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EXISTENCE OF THREE SOLUTIONS TO IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT. We deal with Dirichlet boundary value problems for impulsive differential equations depending on a parameter λ. Under some assumptions, the existence of at least three solutions is obtained by using a critical point theorem.

1. Introduction. In this paper, we are concerned with the existence of three solutions for the following Dirichlet boundary value problems

$$\begin{aligned} -u''(t) &= \lambda f(u(t)), \quad t \neq t_j,\ t \in [0,1],\\ \Delta u'(t_j) &= I_j(u(t_j)), \quad j = 1, 2, \ldots, p,\\ u(0) &= u(1) = 0 \end{aligned} \tag{1.1}$$

where $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $f \in C(R,R)$, $I_j \in C(R,R)$, $j = 1, 2 \ldots, p$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, $u'(t_j^+)$ and $u'(t_j^-)$ denote the right and the left limits, respectively, and $\lambda \in [0, +\infty)$ is a real parameter.

In recent years, a great deal of work has been done in the study of the existence of multiple solutions for impulsive boundary value problems; we refer the reader to [**1, 2, 4, 5**]. These classical tools in literature are fixed-point theorems in cones. It is well known that the critical point theorem is an important tool in dealing with problems for differential equations. We also note that, in the last few years, some researchers have used variational methods to study the existence of solutions for impulsive differential equations boundary value problems

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[**6–11**]. But few researchers have paid more attention to the existence of three solutions for impulsive differential equation boundary value problems by applying critical point theory.

In this paper, our main aim is to establish the existence of at least three solutions for problem (1.1).

2. Preliminaries. To begin with, we introduce some notation. Denote by X the Sobolev space $H_0^1(0,1)$, and consider the inner product

$$(u,v) = \int_0^1 u'(t)v'(t)\,dt$$

and the norm

$$||u|| = \left(\int_0^1 |u'(t)|^2\right)^{1/2}.$$

Hence, X is a separable, reflexive Banach space. For every $u \in X$, we consider the functional $\varphi : X \to R$ defined by

$$\varphi(u) = \frac{1}{2}||u||^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(t)\,dt - \lambda \int_0^1 F(u(t))\,dt,$$

where $F(u(t)) = \int_0^u f(s)\,ds$.

It is clear that φ is differentiable at any $u \in X$ and

$$\varphi'(u)v = \int_0^1 u'(t)v'(t)\,dt + \sum_{j=1}^{p} I_j(u(t_j))v(t_j) - \lambda \int_0^1 f(t,u(t))v(t)\,dt$$

for any $u \in X$. Obviously, φ' is continuous.

Lemma 2.1 [**6**]. *If $u \in X$ is a critical point of the functional φ, then u is a classical solution of problem* (1.1).

We define the norm in $C([0,1])$ by $||u||_\infty = \max_{t\in[0,1]} |u(t)|$.

Lemma 2.2. *Let $u \in X$. Then $||u||_\infty \le ||u||/2$.*

Proof. For $u \in X$, then $u(0) = u(1) = 0$. Hence, for $t \in [0,1]$, we have

$$u(t) = \int_0^t u'(s)\,ds = -\int_t^1 u'(s)\,ds,$$

which implies

$$2|u(t)| \le \int_0^1 |u'(s)|\,ds \le \left(\int_0^1 |u'(s)|^2 ds\right)^{1/2} = \|u\|,$$

which completes the proof. $\square$

Suppose that $E \subset X$. We denote $\overline{E}^{\omega}$ as the weak closure of E, that is, $x \in \overline{E}^{\omega}$ if there exists a sequence $\{x_n\} \subset E$ such that $f(x_n) \to f(x)$ for every $f \in X^*$. To verify our main results, we need the following result.

Theorem 2.1 [**3**, Theorem 2.1]. *Let X be a separable and reflexive real Banach space, and let $\Phi : X \to R$ be a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*. $J : X \to R$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0) = 0$ and that*

(i) $\lim_{||x||\to+\infty}(\Phi(x) - \lambda J(x)) = +\infty$ *for all* $\lambda \in [0, +\infty)$;

Further, assume that there are $r > 0$, $x_1 \in X$ such that

(ii) $r < \Phi(x_1)$;

(iii) $\sup_{x\in\overline{\Phi^{-1}((-\infty,r))}^{\omega}} J(x) < r/(r + \Phi(x_1))J(x_1)$.

Then, for each

$$\lambda \in \Lambda_1 = \left(\frac{\Phi(x_1)}{J(x_1) - \sup_{x\in\overline{\Phi^{-1}((-\infty,r))}^{\omega}} J(x)}, \frac{r}{\sup_{x\in\overline{\Phi^{-1}((-\infty,r))}^{\omega}} J(x)}\right),$$

the equation

$$\Phi'(x) - \lambda J'(x) = 0 \tag{1.2}$$

has at least three solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2 \subseteq \left[0, \frac{hr}{r(J(x_1)/\Phi(x_1)) - \sup_{x \in \overline{\Phi^{-1}((-\infty, r))}^{w}} J(x)}\right)$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, (1.2) has at least three solutions in X whose norms are less than σ.

3. Main results. Let $0 < a < 1$ be such that $\{t_1, \dots, t_p\} \subset [a, 1-a]$.

Theorem 3.1. *Assume that the following conditions hold.*

(C_1) *There exist two positive constants c, d with $c < d/\sqrt{2a}$, such that*

$$\max_{x \in [-c,c]} F(x) < 2c^2 \left\{ 2c^2 + \frac{d^2}{a} + \sum_{j=1}^{p} \int_0^d I_j(t)\,dt \right\}^{-1} \times \left(\frac{2a}{d} \int_0^d F(s)\,ds + (1-2a)F(d) \right);$$

(C_2) *There exist positive constants a_i, $i = 1, 2$, $M > 0$ and $0 < \mu < 2$ such that*

$$F(u) \le a_1 |u|^{\mu} - a_2, \quad \text{for } |u| \ge M;$$

(C_3) $\int_0^u I_j(t)dt \ge 0$.

Furthermore, put

$$\varphi_1 = \frac{\max_{u \in [-c,c]} F(u)}{2c^2}, \qquad (3.1)$$

$$\varphi_2 = \frac{(2a/d)\int_0^d F(s)\,ds + (1-2a)F(d) - \max_{u\in[-c,c]} F(u)}{(d^2/a) + \sum_{j=1}^{p} \int_0^d I_j(t)\,dt},$$

and, for each $h > 1$,

$$b = \frac{2hc^2}{\frac{2c^2[(2a/d)\int_0^d F(s)\,ds + (1-2a)F(d)]}{(d^2/a) + \sum_{j=1}^{p} \int_0^d I_j(t)\,dt} - \max_{u \in [-c,c]} F(u)}. \qquad (3.2)$$

Then, for each

$$\lambda \in \Lambda_1 = \left(\frac{1}{\varphi_2}, \frac{1}{\varphi_1}\right), \tag{3.3}$$

problem (1.1) *admits at least three solutions in* X *and, moreover, for each* $h > 1$*, there exist an open interval* $\Lambda_2 \subseteq [0, b]$ *and a positive real number* σ *such that, for each* $\lambda \in \Lambda_2$*, the problem* (1.1) *admits at least three solutions in* X *whose norms in* X *are less than* σ*.*

Proof. Let

$$\Phi(u) = \frac{1}{2}\int_0^1 (u'(t))^2 dt + \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(t)\,dt, \qquad J(u) = \int_0^1 F(u(t))\,dt.$$

Clearly, $\varphi(u) = \Phi(u) - \lambda J(u)$, and Φ is a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and J is a continuously Gâteaux differentiable functional whose Gâteaux is compact.

Next, in view of assumptions (C_2)–(C_3), we have, for any $u \in X$, $|u| \geq M$ and $\lambda \geq 0$,

$$\begin{aligned}\Phi(u) - \lambda J(u) &= \frac{1}{2}||u||^2 + \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(t)\,dt - \lambda\int_0^1 F(u(t))\,dt \\ &\geq \frac{1}{2}||u||^2 - \lambda[a_1|u|^\mu - a_2] \\ &\geq \frac{1}{2}||u||^2 - \lambda\left[\frac{a_1}{2^\mu}||u||^\mu - a_2\right].\end{aligned}$$

From $0 < \mu < 2$, we obtain, for all $\lambda \in [0, +\infty)$,

$$\lim_{||u||\to\infty}(\Phi(u) - \lambda J(u)) = +\infty.$$

So, condition (i) of Theorem 1.1 is satisfied.

Now, we let

$$u_1(t) = \begin{cases} (d/a)t & t \in [0, a) \\ d & t \in [a, 1-a] \\ (d/a)(1-t) & t \in (1-a, 1], \end{cases} \qquad r = 2c^2.$$

It is clear that $u_1 \in X$, and

$$\Phi(u_1) = \frac{1}{2}||u_1||^2 + \sum_{j=1}^{p} \int_0^{u_1(t_j)} I_j(t)\,dt = \frac{d^2}{a} + \sum_{j=1}^{p} \int_0^{d} I_j(t)\,dt,$$

$$J(u_1) = \frac{2a}{d} \int_0^d F(s)\,ds + (1-2a)F(d).$$

By $c < d/\sqrt{2a}$, we have

$$\Phi(u_1) = \frac{d^2}{a} + \sum_{j=1}^{p} \int_0^d I_j(t)\,dt > \frac{d^2}{a} > 2c^2 = r,$$

which shows that assumption (ii) of Theorem 1.1 is obtained.

Next, we verify that assumption (iii) of Theorem 1.1 holds. From Lemma 2.2, the estimate $\Phi(u) \leq r$ implies that

$$|u(t)|^2 \leq \frac{1}{4}||u||^2 \leq \frac{1}{2}\Phi(u) \leq \frac{1}{2}r, \quad \text{for } t \in [0,1].$$

From the definiteness of r, it follows that

$$\Phi^{-1}(-\infty, r) \subseteq \{x \in X, |x(t)| \leq c,\ t \in [0,1]\}.$$

Thus, for any $u \in X$, we have

$$\sup_{x \in \overline{\Phi^{-1}(-\infty,r)}^{\omega}} J(u) = \sup_{x \in \Phi^{-1}(-\infty,r)} J(u) \leq \max_{u \in [-c,c]} F(u).$$

On the other hand, we get

$$\frac{r}{r+\Phi(u_1)} J(u_1) = \frac{2c^2}{2c^2 + (d^2/a) + \sum_{j=1}^{p} \int_0^d I_j(t)\,dt} \times \left(\frac{2a}{d} \int_0^d F(s)\,ds + (1-2a)F(d) \right).$$

Assumption (C_1) implies that

$$\sup_{x \in \overline{\Phi^{-1}(-\infty,r)}^{\omega}} J(u) < \frac{r}{r+\Phi(u_1)} J(u_1),$$

which shows that condition (iii) of Theorem 1.1 is satisfied.

Note that

$$\begin{aligned}
&\frac{\Phi(u_1)}{J(u_1) - \sup_{x\in\overline{\Phi^{-1}(-\infty,r)}^{\omega}} J(u)} \\
&\qquad\qquad \le \frac{(d^2/a)+\sum_{j=1}^{p}\int_0^d I_j(t)\,dt}{(2a/d)\int_0^d F(s)\,ds+(1-2a)F(d)-\max_{u\in[-c,c]}F(u)} \\
&\qquad\qquad = \frac{1}{\varphi_2}, \\
&\frac{r}{\sup_{x\in\overline{\Phi^{-1}(-\infty,r)}^{\omega}} J(u)} \ge \frac{2c^2}{\max_{u\in[-c,c]}F(u)} = \frac{1}{\varphi_1}.
\end{aligned}$$

By a simple computation, it follows from condition (C_1) that $\varphi_2 > \varphi_1$. Applying Theorem 1.1, for each $\lambda \in \Lambda_1 = (1/\varphi_2, 1/\varphi_1)$, problem (1.1) admits at least three solutions in X.

For each $h > 1$, we easily see that

$$\begin{aligned}
&\frac{hr}{r(J(u_1))/(\Phi(u_1)) - \sup_{x\in\overline{\Phi^{-1}(-\infty,r)}^{\omega}} J(u)} \\
&\qquad \le \frac{2hc^2}{\frac{2c^2[(2a/d)\int_0^d F(s)\,ds+(1-2a)F(d)]}{(d^2/a)+\sum_{j=1}^{p}\int_0^d I_j(t)\,dt} - \max_{u\in[-c,c]}F(u)} \\
&\qquad = b.
\end{aligned}$$

Taking condition (C_1) into account, it forces that $b > 0$. Then, from Theorem 1.1, for each $h > 1$, there exist an open interval $\Lambda_2 \subseteq [0, b]$ and a positive real number σ such that, for $\lambda \in \Lambda_2$, problem (1.1) admits at least three solutions in X, whose norms in X are less than σ. The proof is complete. $\square$

4. Examples. Consider the following problem

$$\begin{aligned}
&-u'' = \lambda f(u), \quad t \in [0,1], \\
&\Delta u'(t_1) = \frac{1}{2}u^{1/3}(t_1), \quad t_1 = \frac{1}{2}, \qquad (4.1)\\
&u(0) = u(1) = 0,
\end{aligned}$$

where

$$f(u)=\begin{cases} e^u & u\le 8,\\ u^{2/3}+e^8-4 & u>8.\end{cases}$$

Then

$$F(u)=\begin{cases} e^u-1 & u\le 8,\\ (3/5)u^{5/3}+(e^8-4)u+(59/5)-7e^8 & u>8.\end{cases}$$

Let $a=1/4$, $c=1$, $d=27$; it follows that

$$\max_{u\in[-1,1]} F(u)=e-1<20.2$$

$$\doteq \frac{2c^2[(2a/d)\int_0^d F(s)\,ds+(1-2a)F(d)]}{2c^2+(d^2/a)+\sum_{j=1}^p\int_0^d I_j(t)\,dt},$$

$$\frac{1}{\varphi_1}=\frac{2}{e-1},\qquad \frac{1}{\varphi_2}\doteq 0.098,$$

which show that all the conditions of Theorem 3.1 are satisfied, so problem (4.1) has at least three solutions for $\lambda\in(0.1,2/e-1)$.

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