \varepsilon_m$ for all $t \in [0,a]$ and $m = 1,2,\dots$.

Now, as ε_m converges to zero from above, we can choose a constant $\gamma > 0$ such that $\varepsilon_m \le \gamma$ for each m. Thus, for all $t \in [0,a]$ and

$m = 1, 2, \dots$, from (4.2) and (4.3) we have

$$
\begin{aligned}
|x_m(t)| &= \left| \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t^i}{i!} + r_m(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_m(s))\, ds \right| \\
&\leq \left(\sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i a^i}{i!} + \gamma \right) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[K|x_m(s)| + K_1 \right] ds.
\end{aligned}
$$

Thus, for $m = 1, 2, \dots$, we have (3.1) holding with: $\rho(t) = |x_m(t)|$; $A = \sum_{i=0}^{\lceil q \rceil - 1} [(A_i a^i)/i!] + \gamma$; $B = K$ and $C = K_1$. Lemma 3.1 is applicable, guaranteeing the existence of a constant $v \geq 0$ such that $|x_m(t)| \leq v$ for all $t \in [0, a]$ and $m = 1, 2, \dots$, where

$$
v := \begin{cases} \left(\gamma + \sum_{i=0}^{\lceil q \rceil - 1} \frac{|x^{(i)}(0)| a^i}{i!} \right) E_q(K a^q) & \\ + \frac{K_1}{K} \left[E_q(K a^q) - 1 \right] & \text{for } K > 0; \\ \left(\gamma + \sum_{i=0}^{\lceil q \rceil - 1} \frac{|x^{(i)}(0)| a^i}{i!} \right) + \frac{K_1 a^q}{\Gamma(q+1)} & \text{for } K = 0; \end{cases} \tag{4.4}
$$

and thus $\{x_m\}$ is uniformly bounded by v on $[0, a]$.

From (4.2) we now see that

$$
\begin{gathered}
|f(t, x_m(t))| \leq \mu := K v + K_1, \\
\text{for all } t \in [0, a] \text{ and } m = 1, 2, \dots .
\end{gathered} \tag{4.5}
$$

Define the sequence of functions $\{z_m\} = \{z_m(t)\}$ by

$$
\begin{gathered}
z_m(t) := \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t^i}{i!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_m(s))\, ds, \\
\text{for all } t \in [0, a],\ m = 1, 2, \dots .
\end{gathered} \tag{4.6}
$$

We thus have $z_m^{(i)}(0) = A_i$ for $m = 1, 2, \dots$ and each $i = 0, \dots, \lceil q \rceil - 1$. Now (4.5) yields

$$
|z_m(t)| \leq \sum_{i=0}^{\lceil q \rceil - 1} \frac{|A_i| a^i}{i!} + \mu \frac{a^q}{\Gamma(q+1)}, \quad \text{for all } t \in [0, a] \text{ and } m = 1, 2, \dots .
$$

Thus, the sequence z_m is uniformly bounded on $[0, a]$.

We claim that $\{z_m\}$ is equicontinuous. For $q \in (0,1]$ and for $m = 1, 2, \ldots$ and any $t_1, t_2 \in [0,a]$ with $t_1 \leq t_2$, (4.5) yields

$$\begin{aligned}
|z_m(t_1) - z_m(t_2)| & \\
&= \bigg| \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f(s, x_m(s))\, ds \\
&\qquad - \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} f(s, x_m(s))\, ds \bigg| \\
&= \bigg| \frac{1}{\Gamma(q)} \int_0^{t_1} \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right] f(s, x_m(s))\, ds \\
&\qquad - \frac{1}{\Gamma(q)} \int_{t_2}^{t_1} (t_2 - s)^{q-1} f(s, x_m(s))\, ds \bigg| \\
&\leq [2(t_2 - t_1)^q + t_1^q - t_2^q] \frac{\mu}{\Gamma(q+1)} \\
&\leq 2(t_2 - t_1)^q \frac{\mu}{\Gamma(q+1)}.
\end{aligned}$$

If $q > 1$, then the polynomial $\sum_{i=0}^{\lceil q \rceil - 1} (A_i t^i)/i!$ has a continuous and uniformly bounded derivative on $[0,a]$ and, as such, there is a constant $L > 0$ with

$$\left| \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t_1^i}{i!} - \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t_2^i}{i!} \right| \leq L |t_1 - t_2|.$$

Thus, for $q > 1$ and for $m = 1, 2, \ldots$ and for any $t_1, t_2 \in [0,a]$ with $t_1 \leq t_2$, (4.5) yields

$$\begin{aligned}
|z_m(t_1) - z_m(t_2)| = \bigg| & \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t_1^i}{i!} - \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t_2^i}{i!} \\
& + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f(s, x_m(s))\, ds \\
& - \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} f(s, x_m(s))\, ds \bigg|
\end{aligned}$$

$$\begin{aligned}
&\le L(t_2-t_1)\\
&\qquad + \left| \frac{1}{\Gamma(q)}\int_0^{t_1} \left[(t_1-s)^{q-1}-(t_2-s)^{q-1}\right] f(s,x_m(s))\,ds \right.\\
&\qquad \left. - \frac{1}{\Gamma(q)}\int_{t_2}^{t_1}(t_2-s)^{q-1} f(s,x_m(s))\,ds \right|\\
&\le L(t_2-t_1)+[2(t_2-t_1)^q+t_1^q-t_2^q]\frac{\mu}{\Gamma(q+1)}\\
&\le (t_2-t_1)\left(L+2\frac{\mu}{\Gamma(q+1)}\right),
\end{aligned}$$

where we have assumed $t_2-t_1 \le 1$ and $q>1$, yielding $(t_2-t_1)^q \le t_2-t_1$. Now, given any $\varepsilon_1>0$ we may choose $\delta_1=\varepsilon_1\min\{1,\Gamma(q+1)/(L\Gamma(q+1)+2\mu)\}$ to ensure $|z_m(t_1)-z_m(t_2)|<\varepsilon_1$ whenever $t_2-t_1<\delta_1$.

Thus, $\{z_m\}$ is uniformly bounded and equicontinuous on $[0,a]$ and the Arzela-Ascoli selection theorem [**13**, Selection theorem 2.3, page 4] ensures that there is a subsequence $\{z_{m_k}\}$ of $\{z_m\}$ and a continuous function $x=x(t)$ such that z_{m_k} tends uniformly to x on $[0,a]$.

From (4.3) and (4.6), we see that

$$|x_{m_k}(t)-z_{m_k}(t)| \le |r_{m_k}(t)|, \quad \text{for all } t\in[0,a],$$

and since r_m converges uniformly to zero on $[0,a]$, it follows that x_{m_k} converges uniformly to x on $[0,a]$. Now, since x_m is uniformly bounded by v on $[0,a]$ we must have $|x(t)|\le v$ for all $t\in[0,a]$.

If we define the set R_v by

$$R_v := \{(t,p): t\in[0,a],\ |p|\le v\}$$

then the uniform continuity of f on R_v yields

$$f(\cdot,x_{m_k}(\cdot)) \longrightarrow f(\cdot,x(\cdot))$$

uniformly on $[0,a]$ and so

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^t (t-s)^{q-1} f(s,x_{m_k}(s))\,ds\\
&\qquad\qquad = \int_0^t (t-s)^{q-1} f(s,x(s))\,ds, \quad \text{for all } t\in[0,a].
\end{aligned}$$

We conclude that $x = x(t)$ satisfies

$$x(t) = \sum_{i=0}^{\lceil q\rceil -1} \frac{A_i t^i}{i!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \quad \text{for all } t \in [0,a],$$

and so our limit function x is a solution to (1.1), (1.2) on $[0,a]$. □

Theorem 4.2 assumes the existence of certain ε-approximate solutions to (1.1), (1.2). This condition, in isolation, may be challenging to verify in practice. The following result furnishes easily-verifiable conditions under which the conditions of Theorem 4.2 will hold.

Theorem 4.3. *Let $f : S \to \mathbf{R}$ be continuous. If there exist non-negative constants K and K_1 such that*

$$|f(t,p)| \le K|p| + K_1, \quad \textit{for all } (t,p) \in S, \tag{4.7}$$

then for each $\varepsilon > 0$ there is a corresponding ε-approximate integral-type solution to (1.1), (1.2) *on* $[0,a]$ *and, as such, there is a solution* $x = x(t)$ *to* (1.1), (1.2) *on* $[0,a]$.

Proof. For each $\varepsilon > 0$ we construct a corresponding ε-approximate integral-type solution to (1.1), (1.2) on $[0,a]$. This ensures that we can choose any sequence of positive constants $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$ as $m \to \infty$ with $\{x_m\} = \{x_m(t)\}$ a corresponding sequence of ε_m-approximate integral-type solutions to (1.1), (1.2) on $[0,a]$. In view of (4.7), we will see that (4.2) will hold and so, by Theorem 4.2, we will see that (1.1), (1.2) will have at least one solution on $[0,a]$.

Let $\delta > 0$ be such that $0 < \delta < a$, and define the function $x(t;\delta)$ for all $t \in [0,a]$ as follows:

$$x(t;\delta) := \begin{cases} \sum_{i=0}^{\lceil q\rceil -1} \frac{A_i t^i}{i!} & \text{for all } t \in [0,\delta]; \\ \sum_{i=0}^{\lceil q\rceil -1} \frac{A_i t^i}{i!} & \\ +\frac{1}{\Gamma(q)} \int_0^{t-\delta} (t-s)^{q-1} f(s, x(s;\delta))\, ds & \text{for all } t \in [\delta, a]. \end{cases}$$

Note that the continuity of f ensures that, for each $\delta > 0$ and all $t \in [0,a]$, the function $x(t;\delta)$ is well defined and continuous for each fixed $\delta > 0$ and all $t \in [0,a]$.

We can show, in a similar manner as in the proof of Theorem 4.2, that $x(t;\delta)$ is uniformly bounded on $[0,a]$ by employing (4.7) and Lemma 3.1 to obtain $|x(t;\delta)| \le v$ for all $t \in [0,a]$ with v defined in (4.4). Also, (4.7) then leads to

$$|f(t, x(t;\delta))| \le \mu := Kv + K_1, \quad \text{for all } t \in [0,a].$$

Now if we define a function $\zeta(t;\delta)$ for all $t \in [0,a]$ by

$$\zeta(t;\delta) := \begin{cases} 0 & \text{for all } t \in [0,\delta]; \\ t-\delta & \text{for all } t \in [\delta,a]; \end{cases}$$

then, for all $t \in [0,a]$, we have: $0 \le \zeta(t) - t \le \delta$; and

$$\begin{aligned} \left| \frac{1}{\Gamma(q)} \int_t^{\zeta(t)} (t-s)^{q-1} f(s, x(s;\delta))\, ds \right| &\le \mu \frac{1}{\Gamma(q)} \int_t^{\zeta(t)} (t-s)^{q-1}\, ds \\ &= \mu \frac{[\zeta(t)-t]^q}{\Gamma(q+1)} \le \mu \frac{\delta^q}{\Gamma(q+1)}. \end{aligned}$$

Thus, for all $t \in [0,a]$, we have

$$\begin{aligned} &\left| x(t;\delta) - \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t^i}{i!} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s;\delta))\, ds \right| \\ &\qquad = \left| \frac{1}{\Gamma(q)} \int_t^{\zeta(t)} (t-s)^{q-1} f(s, x(s;\delta))\, ds \right| \le \mu \frac{\delta^q}{\Gamma(q+1)}. \end{aligned}$$

For any given $\varepsilon > 0$ if we choose δ such that

$$0 < \delta < \min \left\{ a, \left[\frac{\varepsilon \Gamma(q+1)}{\mu} \right]^{1/q} \right\},$$

then we have

$$\mu \frac{\delta^q}{\Gamma(q+1)} < \varepsilon,$$

and so our $x(t;\delta)$ will be a corresponding ε-approximate integral-type solution to (1.1), (1.2) on $[0,a]$. $\square$

Theorem 4.3 addresses [**5**, Remark 6.6]. The following example illustrates the ideas of Theorem 4.3.

Example 4.4. Consider the IVP

$$\begin{aligned} D^q[x - x(0)] &= t + 2x \sin x \\ x(0) &= 1 \end{aligned}$$

with $0 < q \le 1$. The IVP has at least one solution on $[0, a]$ for each $a \in (0, \infty)$.

Proof. In this example we have a special case of (1.1), (1.2) with: $f(t, p) = t + p \sin p$ and $A = 1$. If we choose S to be the strip

$$S := \{(t, p) : t \in [0, a], p \in \mathbf{R}\},$$

then for all $(t, p) \in S$, we have

$$\begin{aligned} |f(t, p)| &= |t + 2p \sin p| \\ &\le 2|p| + a, \end{aligned}$$

and so (4.7) holds with $K = 2$ and $K_1 = a$. Furthermore, $f : S \to \mathbf{R}$ is continuous. Thus, Theorem 4.3 holds and the claim follows. $\square$

It would seem that, in general, the choice of a subsequence of the ε_m-approximate integral-type solution $\{x_m\}$ in the proofs in the preceding results is necessary. However, if it is known that there is, at most, one solution to (1.1), (1.2) on $[0, a]$, then under the conditions of Theorem 4.3, every sequence of ε_m-approximate integral-type solutions $\{x_m\}$ for which $\varepsilon_m \to 0$ must converge uniformly to the solution of (1.1), (1.2). If this is not the case, then there would be at least one ε_m-approximate solution $\{x_m\}$ with $\varepsilon_m \to 0$ which diverges at some point $t \in [0, a]$. Thus, there are at least two subsequences of $\{x_m\}$ which converge to distinct limit functions with both of these functions being a solution to (1.1), (1.2) on $[0, a]$. However, this would contradict the assumption of uniqueness.

The following definition is somewhat complementary to that of Definition 4.1.

Definition 4.5. Define a function $\varphi : [0,a] \times D \subseteq \mathbf{R}^2 \to \mathbf{R}$ an ε-approximate differential-type solution to (1.1) on $[0,a]$ if:

$$D^q\left(\varphi - T_{\lceil q\rceil -1}[\varphi]\right) \text{ is continuous on } [0,a];$$
$$(t,\varphi(t)) \in [0,a]\times D, \text{ for all } t\in[0,a];$$
$$\left|D^q\left(\varphi - T_{\lceil q\rceil -1}[\varphi]\right)(t) - f(t,\varphi(t))\right| \le \varepsilon, \text{ for all } t\in[0,a].$$

Remark 4.6. The definition of an ε-approximate differential-type solution can be relaxed. For example, $D^q(\varphi - T_{\lceil q\rceil -1}[\varphi])$ could have simple discontinuities at a finite number of points W in $[0,a]$, and the final line of Definition 4.5 would then be relaxed to hold for all $t \in [0,a] - W$. However, the simpler definition will suffice for our requirements.

The following result gives an estimate on the difference of two ε-approximate differential-type solutions to (1.1).

Theorem 4.7. *Let $f : [0,a]\times D \subseteq \mathbf{R}^2 \to \mathbf{R}$ be continuous, and let $L > 0$ be a constant such that*

$$|f(t,u) - f(t,v)| \le L|u-v|, \quad \textit{for all } (t,u),(t,v)\in[0,a]\times D. \tag{4.8}$$

Let φ_1 and φ_2 be, respectively, ε_1- and ε_2-approximate differential-type solutions to (1.1) *on $[0,a]$. If $\delta \ge 0$ is a constant such that*

$$\left|\sum_{k=0}^{\lceil q\rceil -1}\left(\varphi_1^{(k)}(0) - \varphi_2^{(k)}(0)\right)\frac{t^k}{k!}\right| \le \delta, \quad \textit{for all } t\in[0,a],$$

then φ_1 and φ_2 satisfy

$$|\varphi_1(t) - \varphi_2(t)| \le \delta E_q(Lt^q) + \frac{\varepsilon_1+\varepsilon_2}{L}\left[E_q(Lt^q) - 1\right], \quad \textit{for all } t\in[0,a]. \tag{4.9}$$

Proof. Since each φ_i is, respectively, an ε_i-approximate differential-type solution to (1.1), we have

$$\left|D^q\left(\varphi_i - T_{\lceil q\rceil -1}[\varphi]\right)(t) - f(t,\varphi_i(t))\right| \le \varepsilon_i, \quad \text{for all } t\in[0,a] \tag{4.10}$$

for $i = 1, 2$. Thus, applying I^q to both sides of (4.10) for $i = 1, 2$, we obtain

$$\left|\varphi_i(t) - \sum_{k=0}^{\lceil q\rceil-1} \frac{\varphi_i^{(k)}(0)t^k}{k!} - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,\varphi_i(s))\,ds\right| \le \frac{\varepsilon_i t^q}{\Gamma(q+1)}, \quad \text{for all } t \in [0,a]. \tag{4.11}$$

Adding together the two inequalities contained in (4.11) and using the inequality $|\alpha - \beta| \le |\alpha| + |\beta|$, we obtain

$$\begin{aligned}\Bigg|(\varphi_1(t) - \varphi_2(t)) - \left(\sum_{k=0}^{\lceil q\rceil-1} \frac{\varphi_1^{(k)}(0)t^k}{k!} - \sum_{k=0}^{\lceil q\rceil-1} \frac{\varphi_2^{(k)}(0)t^k}{k!}\right) \\ - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[f(s,\varphi_1(s)) - f(s,\varphi_2(s))\right]\,ds\Bigg| \\ \le \frac{(\varepsilon_1+\varepsilon_2)t^q}{\Gamma(q+1)}, \quad \text{for all } t \in [0,a].\end{aligned} \tag{4.12}$$

Now let

$$\rho(t) := |\varphi_1(t) - \varphi_2(t)|, \quad \text{for all } t \in [0,a].$$

Substitution of ρ into (4.12) coupled with a rearrangement and use of the inequality $|\alpha| - |\beta| \le |\alpha - \beta|$ then gives for all $t \in [0,a]$

$$\begin{aligned}\rho(t) &\le \left|\sum_{k=0}^{\lceil q\rceil-1} \frac{\varphi_1^{(k)}(0)t^k}{k!} - \sum_{k=0}^{\lceil q\rceil-1} \frac{\varphi_2^{(k)}(0)t^k}{k!}\right| \\ &\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,\varphi_1(s)) - f(s,\varphi_2(s))|\,ds + \frac{(\varepsilon_1+\varepsilon_2)t^q}{\Gamma(q+1)} \\ &\le \delta + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} L\rho(s)\,ds + \frac{(\varepsilon_1+\varepsilon_2)t^q}{\Gamma(q+1)} \\ &= \delta + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[L\rho(s) + (\varepsilon_1+\varepsilon_2)\right]\,ds,\end{aligned}$$

where we have used (4.8). An application of Lemma 3.1 then yields the bound (4.9). $\square$

Remark 4.8. Theorem 4.7 is the best possible in the sense that equality can be attained in (4.9) for nontrivial functions φ_1, φ_2. Consider, for $0 < q \leq 1$, the differential equation

$$D^q\left(x - x(0)\right) = Lx, \quad L > 0,$$

and let φ_1, φ_2, respectively, satisfy

$$\begin{aligned} D^q\left(\varphi_1 - \varphi_1(0)\right) &= L\varphi_1 - \varepsilon_1, \quad \varphi_1(0) = b_1; \\ D^q\left(\varphi_2 - \varphi_2(0)\right) &= L\varphi_2 + \varepsilon_2, \quad \varphi_2(0) = b_2; \end{aligned}$$

with $b_1 \leq b_2$. The above linear equations have solutions

$$\begin{aligned} \varphi_1(t) &= \frac{\varepsilon_1}{L} + \left(b_1 - \frac{\varepsilon_1}{L}\right) E_q(Lt^q) \\ \varphi_2(t) &= -\frac{\varepsilon_2}{L} + \left(b_2 + \frac{\varepsilon_2}{L}\right) E_q(Lt^q) \end{aligned}$$

which satisfy

$$|\varphi_1(t) - \varphi_2(t)| = \frac{\varepsilon_1 + \varepsilon_2}{L}\left[E_q(Lt^q) - 1\right] + (b_2 - b_1)E_q(Lt^q),$$

for all $t \in [0, a]$ and so we have equality in (4.9).

Remark 4.9. If φ_1 is an actual solution to (1.1), then $\varepsilon_1 = 0$ in (4.9). Thus, under the conditions of Theorem 4.7, it follows that $\varphi_2 \to \varphi_1$ as $\varepsilon_2 \to 0$ and $\delta \to 0$.

Let $D \subseteq \mathbf{R}$, and let $f, g : [0, a] \times D \to \mathbf{R}$. Consider the two IVPs

$$D^q\left(x - T_{\lceil q \rceil - 1}[x]\right)(t) = f(t, x(t)), \quad x^{(i)}(0) = \alpha_i; \tag{4.13}$$

$$D^q\left(x - T_{\lceil q \rceil - 1}[x]\right)(t) = g(t, x(t)), \quad x^{(i)}(0) = \beta_i, \tag{4.14}$$

$i = 0, \ldots, \lceil q \rceil - 1$, where α_i and β_i are constants such that each α_i, $\beta_i \in D$. The following result (related to [**6**, Theorems 3.2, 3.3]) relates the solutions of (4.13) and (4.14) in the sense that: if f is close to g, and if each α_i is close to each corresponding β_i, then the solutions to the two problems are also close together.

Theorem 4.10. *Let f, $g : [0,a] \times D \to \mathbf{R}$, and that assume there exist constants $L > 0$, $\varepsilon \geq 0$ and $\delta \geq 0$ such that*

$$|f(t,p) - f(t,q)| \leq L|p-q|, \quad \textit{for all } (t,p),\ (t,q) \in [0,a] \times D;$$
$$|f(t,p) - g(t,p)| \leq \varepsilon, \quad \textit{for all } (t,p) \in [0,a] \times D;$$
$$\left| \sum_{i=0}^{\lceil q \rceil -1} (\alpha_i - \beta_i) \frac{t^i}{i!} \right| \leq \delta, \quad \textit{for all } t \in [0,a].$$

If x_1 and x_2 are, respectively, solutions to (4.13) and (4.14) such that each $(t, x_i(t)) \in [0,a] \times D$ for all $t \in [0,a]$, then

$$|x_1(t) - x_2(t)| \leq \delta E_q(Lt^q) + \frac{\varepsilon}{L} \left[E_q(Lt^q) - 1 \right], \quad \textit{for all } t \in [0,a]. \tag{4.15}$$

Proof. For each $t \in [0,a]$, consider

$$\begin{aligned}
\rho(t) &:= |x_1(t) - x_2(t)| \\
&= \Bigg| \sum_{i=0}^{\lceil q \rceil -1} (\alpha_i - \beta_i) \frac{t^i}{i!} \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s, x_1(s)) - g(s, x_2(s))]\, ds \Bigg| \\
&\leq \delta + \Bigg| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s, x_1(s)) - f(s, x_2(s)) \\
&\qquad\qquad + f(s, x_2(s)) - g(s, x_2(s))]\, ds \Bigg| \\
&\leq \delta + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x_1(s)) - f(s, x_2(s))|\, ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x_2(s)) - g(s, x_2(s))|\, ds \\
&\leq \delta + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} L \rho(s)\, ds + \frac{\varepsilon t^q}{\Gamma(q+1)}.
\end{aligned}$$

We can now apply Lemma 3.1 above to obtain (4.15). ◻

Now consider (1.1), (1.2) together with the sequence of problems for $i = 0, \ldots, \lceil q \rceil - 1$

$$D^q \left(x - T_{\lceil q \rceil -1}[x] \right)(t) = g_k(t, x(t)), \quad x^{(i)}(0) = A_{i,k},\ k = 1, 2, \ldots \tag{4.16}$$

where g_k and $A_{i,k}$ are sequences (in k) with each g_k being continuous on $[0,a] \times D$ and each $A_{0,k} \in D$. As a corollary to Theorem 4.10, the following result assures solutions to (4.16) will converge to solutions to (1.1), (1.2).

Corollary 4.11. *Let f and each g_k be continuous on $[0,a] \times D$. Furthermore, let f and each g_k satisfy*

(4.17)
$$|f(t,p) - f(t,q)| \leq L|p-q| \quad \textit{for all } (t,p),\ (t,q) \in [0,a] \times D;$$
(4.18)
$$|f(t,p) - g_k(t,p)| \leq \varepsilon_k \quad \textit{for all } \ (t,p) \in [0,a] \times D;$$
(4.19)
$$\left| \sum_{i=0}^{\lceil q \rceil - 1} (A_i - A_{i,k}) \frac{t^i}{i!} \right| \leq \delta_k, \quad k = 0, 1 \ldots$$
where $L > 0$ is a constant and both ε_k and δ_k converge to zero. If x_k is a solution of (4.16) *and x is a solution to* (1.1), (1.2) *on* $[0,a]$ *such that each $(t, x(t)), (t, x_k(t)) \in [0,a] \times D$ for all $t \in [0,a]$, then $x_k \to x$ on* $[0,a]$.

Proof. For each k, the conditions of Theorem 4.10 hold with the result following from an application of (4.15). ◻

As an illustration of Corollary 4.11, the following example is presented.

Example 4.12. Let $0 < q < 1$, and consider the sequence of IVPs
$$D^q[x - x(0)] = x + \sum_{i=0}^{k} \frac{t^i}{i!} \cos x$$
$$x(0) = 1 + \frac{1}{k+1}, \quad k = 0, 1, \ldots .$$
The IVP has at least one solution x_k on $[0,a]$ such that x_k converges uniformly on $[0,a]$ to a solution of the IVP
$$D^q[x - x(0)] = x + e^t \cos x$$
$$x(0) = 1.$$

Proof. In this example we have a special case of (4.16) with:

$$g_k(t,p) = p + \sum_{i=0}^{k} \frac{t^i}{i!} \cos p; \qquad A_{0,k} = 1 - \frac{1}{k+1};$$

$f(t,p) = p + e^t \cos p$ and $A_0 = 1$.

Let $D = \mathbf{R}$, and see that f and each g_k are continuous on $[0,a] \times \mathbf{R}$. Now, as $\partial f/\partial p$ is uniformly bounded and continuous on $[0,a] \times \mathbf{R}$, it follows that there is a constant $L > 0$ such that (4.17) holds. In addition, for each k and all $(t,p) \in [0,a] \times \mathbf{R}$, we have

$$\begin{aligned} |f(t,p) - g_k(t,p)| &= \left| \left[e^t - \sum_{i=0}^{k} \frac{t^i}{i!} \right] \cos p \right| \le \left| e^t - \sum_{i=0}^{k} \frac{t^i}{i!} \right| \\ &= \left| \sum_{i=0}^{\infty} \frac{t^i}{i!} - \sum_{i=0}^{k} \frac{t^i}{i!} \right| = \left| \sum_{i=k+1}^{\infty} \frac{t^i}{i!} \right| \le \sum_{i=k+1}^{\infty} \frac{a^i}{i!} \\ &\le \frac{a^{k+1}}{(k+1)!} \sum_{i=0}^{\infty} \frac{a^i}{i!} = \frac{a^{k+1}}{(k+1)!} e^a. \end{aligned}$$

Thus, (4.18) holds with $\varepsilon_k = a^{k+1}/(k+1)! e^a$. Finally, we see that (4.19) holds with $\delta_k = 1/(k+1)$.

Thus, Corollary 4.11 holds and the claim follows. □

5. Successive approximations. The method of successive approximations is a sequential technique that involves recursion and, as such, is different from the sequential approach of Section 4. The idea is to solve an equation of type

$$y = F(y), \quad F \text{ is continuous}; \tag{5.1}$$

by defining a starting "point" y_0 and then introducing a sequence y_m in a recursive fashion defined via

$$y_{m+1} := F(y_m), \quad m = 1, 2, \dots .$$

If y_m converges to some y^*, then y^* will be a solution to (5.1).

The method of successive approximations for ordinary differential equations is credited to Liouville [**24**, page 444] and Picard [**28**]. For the fractional order case of (1.1), (1.2) the method of successive approximations has successfully been introduced in [**6**] and further developed in [**21, 22**]. Therein, the sequence of functions $\{\phi_m\} = \{\phi_m(t)\}$ was defined in the following way:

(5.2)
$$\phi_0(t) := \sum_{i=0}^{\lceil q\rceil-1} \frac{A_i t^i}{i!},$$

(5.3)
$$\phi_{m+1}(t) := \sum_{i=0}^{\lceil q\rceil-1} \frac{A_i t^i}{i!} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,\phi_m(s))\, ds, \ m = 1, 2, \dots .$$

Some of the main results in [**6, 21, 22**] for (1.1), (1.2) involve *continuous* functions $f : R \to \mathbf{R}$, where R is the set

$$R := \{(t,p) \in \mathbf{R}^2 : t \in [0,a],\ |p - A| \leq b\}, \quad b > 0. \tag{5.4}$$

Under additional assumptions on f, such as: a Lipschitz condition (see (6.5)); or weak monotonicity in the second variable; [**6, 21, 22**] showed that the sequence ϕ_m converged uniformly on some subinterval of $[0,a]$ to a solution of (1.1), (1.2).

The assumptions of [**6, 21, 22**] naturally lead to the question: is continuity of f alone enough to guarantee the convergence of the sequence (or subsequence) of the successive approximations (5.2), (5.3) to a solution to (1.1), (1.2)? An example is now presented that answers this question in the negative.

Example 5.1. Let $q > 0$ and $\theta : [0,1] \to [0,\infty)$ be a q-differentiable function that satisfies: $\theta(0) = 0$; $\theta(t) > 0$ for all $t \in (0,1]$; θ is strictly increasing on $[0,1]$ and $D^q\theta$ is continuous on $[0,1]$.

Consider the IVP

$$D^q \left(x - T_{\lceil q\rceil-1}[x]\right)(t) = f(t, x(t)); \tag{5.5}$$

$$x(0) = 0, \dots ,\ x^{(\lceil q\rceil-1)}(0) = 0; \tag{5.6}$$

with f defined by

$$f(t,p) := \begin{cases} 0 & \text{for all } t = 0,\ -\infty < p < \infty; \\ D^q\theta(t) & \text{for all } t \in (0,1],\ p \leq 0; \\ D^q\theta(t) - \frac{D^q\theta(t)}{\theta(t)}p & \text{for all } t \in (0,1],\ 0 < p \leq \theta(t); \\ 0 & \text{for all } t \in (0,1],\ p > \theta(t). \end{cases}$$

Although f is continuous on $[0,1] \times \mathbf{R}$, this property alone is insufficient to guarantee the convergence of the associated sequence (or a subsequence) of successive approximations to a solution of (5.5), (5.6) on any subinterval of $[0,1]$.

Proof. Define the sequence of successive approximations associated with (5.5), (5.6) via

(5.7)
$$\phi_0(t) := 0,$$
(5.8)
$$\phi_{m+1}(t) := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \phi_m(s))\, ds, \quad m = 1, 2, \ldots .$$

Thus, we have

$$\begin{aligned} \phi_1(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \phi_0(s))\, ds = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, 0)\, ds \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} D^q\theta(s)\, ds = I^q \left[{}^C D^q \theta(t) \right] = \theta(t) \end{aligned}$$

for all $t \in [0,1]$. The second iteration is

$$\begin{aligned} \phi_2(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \phi_1(s))\, ds \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \theta(s))\, ds \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[D^q\theta(s) - \frac{D^q\theta(s)}{\theta(s)} \theta(s) \right] ds = 0 \end{aligned}$$

for all $t \in [0,1]$.

Similarly, $\phi_3(t) = \theta(t)$ and $\phi_4(t) = 0$ for all $t \in [0,1]$. For each integer $m \geq 0$, we obtain

$$\phi_{2m}(t) = 0 \quad \text{and} \quad \phi_{2m+1}(t) = \theta(t), \quad \text{for all } t \in [0,1].$$

Thus, $\phi_{2m} \to 0$ and $\phi_{2m+1} \to \theta$ uniformly on $[0,1]$ and so ϕ_m cannot converge to a limit on $[0,1]$ because the limit of subsequences is distinct. Furthermore, neither of the limits of the above subsequences are actually solutions to (5.5), (5.6) because

$$f(t,0) \neq 0, \qquad f(t,\theta(t)) \neq D^q\left(\theta - T_{\lceil q\rceil -1}[\theta]\right)(t), \quad \text{for all } t \in (0,1].$$

It can be verified that a solution to (5.5), (5.6) is $x(t) = \theta(t)/2$ for all $t \in [0,1]$. This particular solution cannot be obtained by the above scheme of successive approximations. $\square$

The above example is an adaptation of a celebrated example formed for the case of ordinary differential equations ($q = 1$) by Müller [**27**] and Reid [**32**, pages 50–51].

The following result is known as "enveloping" of solutions [**14**, page 865], where the sign of the error between the successive approximations and the true solution alternates.

Define and denote the non-negative half-plane P by

$$P := \{(t,p) : t \geq 0,\ p \in \mathbf{R}\}.$$

Theorem 5.2. *Let $f : P \to \mathbf{R}$ be continuous and satisfy*

$$f(t,v) \leq f(t,w), \quad \textit{for all } v \leq w. \tag{5.9}$$

Let $x = x(t)$ be a solution to

$$D^q\left(x - T_{\lceil q\rceil -1}[x]\right)(t) = -f(t,x(t))$$

on $[0,\infty)$ with $x(t) \leq \sum_{i=0}^{\lceil q\rceil -1} x^{(i)}(0)t^i/i!$ for all $t \geq 0$, and consider the sequence of successive approximations $\{\phi_m\} = \{\phi_m(t)\}$ defined by

$$\phi_0(t) := \sum_{i=0}^{\lceil q\rceil -1} \frac{A_i t_i}{i!},$$

$$\phi_{m+1}(t) := \phi_0(t) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,\phi_m(s))\,ds, \quad m = 1,2,\ldots.$$

If $z_m = z_m(t)$ *denotes the "error"*

$$z_m(t) := \phi_m(t) - x(t), \quad \textit{for all } t \geq 0 \textit{ and } m = 0, 1, 2, \dots,$$

then

$$(5.10) \qquad (-1)^m z_m(t) \geq 0, \quad \textit{for all } t \geq 0 \textit{ and } m = 0, 1, 2 \dots .$$

Proof. It can be shown directly that (5.10) holds for $m = 0, 1, 2$.

Assume that (5.10) holds for some even number $m = k \geq 2$, so that $z_k = \phi_k - x \geq 0$ on $[0, \infty)$. Thus, for all $t \in [0, \infty)$, we have

$$\begin{aligned}(-1)^{k+1} z_{k+1}(t) &= -\left[\phi_{k+1}(t) - x(t)\right] \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[f(s, \phi_k(s)) - f(s, x(s))\right] ds \\ &\geq 0\end{aligned}$$

where we have invoked (5.9).

A similar result follows by assuming that (5.10) holds for some odd number $m = k \geq 3$. Combining the above cases we see that (5.10) holds by induction. $\square$

Example 5.3. As an illustration of Theorem 5.2, consider the simple equation with $0 < q < 1$

$$D^q[x - 1] = -x,$$

which has the solution $x(t) = E_q(-t^q)$ for $t \in [0, \infty)$. The successive approximations defined in Theorem 5.2 then become the m-th partial sum of $E_q(-t^q)$, namely,

$$\phi_m(t) = \sum_{k=0}^{m} \frac{(-t^q)^k}{\Gamma(qk+1)}, \quad t \in [0, \infty), \; m = 0, 1, \dots .$$

It has been shown in [**30**, pages 1115–1116] that $E_q(-t^q) \leq 1$ for all $t \geq 0$. All of the conditions of Theorem 5.2 hold, and thus (5.10) holds.

6. Fixed-point approach. This section is concerned with an optimal application of Banach's fixed point theorem [**9**, page 10] to an operator associated with (1.1), (1.2). Banach's theorem is simple, elegant and quite wide-ranging. It applies to "contractive" mappings between complete metric spaces, yielding the existence of a unique fixed-point to the operator involved.

Theorem 6.1 [**9**]. *Let (X, d) be a complete metric space and let $F : X \to X$. If F is contractive in the sense that there exists a positive constant $\sigma < 1$ with*

$$d(Fx, Fy) \leq \sigma d(x, y), \quad \textit{for all } x, y \in X, \tag{6.1}$$

then F has a unique fixed point u, that is, $Fu = u$ for a unique $u \in X$. In addition, $F^m y \to u$ for each $y \in X$, where $F^0 y := y$ and $F^{m+1}y := F(F^m y)$.

The contraction condition (6.1) is sensitive to the metric d in the sense that a mapping may be contractive on a set X under one particular metric but not contractive on X under a different metric [**9**, pages 24–25]. Motivated to optimize this dependency, a new metric is defined shortly that involves the Mittag-Leffler function. This particular metric will be optimal in the sense that it forces the operator involved to be contractive on the whole of $X = C([0, a])$ (the space of continuous functions on $[0, a]$), rather than on a smaller set. In this way, Banach's classical theorem will directly apply and there is no need to firstly obtain existence of a fixed point on a set of type $C([0, h])$, $h < a$, and secondly to extend this solution to (1.1), (1.2) from $[0, h]$ to the whole of $[0, a]$ as in [**18**], nor is there any need to appeal to more abstract versions of Banach's classical theorem as in [**6**].

Remark 6.2. It is well known [**9**, page 10] that, by beginning at an arbitrary $y \in X$, Banach's theorem provides the following estimate on the "error" between the mth iteration $F^m y$ and the fixed point u, namely,

$$d(F^m y, u) \leq \frac{\sigma^m}{1 - \sigma}\, d(y, Fy). \tag{6.2}$$

Let $\beta > 0$ be a constant and $q > 0$. Consider the space of continuous functions $C([0,a])$ coupled with a suitable metric, either

$$d_\beta(x,y) := \max_{t\in[0,a]} \frac{|x(t)-y(t)|}{E_q(\beta t^q)}$$

or

$$d_0(x,y) := \max_{t\in[0,a]} |x(t)-y(t)|, \quad \text{the well known max-metric.} \tag{6.3}$$

The above definition of d_β is a new generalization of Bielecki's metric [**2**], [**9**, pages 25–26], [**10**, pages 153–155], [**33**, page 44].

Some important properties of d_β are now listed.

Lemma 6.3. *If $\beta > 0$ is a constant and $q > 0$, then*:

(i) *d_β is a metric*;

(ii) *d_β is equivalent to the max-metric d_0*;

(iii) *$(C([0,a]), d_\beta)$ is a complete metric space.*

Proofs. (i) If $\beta > 0$ is a constant, then we have $E_q(\beta t^q) > 0$ for all $t \in [0,a]$ and E_q is continuous on $[0,a]$. The three properties of a metric [**4**, page 21] are now easily verified.

(ii) Now, since E_q is continuous and strictly increasing on $[0,a]$, we have

$$\frac{1}{E_q(\beta a^q)} \le \frac{1}{E_q(\beta t^q)} \le 1, \quad \text{for all } t \in [0,a],$$

and so

$$\frac{1}{E_q(\beta a^q)} d_0(x,y) \le d_\beta(x,y) \le d_0(x,y), \quad \text{for all } x, y \in C([0,a]). \tag{6.4}$$

Thus, (6.4) ensures that our metrics are equivalent.

(iii) The completeness of $(C([0,a]), d_\beta)$ now follows from the completeness of $(C([0,a]), d_0)$ and (ii). If $\{x_n\}$ is a Cauchy sequence in $(C([0,a]), d_0)$, then (ii) ensures that $\{x_n\}$ is a Cauchy sequence in $(C([0,a]), d_\beta)$ as

$$\lim_{m,n\to\infty} d_0(x_n, x_m) = 0 \quad \text{implies} \quad \lim_{m,n\to\infty} d_\beta(x_n, x_m) = 0.$$

Furthermore, it can show shown [**16**, Example 3, page 32] that there is a continuous function x on $[0,a]$ such that $\lim_{n\to\infty} d_0(x_n, x) = 0$. As a result of (ii), we then have $\lim_{n\to\infty} d_\beta(x_n, x) = 0$. Hence, our Cauchy sequence x_n in $(C([0,a]), d_\beta)$ is convergent and the limit is a continuous function on $[0,a]$. Thus, $(C([0,a]), d_\beta)$ is a complete metric space. $\square$

Define and denote the strip S by

$$S := \{(t,p) : t \in [0,a],\ p \in \mathbf{R}\}.$$

The main result of this section now follows.

Theorem 6.4. *Let $f : S \to \mathbf{R}$ be continuous. If there is a positive constant L such that*

$$|f(t,u) - f(t,v)| \le L|u-v|, \quad \text{for all } (t,u), (t,v) \in S, \tag{6.5}$$

then the IVP (1.1), (1.2) *has a unique solution on* $[0,a]$. *In addition, if a sequence of functions $\{x_i\}$ is defined inductively by choosing any $x_0 \in C([0,a])$ and setting*

$$x_{i+1}(t) = \sum_{k=0}^{\lceil q \rceil - 1} \frac{A_k t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_i(s))\, ds, \tag{6.6}$$
$$t \in [0,a], \quad i = 1, 2, \dots ,$$

then the sequence $\{x_i\}$ converges uniformly on $[0,a]$ to the unique solution x of (1.1), (1.2).

Proof. Since f is a continuous function on S, (6.6) is well defined. Let $L > 0$ be the constant defined in (6.5), and let $\beta := L\gamma$ where $\gamma > 1$ is an arbitrary constant. Consider the complete metric space $(C([0,a]), d_\beta)$, and let F be defined by

$$[Fx](t) := \sum_{k=0}^{\lceil q \rceil - 1} \frac{A_k t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \quad t \in [0,a]. \tag{6.7}$$

By Lemma 2.1, showing the existence of fixed-points of F is equivalent to showing the existence of solutions to the IVP (1.1), (1.2). Thus, we want to prove that there exists a unique x such that $Fx = x$. To do this, we show that $F : C([0,a]) \to C([0,a])$ is a contractive map with contraction constant $\sigma = 1/\gamma < 1$ under the d_β metric and Banach's fixed-point theorem will then apply.

We now show that F is contractive with respect to d_β. For any x, $y \in C([0,a])$, consider

$$
\begin{aligned}
d_\beta(Fx, Fy) &:= \max_{t\in[0,a]} \frac{|[Fx](t) - [Fy](t)|}{E_q(\beta t^q))} \\
&\le \max_{t\in[0,a]} \left[\frac{1}{E_q(\beta t^q)} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,x(s)) \right. \\
&\qquad\qquad \left. - f(s,y(s))|\, ds \right] \\
&\le \max_{t\in[0,a]} \left[\frac{1}{E_q(\beta t^q)} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} L|x(s) - y(s)|\, ds \right] \\
&= L \max_{t\in[0,a]} \left[\frac{1}{E_q(\beta t^q)} \frac{1}{\Gamma(q)} \right. \\
&\qquad \left. \times \int_0^t (t-s)^{q-1} E_q(\beta s^q) \frac{|x(s) - y(s)|}{E_q(\beta s^q)}\, ds \right] \\
&\le L d_\beta(x,y) \max_{t\in[0,a]} \left[\frac{1}{E_q(\beta t^q)} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_q(\beta s^q)\, ds \right] \\
&= L d_\beta(x,y) \max_{t\in[0,a]} \left[\frac{1}{E_q(\beta t^q)} I^q \left({}^C D^q \left(\frac{E_q(\beta t^q)}{\beta} \right) \right) \right] \\
&= L d_\beta(x,y) \max_{t\in[0,a]} \left[\frac{1}{E_q(\beta t^q)} \left(\frac{E_q(\beta t^q)) - 1}{\beta} \right) \right] \\
&= \frac{d_\beta(x,y)}{\gamma} \max_{t\in[0,a]} \left[1 - \frac{1}{E_q(\beta t^q)} \right] \\
&= \frac{d_\beta(x,y)}{\gamma} \left[1 - \frac{1}{E_q(\beta a^q)} \right] \\
&\le \frac{d_\beta(x,y)}{\gamma}
\end{aligned}
$$

where we have used (6.5) and $\beta = L\gamma$ above. As $\gamma > 1$, we see that F is a contractive map with contraction constant $\sigma = 1/\gamma$ and Banach's fixed-point theorem applies, yielding the existence of a unique fixed-point x of F. In addition, from Banach's theorem, the sequence $\{x_i\}$

defined in (6.6) converges uniformly in the norm $\|\cdot\|_\beta$, and thus the sequence $\{x_i\}$ converges uniformly in the max-norm $\|\cdot\|_0$ to that fixed-point x. This completes the proof. $\square$

The statement of Theorem 6.4 is not new, but the application of the metric d_β is novel and optimizes the proof. For example, if the max-metric d_0 was used in the proof (as in [**17**]), then Banach's theorem would only be contractive on $C([0,h])$ with $h < a$ and then a solution to (1.1), (1.2) on $[0,h]$ would need to be systematically extended to the whole interval $[0,a]$. The proof of Theorem 6.4 illustrates that the approach of [**17**] of existence-extension is unnecessary. In addition, the proof of Theorem 6.4 demonstrates that, invoking more abstract versions of Banach's theorem is unnecessary: the basic theorem of Banach will suffice. Theorem 6.4 addresses [**5**, Remark 6.10].

In view of Remark 6.2, the approach in the proof of Theorem 6.4 can be used to evaluate the rate of convergence of iterates. If x, $x_0 \in C([0,a])$ and $\beta := L\gamma$ with $\gamma > 1$, then (6.2) yields

$$d_\beta(F^m x_0, x) \le \frac{\gamma^{-m}}{1-\gamma^{-1}} d_\beta(x_0, Fx_0), \quad m = 1, 2, \dots,$$

and so

$$\|F^m x_0 - x\|_0 \le E(L\gamma t^q)\frac{\gamma^{-m}}{1-\gamma^{-1}}\|x_0 - Fx_0\|_0, \quad m = 1, 2, \dots, \tag{6.8}$$

where $\|\cdot\|_0$ is the norm induced by the max-metric (6.3). The choice $\gamma := m/La$ yields a nice evaluation of the rate of convergence in (6.8), namely,

$$\|F^m x_0 - x\|_0 \le E_q\left(ma^{q-1}\right)\left(\frac{La}{m}\right)^m \frac{m}{m-La}\|x_0 - Fx_0\|_0,$$
$$m = 1, 2, \dots .$$

The following result illustrates the dependency of solutions to the IVP (1.1), (1.2) with respect to initial values.

Theorem 6.5. *The solution supplied under the conditions of Theorem* 6.4 *is Lipschitz continuous in A, uniformly in t. In addition, for* *any two sets of initial conditions A_i, $B_i \in \mathbf{R}$,*

$$|x(t; A_0, \dots, A_{\lceil q\rceil-1}) - x(t; B_0, \dots, B_{\lceil q\rceil-1})| \leq E_q(Lt^q) \sum_{i=0}^{\lceil q\rceil-1} |A_i - B_i| \frac{a^i}{i!}, \quad \textit{for all } t \in [0, a].$$

Proof. Using (6.5) in a standard fashion, we obtain the estimate

$$|x(t; A_0, \dots, A_{\lceil q\rceil-1}) - x(t; B_0, \dots, B_{\lceil q\rceil-1})| \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} L |x(s; A_0, \dots, A_{\lceil q\rceil-1}) - x(s; B_0, \dots, B_{\lceil q\rceil-1})| \, ds + \sum_{i=0}^{\lceil q\rceil-1} |A_i - B_i| \frac{a^i}{i!},$$

for all $t \in [0, a]$. An application of Lemma 3.1 yields the desired result. □

7. Degree theory approach. In this section a more modern approach is taken to the existence of solutions to (1.1), (1.2) than in previous sections. The aim is to generate sufficient conditions, in as much generality as possible, that guarantee at least one solution to (1.1), (1.2) exists on $[0, a]$. The method of the Leray-Schauder degree [**23**, Chapter 4, pages 54–71] is used to do this.

Theorem 7.1. *Let $f : S \to \mathbf{R}$ be continuous, let $h : [0, a] \to \mathbf{R}$ be continuous, and let $g : [0, \infty) \to [0, \infty)$ be continuous and non-decreasing. If*

$$|f(t, p)| \leq h(t) g\left(\left|p - \sum_{i=0}^{\lceil q\rceil-1} \frac{A_i t^i}{i!}\right|\right), \quad \textit{for all } (t, p) \in S, \tag{7.1}$$

and there exists a constant $b > 0$ such that

$$\frac{b}{g(b)} > \max_{t \in [0,a]} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) \, ds \tag{7.2}$$

then (1.1), (1.2) *has at least one solution* x *on* $[0, a]$ *such that* $|x(t)| < b$ *for all* $t \in [0, a]$.

Proof. We apply basic Leray-Schauder degree theory.

Consider the normed space $(C([0, a]), \|\cdot\|_0)$ and define the closed subset $\overline{\Omega} \subset C([0, a])$ by

$$\overline{\Omega} := \left\{ x \in C([0, a]) : \max_{t\in[0,a]} \left| x(t) - \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t^i}{i!} \right| \le b \right\}.$$

Let F be defined as in (6.7), and note that F is well-defined on $\overline{\Omega}$. We claim that $F : \Omega \to C([0, a])$ is a compact operator in the sense that: F is continuous; $\overline{F(\Omega)}$ is a compact set.

We show that F is continuous on Ω. Our approach follows that of [**6**, page 235]. Function f is continuous on the compact set R defined in (5.4) and so f is uniformly continuous on R. This means that, given any $\varepsilon > 0$, we can choose a $\delta = \delta(\varepsilon) > 0$ such that

$$|f(t, u) - f(t, v)| < \frac{\varepsilon}{a^q}\Gamma(q + 1) \quad \text{whenever } |u - v| < \delta.$$

Thus, for x, $y \in \Omega$, consider

$$\begin{aligned} |[Fx](t) - [Fy](t)| &= \left| \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} [f(s, x(s)) - f(s, y(s))]\, ds \right| \\ &< \frac{\varepsilon \Gamma(q + 1)}{a^q} \left| \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1}\, ds \right| = \frac{\varepsilon t^q}{a^q} \le \varepsilon. \end{aligned}$$

Thus, for any given $\varepsilon > 0$,

$$\|Fx - Fy\|_0 < \varepsilon \quad \text{whenever } \|x - y\|_0 < \delta$$

with δ chosen above.

To show that $\overline{F(\Omega)}$ is a compact set, we show that $F(\Omega)$ is equicontinuous and uniformly bounded. The Arzela-Ascoli theorem [**19**, page 104] ensures $\overline{F(\Omega)}$ will then be a compact set.

Since f is continuous on the compact rectangle R, there exists a constant $M > 0$ such that

$$|f(t, p)| \le M, \quad \text{for all } (t, p) \in R.$$

Let $x \in \Omega$, and consider

$$\left\| Fx - \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t^i}{i!} \right\|_0 = \max_{t \in [0,a]} \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \right| \leq \frac{M a^q}{\Gamma(q+1)}.$$

Thus, $F(\Omega)$ is uniformly bounded.

Furthermore, for any $0 \leq t_1 \leq t_2 \leq a$, we have that, for any given $\varepsilon > 0$, we can choose a δ such that, if $|t_2 - t_1| < \delta$, then $|[Fx](t_1) - [Fx](t_2)| < \varepsilon$. The proof of this step is virtually identical to an analogous step in the proof of Theorem 4.2 and so is omitted.

Now consider the family of mappings

$$H(x, \lambda) := I(x) - \lambda F(x), \quad \text{for all } (\lambda, x) \in [0,1] \times \overline{\Omega}$$

where I is the identity map. We claim that $H \neq 0$ on $[0,1] \times \partial\Omega$, ensuring that the Leray-Schauder topological degree of H on Ω relative to 0 is well defined.

Suppose $H(\lambda, x) = 0$ for some fixed $\lambda \in [0,1]$ and some $x \in \partial\Omega$. Thus, we have

$$\begin{aligned}
b &= \max_{t \in [0,a]} \left| H(x(t), \lambda) - \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i t^i}{i!} \right| \\
&= \max_{t \in [0,a]} \left| \frac{\lambda}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \right| \\
&\leq \max_{t \in [0,a]} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) g\left(\left| x(s) - \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i s^i}{i!} \right| \right) ds \\
&\leq \max_{t \in [0,a]} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) g\left(\max_{s \in [0,a]} \left| x(s) - \sum_{i=0}^{\lceil q \rceil - 1} \frac{A_i s^i}{i!} \right| \right) ds \\
&\leq g(b) \max_{t \in 0,a]} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\, ds < b,
\end{aligned}$$

where we have used (7.2) and the assumption that g is nondecreasing. We have reached a contradiction.

Thus, the following degree calculations are well defined, and the invariance of the homotopy property of the Leray-Schauder degree [**23**, page 70] may be applied

$$\begin{aligned}\deg_{\mathrm{LS}}(H(\cdot,\lambda),\Omega,0) &= \deg_{\mathrm{LS}}(H(\cdot,1),\Omega,0) = \deg_{\mathrm{LS}}(H(\cdot,0),\Omega,0)\\ &= \deg_{\mathrm{LS}}\left(I - \sum_{i=0}^{\lceil q\rceil -1}\frac{A_i t^i}{i!},\Omega,0\right) = 1,\end{aligned}$$

with the value of 1 obtained since $\sum_{i=0}^{\lceil q\rceil -1} A_i t^i/i \in \Omega$. The above calculations show that, for each $\lambda \in [0,1]$, there is an $x \in \Omega$ such that $H(x,\lambda) = 0$. Thus, $H(x,1) = 0$ has at least one solution $x \in \Omega$, with this problem being equivalent to finding fixed points of F. Thus, our solution to (1.1), (1.2) exists as claimed. □

Remark 7.2. Condition (7.2) will hold, for example, if

$$\frac{b}{g(b)} > \frac{a^q}{\Gamma(q+1)}\max_{t\in[0,a]} h(t). \tag{7.3}$$

The following example illustrates Theorem 7.1.

Example 7.3. Consider the IVP

$$\begin{aligned}D^{1/2}[x - x(0)] &= te^x\\ x(0) &= 0.\end{aligned}$$

The IVP has at least one solution on $[0,1/4]$.

Proof. In this example we have a special case of (1.1), (1.2) with: $f(t,p) = te^p$; $A_0 = 0$ and $q = 1/2$. Choose the strip

$$S_{1/4} := \{(t,p) : t \in [0,1/4],\ p \in \mathbf{R}\}.$$

Note that, for all $(t,p) \in S_{1/4}$, we have

$$|f(t,p)| = |te^p| \le te^{|p|}$$

and so (7.1) will hold with: $h(t) = t$ and $g(|p|) = e^{|p|}$. Now choose $b = 1/2$ and, in view of (7.3) and its context, we see that

$$\begin{aligned}\frac{a^q}{\Gamma(q+1)} \max_{t\in[0,a]} h(t) &= \frac{(1/4)^{1/2}}{\Gamma(3/2)} \max_{t\in[0,1/4]} t\\ &= \frac{(1/4)^{3/2}}{\Gamma(3/2)}\\ &= \frac{1}{4\sqrt{2\pi}} \approx 0.1.\end{aligned}$$

Also,

$$\frac{b}{g(b)} = \frac{1/2}{e^{1/2}} \approx 0.3.$$

Thus, (7.3) holds and Theorem 7.1 may be applied. □

In a similar style to Theorem 7.1 the following result ensures the existence of at least one "local" solution to (1.1), (1.2).

Theorem 7.4. *Let $f : S \to \mathbf{R}$ be continuous, $h : [0, a] \to \mathbf{R}$ continuous and $g : [0,\infty) \to [0,\infty)$ continuous and non-decreasing. If* (7.1) *holds, then there is a $\delta \le a$ such that* (1.1), (1.2) *has at least one solution x on* $[0, \delta]$.

Proof. The proof is very similar to that of Theorem 7.1 and so is only outlined. Given g and h, choose a $b > 0$ and $\delta \le a$ sufficiently small so that

$$\frac{b}{g(b)} > \max_{t\in[0,\delta]} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\, ds,$$

and then apply similar principles as in the proof of Theorem 7.1 in the normed space $(C([0,\delta]), \|\cdot\|_0)$ and to the operator F on Ω, but with $t \in [0, a]$ replaced by $t \in [0, \delta]$. □

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