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ON EXISTENCE AND UNIQUENESS OF THE MILD SOLUTION FOR FRACTIONAL SEMILINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we consider the existence and uniqueness of the mild solution for the fractional integro-differential equation

$$\frac{d^a x(t)}{dt^a} = Ax(t) + g(t, x(t)) + \int_{t_0}^{t} f(t, s, x(s))\, ds,$$

where $0 < a \leq 1$, g and f are given functions.

1. Introduction. Let d^α/dt^α denote the Caputo fractional derivative of order α, for $0 < \alpha \leq 1$. We consider the following integro-differential equation

$$(1)\qquad \begin{cases} \dfrac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + g(t, x(t)) \\ \qquad\qquad + \displaystyle\int_{t_0}^{t} f(t, s, x(s))\, ds \quad t > t_0 \geq 0, \\ x(t_0) = x_0 \in X \end{cases}$$

where A is a generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ on the Banach space X, $f : D \times X \to X$ and $g : I_h \times X \to X$ is continuous in t, for

$$I_h := [t_0, t_0 + h] \quad \text{and} \quad D := \{(t, s) : t_0 \leq s \leq t \leq t_0 + h\}, \quad h > 0.$$

Using a fixed point theorem, we prove the existence and uniqueness of a mild solution for equation (1). The nonlinearities $g(t, x(t))$ and $f(t, s, x(s))$ are assumed to satisfy some conditions, given later.

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The problem of existence and uniqueness of a solution for fractional differential equations has been considered by many authors during the past three decades (see [**1**–**4, 6, 8**]). For example, the case $A = f = 0$ has been investigated by Delbosco and Rodino [**1**]. Kilbas, Bonilla and Trujillo [**4**] consider the same problem ($A = f = 0$). They proved existence and uniqueness theorems in terms of a Lipschitz function g on the space of summable functions by using the successive approximation method. Yu and Gao [**8**] developed more general conditions in terms of a non-Lipschitz function g. Furati and Tatar [**2**] considered the case $A = 0$ in which the nonlinearities involve power functions in t, s and x. Momani, Jameel and Al-Azawi [**6**] obtained local and global uniqueness theorems of the above fractional integro-differential equation ($A = 0$) using Bihari's inequality in the case of non-Lipschitz functions. Recently, Jaradat, Al-Omeri and Momani [**3**] proved local existence and uniqueness of a mild solution for a fractional semilinear initial value problem for a locally Lipschitz function g in terms of some kernel operators.

In this paper, the local existence and uniqueness of the mild solution of the integro-differential equation (1) are proved, and the result is extended in a global sense.

The paper is organized as follows. In Section 2, some definitions and lemmas are recalled in order to be used for proving the main result. Section 3 contains the main results and proofs.

2. Preliminaries. In this section, some definitions and lemmas are presented to be used later in Section 3.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbf{R}$, if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1 \in C[0, \infty)$, and it is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu$, $n \in \mathbf{N}$.

Definition 2.2. A function $f \in C_\mu$, $\mu \geq -1$, is said to be a (Riemann-Liouville) fractional integrable of order $\alpha > 0$ if

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds < \infty \quad \text{for } t > 0;$$

and if $\alpha = 0$, then $I^0 f(t) := f(t)$.

Next, we introduce the Caputo fractional derivative.

Definition 2.3. The fractional derivative in the Caputo sense is defined as

$$\frac{d^{\alpha}f(t)}{dt^{\alpha}} := I^{n-\alpha}\left(\frac{d^{n}f(t)}{dt^{n}}\right) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-s)^{n-\alpha-1}\left(\frac{d^{n}f(s)}{ds^{n}}\right)ds$$

for $n-1<\alpha\le n$, $n\in\mathbf{N}$, $t>0$ and $f\in C_{-1}^{n}$.

The properties of the above operators and common symbols can be found in [**5, 7**].

The proof of existence and uniqueness of equation (1) is based on the following well-known "mild solution" (see [**3**, Definition 1.3]).

Definition 2.4. A continuous solution $x(t)$ of the integral equation

(2)

$$x(t) = T(t-t_0)x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}T(t-s)\left[g(s,x(s))+\left(\int_{t_0}^{s}f(s,r,x(r))\,dr\right)\right]ds$$

is called a mild solution of (1).

Applying the integral operator I^{α} to both sides of equation (1), and using some basic properties in the fractional calculus, one can show the following (see [**6**, Lemma 2.2]).

Lemma 2.5. *The initial value problem* (1) *is equivalent to the integral equation*

(3)

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}Ax(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left[g(s,x(s))+\left(\int_{t_0}^{s}f(s,r,x(r))\,dr\right)\right]ds,$$

for any $t\in[t_0,t_0+h]$, $t_0\ge 0$.

To proceed, we need the following assumptions.

(A1) $T(\cdot)$ is a C_0-semigroup generated by the operator A on X which satisfies $M = \max_{t\in I_h} \|T(t-t_0)\|_{B(X)}$, where $B(X)$ is the Banach space of all bounded linear operators on X.

(A2) The functions $f : D \times X \to X$ and $g : I_h \times X \to X$ are continuous, and there exist a nondecreasing fractional integrable (of order $\alpha + 1$) function $k_1 \in C(D; \mathbf{R}^+)$ and a fractional integrable (of order α) function $k_2 \in C(I_h; \mathbf{R}^+)$, such that

$$\|f(t, s, x(s)) - f(t, s, y(s))\| \le k_1(t, s) \|x(s) - y(s)\|$$

and

$$\|g(t, x(t)) - g(t, y(t))\| \le k_2(t) \|x(t) - y(t)\|$$

where $0 < \alpha \le 1$. The function k_1 is nondecreasing in the first argument holding the second argument fixed, i.e., $k_1(t_1, s) \le k_1(t_2, s)$ for $t_1 \le t_2$.

(A3) The functions f and g are as in assumption (A2) with $k_1(t, s) = k_1$ and $k_2(t) = k_2$. The constants k_1 and k_2 depend upon $C > 0$ such that $\|x(t)\| \le C$ and $\|y(t)\| \le C$ for any $t \in I_h$. Moreover, we assume that

$$\max_{t\in I_h} \int_{t_0}^{t} |f(t, s, 0)| \, ds = b_1 < \infty \quad \text{and} \quad \max_{t\in I_h} |g(t, 0)| = b_2 < \infty.$$

We need the following lemma later.

Lemma 2.6. *Assume that $a_i > 0$ for $i = 1, 2, 3$, and that $0 < \alpha \le 1$. The function*

$$f(x) = a_1 - x^\alpha(a_2 + a_3 x)$$

is nonincreasing for $x \ge 0$, and there is an interval $[0, a)$, $a > 0$, such that $0 \le f \le a_1$.

3. Main results. In this section we introduce the main theorems on existence and uniqueness of the mild solution of the integro-differential equation (1). The first of our main results is as follows.

Theorem 3.1. *Under assumptions* (A1) *and* (A2), *for* $x_0 \in X$ *the integral equation* (1) *has a unique mild solution* $x \in C(I_h; X)$ *provided that* $I^\alpha[k_2(t) + I^1 k_1(t,t)] < qM^{-1}$, $0 < q < 1$.

Proof. Let $x_0 \in X$ be fixed. Define an operator $G : C(I_h; X) \to C(I_h; X)$ for $t \in I_h$ by

(4)

$$(Gx)(t) = T(t-t_0)x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}T(t-s)\left[g(s,x(s)) + \left(\int_{t_0}^{s} f(s,r,x(r))dr\right)\right]ds.$$

By the assumptions, the operator G is well defined on $C(I_h; X)$. Now we use the fixed point theorem (for contraction mapping) to prove the existence of a solution $x(t) \in X$. Let x, $y \in C(I_h; X)$, then, by using assumptions (A1) and (A2), we get

$$\begin{aligned}
&\|(Gx)(t) - (Gy)(t)\| \\
&\quad\le \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|g(s,x(s)) - g(s,y(s))\|\,ds \\
&\qquad + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left(\int_{t_0}^{s}\|f(s,r,x(r)) - f(s,r,y(r))\|\,dr\right)ds \\
&\quad\le \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}k_2(s)\|x(s) - y(s)\|\,ds \\
&\qquad + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}\int_{t_0}^{s}(t-s)^{\alpha-1}k_1(s,r)\|x(r) - y(r)\|\,dr\,ds.
\end{aligned}$$

We use the change in the order of the second integral to get

$$\begin{aligned}
&\|(Gx)(t) - (Gy)(t)\| \\
&\qquad\le \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}k_2(s)\,\|x(s) - y(s)\|\,ds
\end{aligned}$$

$$+\frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}\|x(r)-y(r)\|\left(\int_{r}^{t}(t-s)^{\alpha-1}k_1(s,r)\,ds\right)dr$$

$$\leq \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}k_2(s)\,\|x(s)-y(s)\|\;ds$$

$$+\frac{M}{\Gamma(\alpha+1)}\int_{t_0}^{t}(t-s)^{\alpha}k_1(t,s)\,\|x(s)-y(s)\|\;ds$$

$$=\frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left[k_2(s)+\frac{(t-s)k_1(t,s)}{\alpha}\,\|x(s)-y(s)\|\right]ds$$

$$\leq M\left(I^{\alpha}k_2(t)+I^{\alpha+1}k_1(t,t)\right)\|x-y\|_{\infty}\leq q\|x-y\|_{\infty}.$$

Therefore, G has a unique fixed point $x=G(x)\in C(I_h;X)$, which is a solution of (2), and hence it is a mild solution of (1). □

We now prove the existence and uniqueness theorems for the case that f and g satisfy Lipschitz conditions locally in $x(t)$ and uniformly in t on a compact interval.

Theorem 3.2. *Under assumptions* (A1) *and* (A3) *for* $x_0\in X$*, the integral equation* (1) *has a unique mild solution* $x\in C([t_0,t_1];X)$*, for some* $t_1\leq t_0+h$*.*

Proof. We define a closed subset B given by

$$B=\{x\in C\,(I;X):\ \text{for }\ \|x(t)\|\leq L=2M\,\|x_0\|\,,\ \text{for }t_0\leq t\leq t_1\}$$

of $C([t_0,t_1];X)\subseteq C([t_0,t_0+h];X)$, where $t_1=t_0+\delta$,

$$\varepsilon^{\alpha}\left(\frac{Bk_2+b_2+b_1}{\Gamma(\alpha+1)}+\frac{Bk_1}{\Gamma(\alpha+2)}\varepsilon\right)<\|x_0\|\,,$$

and the above constants, as in assumptions (A1) and (A3), depend only on $h>0$ and $t_0\geq 0$. It is clear that B is a closed subset of the Banach space $C([t_0,t_1];X)$. Moreover, it is a nonempty subset, since $x=\beta Mx_0\in B$ and $0\leq\beta\leq 2$.

Now we prove that the operator G defined by (4) maps the set B into itself. Let $x \in B$, then by assumption (A1) we get

$$
\begin{aligned}
&\|(Gx)(t)\| \\
&\leq M\|x_0\| + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|g(s,x(s))\|\,ds \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\int_{t_0}^{s}\|f(s,r,x(r))\|\,dr\,ds \\
&= M\|x_0\| + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|g(s,x(s))-g(s,0)+g(s,0)\|\,ds \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}\int_{t_0}^{s}(t-s)^{\alpha-1}\|f(s,r,x(r))-f(s,r,0)+f(s,r,0)\|\,dr\,ds \\
&\leq M\|x_0\| + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|g(s,x(s))-g(s,0)\|\,ds \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}\int_{t_0}^{s}(t-s)^{\alpha-1}\|f(s,r,x(r))-f(s,r,0)\|\,dr\,ds \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|g(s,0)\|\,ds \\
&\quad + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left(\int_{t_0}^{s}\|f(s,r,0)\|\,dr\right)ds \\
&\leq M\|x_0\| + \frac{k_2 M}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|x(s)\|\,ds \\
&\quad + \frac{Mk_1}{\Gamma(\alpha)}\int_{t_0}^{t}\int_{t_0}^{s}(t-s)^{\alpha-1}\|x(r)\|\,dr\,ds
\end{aligned}
$$

$$+ MI^{\alpha}\|g(t,0)\| + MI^{\alpha}\left(\int_{t_0}^{t}\|f(t,s,0)\|\ ds\right).$$

By assumption (A3) and the properties of the subset B, one can get

$$\begin{aligned}\|(Gx)(t)\| &\le M\|x_0\| + \frac{MLk_2}{\Gamma(\alpha+1)}(t-t_0)^{\alpha} \\ &\quad + \frac{MLk_1}{\Gamma(\alpha+2)}(t-t_0)^{\alpha+1} + \frac{Mb_2}{\Gamma(\alpha+1)}(t-t_0)^{\alpha} \\ &\quad + MI^{\alpha}\left(\int_{t_0}^{t}\|f(t,s,0)\|\ ds\right) \\ &\le M\|x_0\| + \frac{MLk_2}{\Gamma(\alpha+1)}(t-t_0)^{\alpha} \\ &\quad + \frac{MLk_1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha+1} \\ &\quad + \frac{Mb_2}{\Gamma(\alpha+1)}(t-t_0)^{\alpha} + \frac{Mb_1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha} \\ &= M\|x_0\| + M(t-t_0)^{\alpha}\left(\frac{Lk_2+b_2+b_1}{\Gamma(\alpha+1)} + \frac{Lk_1}{\Gamma(\alpha+2)}(t-t_0)\right)\end{aligned}$$

In view of the definition of t_1 and Lemma 2.6, there exists a positive number ε such that equation (3) holds, where we put

$$a_1 = \|x_0\|, \quad a_2 = (Lk_2+b_2+b_1)/\Gamma(\alpha+1), \quad a_3 = Lk_1/\Gamma(\alpha+2).$$

Hence, we have

$$\|(Gx)(t)\| \le 2M\|x_0\|.$$

As in Theorem 3.1, the operator G has a unique fixed point $x \in B$ such that $x(t)$ is the solution of equation (1). □

We close this article with the following result.

Theorem 3.3. *Let $D = \{(t,s) : t_0 \le s \le t < \infty\}$. Under assumptions* (A1) *and* (A3)*, the integro-differential equation* (1) *has a unique mild solution $x(t)$ such that $t \in [0,T)$ for some $T \le +\infty$.*

Proof. By Theorem 3.2, there is a unique mild solution $x_1(t)$ on the interval $[0, t_1]$. Again, applying Theorem 3.2 with initial condition $x_1(t_1)$, there is a solution $x_2(t)$ on the interval $[t_1, 2t_1]$. Continuing in this manner, one can get a solution $x(t) = x_k(t)$ for $t \in [(k-1)t_1, kt_1]$, $k \geq 1$, which is unique, since otherwise there are solutions $x(t)$ and $y(t)$ on the interval $[(k-1)t_1, kt_1]$, for some k, contradicting the uniqueness part of Theorem 3.2. Hence, the above solution is unique.

We now prove that the interval on which this solution exists can be globally extended. Let $[0, T)$ be the maximal existence interval of the solution $x(t)$ of equation (1) such that $T < \infty$, and let (t_n) be a sequence that converges to T. If $\lim_{n\to\infty} \|x(t_n)\|$ exists, then $\|x(t_n)\|$ is bounded for all n. Hence, by using Theorem 3.2 for $\varepsilon > 0$, one can extend the solution $x(t)$ on $[0, t_n + \varepsilon]$, where $t_n + \varepsilon \geq T$ and n is sufficient large. This contradicts the maximality of T. Hence, $\lim_{t\to T} \|x(t)\|$ does not exist, which implies the result. □

Remark 1. By Lemma 2.5, the above results are also satisfied by replacing the integral equation (3) instead of equation (1).

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