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ON SOME NEW GRONWALL–BELLMAN–OU-IANG TYPE INTEGRAL INEQUALITIES TO STUDY CERTAIN EPIDEMIC MODELS

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ABSTRACT. We present some new nonlinear Gronwall–Bellman–Ou-Iang type integral inequalities related to Pachpatte's inequality. [**11**]. These inequalities generalize former results and can be used as handy tools to study the qualitative behavior as well as of certain quantitative properties of solutions of certain epidemic models and of certain differential equations.

1. Introduction. It is well known that the integral inequalities involving functions of one, and more than one, independent variable which provide explicit bounds on unknown functions, have proved to be very useful and important devices in the study of many qualitative behaviors as well as quantitative properties of solutions of differential and integral equations. In recent years, these inequalities have been greatly enriched by the recognition of their potential and intrinsic worth in many applications of the applied sciences. In the past few years, a number of integral inequalities have been established by many scholars, which were motivated by certain applications. For example, we refer the reader to references [**1, 2, 4–7, 12**] and some of the references cited therein.

In a paper published in 1981, [**9**] studied the qualitative behavior of solutions of the equation

$$u(t) = k\left[p(t) - \int_0^t A(t-s)u(s)\,ds\right]\left[q(t) + \int_0^t a(t-s)u(s)\,ds\right]. \tag{1.1}$$

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This equation arose in the study of the spread of an infections disease that did not induce permanent immunity. For detailed definitions of the various functions arising in (1.1), see [**8, 9**] and some of the references cited therein.

Throughout the paper, R denotes the set of real numbers, $R_+ = [0, \infty)$, and $C(M, N)$ denotes the class of all nondecreasing continuous functions from M to N.

In 1956 Bihari proved the following useful nonlinear inequality:

Theorem 1.1 [**12**, page 107]. *Let $u(t)$ and $f(t)$ be nonnegative continuous functions defined on R_+. Let $W(u) \in C(R_+, R_+)$ and $W(u) > 0$ on $(0, \infty)$. If*

$$u(t) \leq c + \int_0^t f(s) W(u(s))\, ds \quad \textit{for all } t \in R_+, \tag{1.2}$$

where c is a nonnegative constant, then

$$u(t) \leq G^{-1}\left(G(c) + \int_0^t f(s)\, ds\right) \quad \textit{for all } 0 \leq t \leq t_1, \tag{1.3}$$

where

$$G(t) = \int_{r_0}^{r} \frac{ds}{W(s)} \quad \textit{for all } r;\ r_0 > 0, \tag{1.4}$$

and G^{-1} is the inverse function of G and $t_1 \in R_+$ is chosen so that $G(c) + \int_0^t f(s)\, ds \in \operatorname{Dom}(G^{-1})$ for all $t \in R_+$ lying in the interval $0 \leq t \leq t_1$.

In 1995, Pachpatte [**11**] proved the following useful integral inequality and studied the qualitative behavior of solutions of (1.1).

Theorem 1.2 [**11**]. *Let $u(t); f(t); g(t) \in C(R_+, R_+)$ and $c_1; c_2 \in R_+$ be constants. If*

$$u(t) \leq \left[c_1 + \int_0^t f(s) u(s)\, ds\right]\left[c_2 + \int_0^t g(s) u(s)\, ds\right], \tag{1.5}$$

and

$$c_1c_2\int_0^t\left[g(s)\int_0^s f(r)\,dr+f(s)\int_0^s g(r)\,dr\right]Q(s)\,ds<1\quad \textit{for all } t\in R_+,$$

then

$$(1.6)\qquad u(t)\le\frac{c_1c_2Q(t)}{[1-c_1c_2\int_0^t[g(s)\int_0^s f(r)\,dr+f(s)\int_0^s g(r)\,dr]Q(s)\,ds]},$$

where $Q(t)=\exp(\int_0^t[c_1g(s)+c_2f(s)]\,ds)$ *for all* $t\in R_+$.

In 1981, Gripenberg [**9**] studied the existence of a unique, bounded, continuous and nonnegative solution of (1.1) for all $t\in R_+$ under appropriate assumptions on $A(t-s)$ and $a(t-s)$ and also obtained sufficient conditions for the convergence of the solution to a limit when $t\to\infty$. In 1995, Pachpatte [**11**] studied the boundedness, asymptotic behavior, and growth of the solutions of (1.1) under some suitable conditions of the functions involved in (1.1). Aside from the various physical meanings of the functions arising in (1.1), we believe that, the equations like (1.1) are of great interest in their own right and that further investigation of the qualitative behavior of their solutions even under the usual hypotheses on the functions in (1.1) are much more interesting.

The main purpose of this paper is to establish explicit bounds on Theorem 1.2 and similar inequalities which can be used to study the qualitative behavior of solutions of (1.1); some applications of our results are also given.

In 1995, Pachpatte [**10**] stated the following useful inequality related to certain integral inequalities arising in the theory of differential equations.

Theorem 1.3 [**12**, page 255]. *Let* $u(t);f(t);g(t)\in C(R_+,R_+)$ *and* $c_1;c_2\in R_+$ *be constants. If*

(1.7)

$$u^2(t)\le\left[c_1^2+2\int_0^t f(s)u(s)\,ds\right]\left[c_2^2+2\int_0^t g(s)u(s)\,ds\right]\quad \textit{for all } t\in R_+,$$

then

$$(1.8)\quad u(t) \le p(t)\exp\left(2\int_0^t \left[g(s)\int_0^s f(r)\,dr + f(s)\int_0^s g(r)\,dr\right] ds\right) \quad \text{for all } t \in R_+,$$

where

$$(1.9)\quad p(t) = \left[c_1c_2 + c_1^2\int_0^t g(s)\,ds + c_2^2\int_0^t f(s)\,ds\right] \quad \text{for all } t \in R_+.$$

Remark 1.1. It is interesting to note that, in the special case when $g(t) = 0$ and $c_2 = 1$ or $f(t) = 0$ and $c_1 = 1$, the inequality given in Theorem 1.2 reduces to the well-known Gronwall's inequality [**3**, page 31].

Remark 1.2. It is interesting to note that, in the special case when $g(t) = 0$ and $c_2 = 1$ or $f(t) = 0$ and $c_1 = 1$, the inequality given in Theorem 1.3 reduces to the well-known Ou-Iang inequality [**12**, page 233] (see Corollary 2.2).

2. Main results. In this section we state and prove new integral inequalities related to the integral inequality which was established in Theorem 1.2 and which can be used directly in the study of the qualitative behavior of solutions of (1.1).

Theorem 2.1. *Let* $u(t); f(t) \in C(R_+, R_+)$ *and* $c \in R_+$ *is a constant. If*

$$(2.1)\quad u(t) \le \left[c + \int_0^t f(s)u(s)\,ds\right]^2 \quad \text{for all } t \in R_+,$$

then we have the following:

(a)
(2.2)

$$u(t) \le \frac{c^2\exp(2c\int_0^t f(s)\,ds)}{[1 - 2c^2\int_0^t [f(s)\int_0^s f(r)\,dr]\exp(2c\int_0^s f(r)\,dr)\,ds]} \quad \text{for all } t \in R_+,$$

such that $2c^2 \int_0^t [f(s) \int_0^s f(r)\,dr] \exp(2c \int_0^s f(r)\,dr)\,ds < 1$ *for all* $t \in R_+$.

(b)

$$u(t) \le \frac{c^2}{[1 - c\int_0^t f(s)\,ds]^2} \quad \textit{for all } t \in R_+, \tag{2.3}$$

such that $c\int_0^t f(s)\,ds < 1$ *for all* $t \in R_+$.

Proof. (a) The desired inequality in (2.2) follows by setting $g(t) = f(t)$ and $c_1 = c_2 = c$ in Theorem 1.2.

(b) Define a function $v(t)$ by

$$v(t) = u^{1/2}(t), \quad v(0) = c; \tag{2.4}$$

then (2.1) shows that

$$v(t) = c + \int_0^t f(s) v^2(s)\,ds \quad \text{for all } t \in R_+. \tag{2.5}$$

Now, by application of Theorem 1.1 where $W(u(s)) = v^2(s)$, we obtain

$$v(t) \le G^{-1}\left(G(c) + \int_0^t f(s)\,ds\right) \Longrightarrow G(v(t)) - G(c) \le \int_0^t f(s)\,ds, \tag{2.6}$$

but from (1.4) we have $G(v(t)) - G(v(0)) = G(v(t)) - G(c) = \int_c^{v(t)} (ds/s^2) = (1/c) - (1/v(t))$; then from (2.6) we have

$$v(t) \le \frac{c}{1 - c\int_0^t f(s)\,ds} \quad \text{for all } t \in R_+. \tag{2.7}$$

The desired inequality in (2.3) now follows by using (2.7) in (2.4). The proof is complete. $\square$

Remark 2.1. It is interesting to note that result (a) of Theorem 2.1 is one of the corollaries of Theorem 1.2; also, result (b) of Theorem 2.1 is one of the corollaries of Theorem 1.1.

Corollary 2.1. *From Theorem* 2.1 *we note that, the estimate in* (b) *is stronger than the estimate in* (a).

Proof. To see this, let $c = 1/2$ and $f(t) = 1/(t+1)$, the inequality in (a) yields

$$u(t) \leq \frac{t+1}{2[2+t-(t+1)\ln(t+1)]},$$

such that

$$t \in J_1 = \left[0, e^{(t+2)/(t+1)} - 1\right] \subseteq R_+.$$

But the inequality in (b) yields

$$u(t) \leq \frac{1}{[2-\ln(t+1)]^2},$$

such that

$$t \in J_2 = [0, e^2 - 1] \subseteq R_+,$$

and since $e^{(t+2)/(t+1)} \leq e^2$ for all $t \in R_+$ then, we have $J_1 \subseteq J_2$. The proof is complete. □

Now, we prove the following generalizations of Theorem 2.1.

Theorem 2.2. *Let* $u(t); f(t) \in C(R_+, R_+)$ *and* $h(t)$ *be a continuous, positive and nondecreasing function. If*

$$u(t) \leq \left[h(t) + \int_0^t f(s)u(s)\,ds\right]^2 \quad \text{for all } t \in R_+, \tag{2.8}$$

and $Q_1(t) = \exp(2\int_0^t h(s)f(s)\,ds)$ *for all* $t \in R_+$, *then we have the following*:

(a)

$$u(t) \leq \frac{h^2(t)Q_1(t)}{[1 - 2\int_0^t [h(s)f(s)\int_0^s h(r)f(r)\,dr]Q_1(s)\,ds]} \quad \text{for all } t \in R_+, \tag{2.9}$$

such that $2\int_0^t [h(s)f(s)\int_0^s h(r)f(r)\,dr]Q_1(s)\,ds < 1$ *for all* $t \in R_+$.

(b)

$$u(t) \le \frac{h^2(t)}{[1 - \int_0^t h(s)f(s)\,ds]^2} \quad \textit{for all } t \in R_+, \tag{2.10}$$

such that $\int_0^t h(s)f(s)\,ds < 1$ *for all* $t \in R_+$.

Proof. Let $u(t) = n(t)h^2(t)$. Then, from (2.8), we have

$$n(t) \le \left[1 + \int_0^t f(s)h(s)n(s)\,ds\right]^2 \quad \text{for all } t \in R_+. \tag{2.11}$$

(a) Now, by application of part (a) of Theorem 2.2 on (2.11) where $c = 1$, we obtain

$$n(t) \le \frac{Q_1(t)}{[1 - 2\int_0^t [h(s)f(s)\int_0^s h(r)f(r)\,dr]Q_1(s)\,ds]} \quad \text{for all } t \in R_+. \tag{2.12}$$

The desired inequality in (2.9) now follows by using (2.12) in (2.11).

(b) Now, by application of part (b) of Theorem 2.2 on (2.11) where $c = 1$, we obtain

$$n(t) \le \frac{1}{[1 - \int_0^t h(s)f(s)\,ds]^2} \quad \text{for all } t \in R_+. \tag{2.13}$$

The desired inequality in (2.10) now follows by using (2.13) in (2.11). The proof is complete. □

Remark 2.2. It is interesting to note that, in the special case when $h(t) = c > 0$, the inequalities given, in parts (a) and (b) of Theorem 2.2 reduce to the inequalities given in parts (a) and (b) of Theorem 2.1, respectively.

Theorem 2.3. *Let* $u(t); f(t) \in C(R_+, R_+)$, *and let* $p; c \in R_+$ *be constants such that* $p \ne 1$. *If*

$$u(t) \le \left[c + \int_0^t f(s)u(s)\,ds\right]^p \quad \textit{for all } t \in R_+, \tag{2.14}$$

then

$$u(t) \leq \left[c^q + q\int_0^t f(s)\,ds\right]^{p/q} \quad \textit{for all } t \in R_+, \tag{2.15}$$

where $p + q = 1$.

Proof. Define a function $v(t)$ by

$$v(t) = u^{1/p}(t); \quad v(0) = c \text{ for all } t \in R_+; \tag{2.16}$$

then (2.14) shows that

$$v(t) \leq c + \int_0^t f(s)v^p(s)\,ds \quad \text{for all } t \in R_+. \tag{2.17}$$

Now, by application of Theorem 1.1 where $W(v(s)) = v^p(s)$ and using (2.16), we obtain

$$\begin{aligned} v(t) &\leq G^{-1}\left(G(c) + \int_0^t f(s)\,ds\right) \Longrightarrow G(v(t)) - G(c) \\ &\leq \int_0^t f(s)\,ds \quad \text{for all } t \in R_+, \end{aligned} \tag{2.18}$$

and from (1.4) we have $G(v(t)) - G(c) = \int_c^{v(t)} (ds/s^p) = [v^q(t) - c^q]/q$; thus, from (2.18) we obtain

$$v(t) \leq \left[c^q + q\int_0^t f(s)\,ds\right]^{1/q} \quad \text{for all } t \in R_+. \tag{2.19}$$

The desired inequality in (2.15) now follows by using (2.19) in (2.16). The proof is complete. □

Corollary 2.2. *Let* $p = 1/2$, $c = C^2$ *and* $f(t) = 2F(t)$ *in Theorem* 2.3. *We have the following*:

If

$$u^2(t) \leq C^2 + 2\int_0^t F(s)u(s)\,ds \quad \textit{for all } t \in R_+,$$

then

$$u(t) \leq C + \int_0^t F(s)\,ds \qquad \textit{for all } t \in R_+,$$

which is Ou-Iang's inequality [**12**, page 233].

Remark 2.3. It is interesting to note that the result of Theorem 2.3 is one of the corollaries of Theorem 1.1.

Remark 2.4. It is interesting to note that, in the special case when $p = 2$, the inequality given in Theorem 2.3 reduces to part (b) of Theorem 2.2.

Corollary 2.3. *Let $c = 0$ in Theorem* 2.3. *We have the following*: *If*

$$u(t) \leq \left[\int_0^t f(s)u(s)\,ds\right]^p \qquad \textit{for all } t \in R_+,$$

then

$$u(t) \leq \left[(1-p)\int_0^t f(s)\,ds\right]^{p/(1-p)} \qquad \textit{for all } t \in R_+,$$

for $0 \leq p < 1$ and

$$u(t) \equiv 0,$$

for $p > 1$.

Theorem 2.4. *Let $u(t); f(t); g(t) \in C(R_+, R_+)$, and let $c_1; c_2; p \in [1, \infty)$ be constants. If*

$$u^p(t) \leq \left[c_1 + \int_0^t f(s)u(s)\,ds\right]\left[c_2 + \int_0^t g(s)u(s)\,ds\right] \qquad \textit{for all } t \in R_+, \tag{2.20}$$

and

$$\frac{[c_1c_2]^{1/p}}{p}\int_0^t \left[g(s)\int_0^s f(r)\,dr + f(s)\int_0^s g(r)\,dr\right]Q_2(s)\,ds < 1 \qquad \textit{for all } t \in R_+,$$

then

$$u(t) \le \frac{[c_1c_2]^{1/p}Q_2(t)}{\left[1-[c_1c_2]^{1/p}/p\int_0^t \left[g(s)\int_0^s f(r)\,dr + f(s)\int_0^s g(r)\,dr\right] Q_2(s)\,ds\right]} \quad \text{for all } t\in R_+, \tag{2.21}$$

where $Q_2(t)=\exp((1/p)\int_0^t[c_1g(s)+c_2f(s)]\,ds)$ *for all* $t\in R_+$.

Proof. Define a function $n(t)$ by

$$n(t)=\left[c_1+\int_0^t f(s)u(s)\,ds\right]\left[c_2+\int_0^t g(s)u(s)\,ds\right] \quad \text{for all } t\in R_+.$$

Then, from (2.20), we have

$$n(0)=c_1c_2; \qquad u(t)\le n^{1/p}(t) \quad \text{for all } t\in R_+. \tag{2.22}$$

Differentiating $n(t)$ with respect to t and using (2.22) and the monotonicity of $n(t)$, we deduce

$$\begin{aligned}\frac{dn(t)}{dt} &\le [c_1g(t)+c_2f(t)]\,n^{1/p}(t)\\ &\quad+\left[f(t)\int_0^t g(s)\,ds+g(t)\int_0^t f(s)\,ds\right]n^{2/p}(t) \quad \text{for all } t\in R_+,\end{aligned}$$

but $n^{2/p}(t)>0$, and then we have

$$\begin{aligned}n^{-2/p}(t)\left[\frac{dn(t)}{dt}\right] &\le [c_1g(t)+c_2f(t)]n^{-1/p}(t)\\ &\quad+\left[f(t)\int_0^t g(s)\,ds+g(t)\int_0^t f(s)\,ds\right] \quad \text{for all } t\in R_+.\end{aligned} \tag{2.23}$$

Since $p\ge 1$, then $p+1\ge 2 \Rightarrow (p+1)/p\ge 2/p$; thus, we have $-(p+1)/p\le -2/p$ and, from the monotonicity of $n(t)$ and the fact that $n(t)\ge 1$, we deduce

$$n^{-(p+1)/p}(t)\left[\frac{dn(t)}{dt}\right]\le n^{-2/p}(t)\left[\frac{dn(t)}{dt}\right] \quad \text{for all } t\in R_+. \tag{2.24}$$

Now, from (2.23) and (2.24), we obtain

$$n^{-(p+1)/p}(t)\left[\frac{dn(t)}{dt}\right] \leq \left[c_1 g(t) + c_2 f(t)\right] n^{-1/p}(t) + \left[f(t)\int_0^t g(s)\,ds + g(t)\int_0^t f(s)\,ds\right] \quad \text{for all } t \in R_+, \tag{2.25}$$

and letting $n^{-1/p}(t) = m(t) \Rightarrow m(0) = n^{-1/p}(0) = [c_1c_2]^{-1/p}$, then we have

$$n^{-(p+1)/p}(t)\left[\frac{dn(t)}{dt}\right] = -p\left[\frac{dm(t)}{dt}\right] \quad \text{for all } t \in R_+;$$

thus, from (2.25), we have

$$\frac{dm(t)}{dt} + \frac{1}{p}\left[c_1 g(t) + c_2 f(t)\right] m(t) \geq -\frac{1}{p}\left[f(t)\int_0^t g(s)\,ds + g(t)\int_0^t f(s)\,ds\right] \quad \text{for all } t \in R_+.$$

The above inequality implies the estimation for $m(t)$ such that

$$m(t) \geq \frac{\left[[c_1c_2]^{-1/p} - (1/p)\int_0^t \left[f(s)\int_0^s g(r)dr + g(s)\int_0^s f(r)\,dr\right] Q_2(s)ds\right]}{Q_2(t)},$$

but $m(t) = n^{-1/p}(t)$; thus, we have

$$n^{1/p}(t) \leq \frac{[c_1c_2]^{1/p} Q_2(t)}{\left[1 - \frac{[c_1c_2]^{1/p}}{p}\int_0^t \left[f(s)\int_0^s g(r)\,dr + g(s)\int_0^s f(r)\,dr\right] Q_2(s)\,ds\right]} \quad \text{for all } t \in R_+. \tag{2.26}$$

The desired inequality in (2.21) follows by using (2.26) in (2.22). The proof is complete. □

Remark 2.5. It is interesting to note that, in the special case when $p = 1$, $g(t) = 0$ and $c_2 = 1$ or $p = 1$, $f(t) = 0$ and $c_1 = 1$, the inequality given in Theorem 2.4 reduces to the well-known Gronwall's inequality [**3**, page 31].

Remark 2.6. It is interesting to note that, in the special case when $p = 1$, the inequality given in Theorem 2.4 reduces to Pachpatte's inequality given in Theorem 1.2.

Remark 2.7. It is interesting to note that, in the special case when $p = 1$, $c_1 = c_2 = c$ and $g(t) = f(t)$, the inequality given in Theorem 2.4 reduces to the inequality given in part (a) of Theorem 2.1.

Corollary 2.4. *Letting $p = 2$, $f(t) = 2F(t)$, $g(t) = 0$, $c_1 = c^2 \geq 1$ and $c_2 = 1$ or $p = 2$, $f(t) = 0$, $g(t) = 2F(t)$, $c_1 = 1$ and $c_2 = c^2 \geq 1$ in Theorem 2.4, we have the following*:

If

$$u^2(t) \leq c^2 + 2\int_0^t F(s)u(s)\,ds \quad \text{for all } t \in R_+,$$

then

$$u(t) \leq c\exp\left(\int_0^t F(s)\,ds\right) \quad \text{for all } t \in R_+.$$

Corollary 2.5. *If we replace p by 2, c_1 by $c_1^2 > 1$, c_2 by $c_2^2 > 1$, $f(t)$ by $2f(t)$ and $g(t)$ by $2g(t)$ in Theorem 2.4, we have the following*:

If

$$u^2(t) \leq \left[c_1^2 + 2\int_0^t f(s)u(s)\,ds\right]\left[c_2^2 + 2\int_0^t g(s)u(s)\,ds\right] \quad \text{for all } t \in R_+,$$

and

$$2c_1c_2\int_0^t\left[g(s)\int_0^s f(r)\,dr + f(s)\int_0^s g(r)\,dr\right]Q_3(s)\,ds < 1 \quad \text{for all } t \in R_+,$$

then

$$u(t) \leq \frac{c_1c_2Q_3(t)}{\left[1 - 2c_1c_2\int_0^t\left[g(s)\int_0^s f(r)\,dr + f(s)\int_0^s g(r)\,dr\right]Q_3(s)\,ds\right]} \quad \text{for all } t \in R_+,$$

where $Q_3(t) = \exp(\int_0^t [c_1^2 g(s) + c_2^2 f(s)]\,ds)$ for all $t \in R_+$.

Remark 2.8. Under the conditions of Theorem 2.4, we observe that our results in Corollary 2.4 and Corollary 2.5 are stronger than Ou-Iang's inequality [**12**, page 233] (see Corollary 2.2) and Pachpatt's inequality [**10**, Theorem 1.3], respectively.

3. Some applications. In this section, we use the results which were obtained in Theorems 2.1 and 2.2 to study the boundedness, asymptotic behavior and growth of solutions of (1.1), under some suitable conditions on the functions involved in (1.1), which are different from Pachpatte's conditions in [**11**].

In what follows, we assume that $u(t); p(t); q(t); A(t-s); a(t-s) \in C(R_+, R_+)$ and $k \in R_+$ in (1.1) is a constant and restrict our consideration to solutions of (1.1) which exist on R_+.

Theorem 3.1. *Consider* (1.1)*, and let*

$$\max(k|p(t)|, |q(t)|) = c \quad \text{for all } t \in R_+, \tag{3.1}$$

$$\max(|A(t-s)|, |a(t-s)|) = \varepsilon f(t) \quad \text{for all } 0 \le s \le t;\ s, t \in R_+, \tag{3.2}$$

where $c; \varepsilon \in R_+$ *are constants and* $f(t) \in C(R_+, R_+)$*, and we have the following*:

(a1) *If* $2c^2\varepsilon^2 \int_0^t [f(s) \int_0^s f(r)\,dr] \exp(2c\varepsilon \int_0^s f(r)\,dr)\,ds < 1$ *for all* $t \in R_+$*, and*

$$\frac{c^2 \exp(2c\varepsilon \int_0^t f(s)\,ds)}{\left[1 - 2c^2\varepsilon^2 \int_0^t \left[f(s) \int_0^s f(r)\,dr\right] \exp\left(2c\varepsilon \int_0^s f(r)\,dr\right)\,ds\right]} < \infty \quad \text{for all } t \in R_+, \tag{3.3}$$

then every solution $u(t)$ *of* (1.1) *existing on* R_+ *is bounded.*

(a2) *If* $c\varepsilon \int_0^t f(s)ds < 1$ *for all* $t \in R_+$ *and*

$$\frac{c^2}{[1 - c\varepsilon \int_0^t f(s)ds]^2} < \infty \quad \text{for all } t \in R_+, \tag{3.4}$$

then every solution $u(t)$ *of* (1.1) *existing on* R_+ *is bounded.*

Proof. (a1) From (1.1) and using hypotheses (3.1) and (3.2), it is easy to observe that

$$|u(t)| \leq \left[c + \varepsilon \int_0^t f(s)|u(s)|\,ds\right]^2 \quad \text{for all } t \in R_+. \tag{3.5}$$

Now, by application of part (a) from Theorem 2.1 and using hypotheses (3.3), (3.4) and (3.5), we obtain

$$|u(t)| \leq \frac{c^2 \exp\left(2c\varepsilon \int_0^t f(s)\,ds\right)}{\left[1 - 2c^2\varepsilon^2 \int_0^t \left[f(s)\int_0^s f(r)\,dr\right] \exp\left(2c\varepsilon \int_0^s f(r)\,dr\right)\,ds\right]} < \infty \quad \text{for all } t \in R_+. \tag{3.6}$$

The estimation in (3.6) implies the boundedness of solution $u(t)$ of (1.1) on R_+.

(a2) Now, by application of part (b) from Theorem 2.1 and using hypotheses (3.4) and (3.5), we obtain

$$|u(t)| \leq \frac{c^2}{\left[1 - c\varepsilon \int_0^t f(s)\,ds\right]^2} < \infty \quad \text{for all } t \in R_+. \tag{3.7}$$

The estimation in (3.7) implies the boundedness of solution $u(t)$ of (1.1) on R_+. The proof is complete. □

Theorem 3.2. *Consider* (1.1). *Let* (3.2) *hold and*

$$\max\left(k|p(t)|, |q(t)|\right) = \delta h(t) \quad \textit{for all } t \in R_+, \tag{3.8}$$

where δ is a positive real constant and $h(t) \in C(R_+, R_+)$. We have the following:

(b1) *If* $2\varepsilon^2 \int_0^t [h(s)f(s)\int_0^s h(r)f(r)\,dr]Q_4(s)\,ds < 1$ *for all* $t \in R_+$, *and*

$$\frac{h^2(t)Q_4(t)}{\left[1 - 2\varepsilon^2 \int_0^t \left[h(s)f(s)\int_0^s h(r)f(r)\,dr\right] Q_4(s)\,ds\right]} < \infty \quad \textit{for all } t \in R_+, \tag{3.9}$$

where $Q_4(t) = \delta^2 \exp(2\varepsilon\delta \int_0^t h(s)f(s)\,ds)$, *then every solution* $u(t)$ *of* (1.1) *existing on* R_+ *is bounded.*

(b2) *If* $\varepsilon\delta \int_0^t h(s)f(s)\,ds < 1$ *for all* $t \in R_+$, *and*

$$\frac{\delta^2 h^2(t)}{\left[1 - \varepsilon\delta \int_0^t h(s)f(s)\,ds\right]^2} < \infty \quad \text{for all } t \in R_+, \tag{3.10}$$

then every solution $u(t)$ *of* (1.1) *existing on* R_+ *is bounded.*

Proof. (b1) From (1.1) and using hypotheses (3.2) and (3.8), it is easy to observe that

$$|u(t)| \leq \left[\delta h(t) + \varepsilon \int_0^t f(s)|u(s)|\,ds\right]^2 \quad \text{for all } t \in R_+. \tag{3.11}$$

Now, by application of part (a) from Theorem 2.2 and using hypotheses (3.9), (3.10) and (3.11), we obtain

$$|u(t)| \leq \frac{h^2(t)Q_4(t)}{\left[1 - 2\varepsilon^2 \int_0^t \left[h(s)f(s) \int_0^s h(r)f(r)\,dr\right] Q_4(s)\,ds\right]} < \infty \quad \text{for all } t \in R_+. \tag{3.12}$$

The estimation in (3.12) implies the boundedness of solution $u(t)$ of (1.1) on R_+.

(b2) Now, by application of part (b) from Theorem 2.1 and using hypotheses (3.10) and (3.11), we obtain

$$|u(t)| \leq \frac{\delta^2 h^2(t)}{\left[1 - \delta\varepsilon \int_0^t h(s)f(s)\,ds\right]^2} < \infty \quad \text{for all } t \in R_+. \tag{3.13}$$

The estimation in (3.13) implies the boundedness of solution $u(t)$ of (1.1) on R_+. The proof is complete.

Theorem 3.3. *Consider* (1.1). *Let* (3.2) *hold and*

$$\max\left(k|p(t)|, |q(t)|\right) = \delta e^{-\lambda t} \quad \text{for all } t \in R_+, \tag{3.14}$$

where δ is as defined in Theorem 3.2 and $\lambda \in R_+$ is a constant. We have the following:

(c1) *If* $2\varepsilon^2 \int_0^t [f(s)e^{-\lambda s} \int_0^s f(r)e^{-\lambda r}\, dr] Q_5(s)\, ds < 1$ *for all* $t \in R_+$, *and*

$$E_1(t) = \frac{Q_5(t)}{\left[1 - 2\varepsilon^2 \int_0^t \left[f(s)e^{-\lambda s} \int_0^s f(r)e^{-\lambda r}\, dr\right] Q_5(s)\, ds\right]} < \infty \quad \text{for all } t \in R_+, \tag{3.15}$$

where $Q_5(t) = \delta^2 \exp(2\delta\varepsilon \int_0^s f(r)e^{-\lambda r}\, dr)$, *then all solutions of* (1.1) *approach zero as* $t \to \infty$.

(c2) *If* $\delta\varepsilon \int_0^t f(s)e^{-\lambda s}\, ds < 1$ *for all* $t \in R_+$, *and*

$$E_2(t) = \frac{\delta^2}{\left[1 - \delta\varepsilon \int_0^t f(s)e^{-\lambda s}\, ds\right]^2} < \infty \quad \text{for all } t \in R_+, \tag{3.16}$$

then all solutions of (1.1) *approach zero as* $t \to \infty$.

Proof. (c1) From (1.1) and using hypotheses (3.2) and (3.14), it is easy to observe that

$$|u(t)| \le \left[\delta e^{-\lambda t} + \varepsilon \int_0^t f(s)|u(s)|\, ds\right]^2 \quad \text{for all } t \in R_+. \tag{3.17}$$

Now, by application of part (a) from Theorem 2.2 to (3.17), we obtain

$$|u(t)| \le E_1(t) e^{-2\lambda t} \quad \text{for all } t \in R_+, \tag{3.18}$$

where $E_1(t)$ is defined by (3.15). From hypothesis (3.15), inequality (3.18) yields the desired result.

(c2) Now by application of part (b) from Theorem 2.2 to (3.17), we obtain

$$|u(t)| \le E_2(t) e^{-2\lambda t} \quad \text{for all } t \in R_+, \tag{3.19}$$

where $E_2(t)$ is defined by (3.16). From hypothesis (3.16), inequality (3.19) yields the desired result.The proof is complete. $\square$

Theorem 3.4. *Consider* (1.1). *Letting* (3.2) *hold and*

$$\max\left(k|p(t)|, |q(t)|\right) = \delta e^{\lambda t} \quad \textit{for all } t \in R_+, \tag{3.20}$$

where δ *and* λ *are as defined in Theorem* 3.3, *we have the following*:

(d1) *If* $2\varepsilon^2 \int_0^t [f(s)e^{\lambda s} \int_0^s f(r)e^{\lambda r}\,dr] Q_6(s)\,ds < 1$ *for all* $t \in R_+$, *and*

$$E_3(t) = \frac{Q_6(t)}{\left[1 - 2\varepsilon^2 \int_0^t \left[f(s)e^{\lambda s} \int_0^s f(r)e^{\lambda r}\,dr\right] Q_6(s)\,ds\right]} < \infty \quad \textit{for all } t \in R_+, \tag{3.21}$$

where $Q_6(s) = \delta^2 \exp(2\delta\varepsilon \int_0^s f(r)e^{\lambda r}\,dr)$; *then all solutions of* (1.1) *are slowly growing.*

(d2) *If* $\delta\varepsilon \int_0^t f(s)e^{\lambda s}\,ds < 1$ *for all* $t \in R_+$, *and*

$$E_4(t) = \frac{\delta^2}{\left[1 - \delta\varepsilon \int_0^t f(s)e^{\lambda s}\,ds\right]^2} < \infty \quad \textit{for all } t \in R_+, \tag{3.22}$$

then all solutions of (1.1) *are slowly growing.*

Proof. (d1) From (1.1) and using hypotheses (3.2) and (3.20), it is easy to observe that

$$|u(t)| \le \left[\delta e^{\lambda t} + \varepsilon \int_0^t f(s)|u(t)|\,ds\right]^2 \quad \text{for all } t \in R_+. \tag{3.23}$$

Now, by application of part (a) from Theorem 2.2 to (3.23), we obtain

$$|u(t)| \le E_3(t)e^{2\lambda t} \quad \text{for all } t \in R_+, \tag{3.24}$$

where $E_3(t)$ is defined by (3.21). From hypothesis (3.21), inequality (3.24) demonstrates that the solution of (1.1) grows more slowly than any positive exponential.

(d2) Now, by application of part (b) from Theorem 2.2 to (3.23), we obtain

$$|u(t)| \le E_4(t)e^{2\lambda t} \quad \text{for all } t \in R_+, \tag{3.25}$$

where $E_4(t)$ is defined by (3.22). From hypothesis (3.22), inequality (3.25) demonstrates that the solution of (1.1) grows more slowly than any positive exponential. The proof is complete. $\square$

In concluding this paper we note that the inequalities established in Section 2 can be used as basic tools in the study of certain classes of nonlinear differential equations as well as of certain integral equations.

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