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UNIQUE SOLVABILITY OF NUMERICAL METHODS FOR STIFF DELAY-INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper deals with the unique solvability of numerical methods for stiff delay-integro-differential equations (DIDEs). Several unique solvability conditions of the extended general linear methods for DIDEs are derived. The conclusions obtained are applied to some common numerical methods such as the extended linear multistep methods and the extended Runge-Kutta methods. In the end, concrete examples illustrate the utility of the theory.

1. Introduction. Delay-integro-differential equations (DIDEs) arise widely in the mathematical modelings of physical and biological phenomena. Significant advances in the research of theoretical solutions and numerical solutions for such equations have been made in recent years (see, e.g., [**8, 9, 10**]). A survey of the related results refers to Brunner's monograph (cf. [**1**]). The existing research deals mainly with stability, dissipativity, convergence and computational implementation of the numerical methods. When numerically computing a stiff DIDE, generally speaking, an implicit algebraic equation needs to be solved. In order to obtain highly effective numerical methods, the concept of *algebraic stability* is often used. The algebraic stability, unfortunately, cannot guarantee the existence of numerical solutions. For example, Crouzeix, Hundsdorffer and Spijker [**3**] constructed a counterexample, which shows that the fourth order Runge-Kutta method

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$$
\begin{array}{c|ccc}
\frac{1}{2}+\frac{\sqrt{6}}{6} & \frac{1}{8} & \frac{1}{8}-\frac{\sqrt{6}}{6} & \frac{1}{4}+\frac{\sqrt{6}}{3} \\
\frac{1}{2}-\frac{\sqrt{6}}{6} & \frac{1}{8}+\frac{\sqrt{6}}{6} & \frac{1}{8} & \frac{1}{4}-\frac{\sqrt{6}}{3} \\
\frac{1}{2} & \frac{1}{8}-\frac{\sqrt{6}}{6} & \frac{1}{8}+\frac{\sqrt{6}}{6} & \frac{1}{4} \\
\hline
 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2}
\end{array}
$$

is algebraically stable but has no solution.

What is the condition to guarantee the unique solvability of the implicit algebraic equation in the numerical methods for stiff DIDEs? This is an important problem for numerical methods. For the unique solvability of the numerical methods applied to stiff ordinary differential equations (ODEs), as we know, there have been a lot of results. Among these findings, unique solvability of Runge-Kutta methods were studied in the papers [**2**–**5**] and their references. Several very general results of multivalue multiderivative methods for solving stiff ODEs have been obtained by Li [**6**].

However, so far, no result deals with the unique solvability of numerical methods for stiff DIDEs. In view of this, in the present paper, we will extend the related research to stiff DIDEs and hence derive the unique solvability conditions of the general linear methods for stiff DIDEs. Also, the derived conclusions can cover many common numerical methods such as the extended linear multistep methods and the extended Runge-Kutta methods. In the end, some concrete examples are given to illustrate the utility of the obtained theory.

2. DIDEs of the class GRI$(\alpha,\beta,\sigma,\gamma)$ and their numerical methods. Let X denote a real or complex Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$, and let $f:\ [t_0,+\infty)\times X\times X\times X\to X$ and $g:\ [t_0,+\infty)\times \mathbf{D}\times X\to X$ be two sufficiently smooth functions, where $\mathbf{D}=\{(t,v): t\in[t_0,+\infty), v\in[t-\tau,t]\}$.

Consider the system of stiff DIDEs with constant delay $\tau>0$ and initial function $\varphi(t)$:

(2.1)
$$
\begin{cases} y'(t)=f(t,y(t),y(t-\tau),\int_{t-\tau}^{t} g(t,v,y(v))\,dv) & t\in[t_0,+\infty), \\ y(t)=\varphi(t) & t\in[t_0-\tau,t_0], \end{cases}
$$

where the functions f, g are assumed to satisfy the following conditions

$$\begin{aligned}(2.2)\quad &\Re\langle f(t,x,y,z)-f(t,\widetilde{x},\widetilde{y},\widetilde{z}),x-\widetilde{x}\rangle\\ &\quad\le \alpha\|x-\widetilde{x}\|^2+\beta\|y-\widetilde{y}\|^2+\sigma\|z-\widetilde{z}\|^2,\quad x,y,z,\widetilde{x},\widetilde{y},\widetilde{z}\in X,\end{aligned}$$

$$(2.3)\quad \|g(t,v,x)-g(t,v,\widetilde{x})\|\le\gamma\|x-\widetilde{x}\|,\qquad (t,v)\in\mathbf{D},\ \widetilde{x}\in X$$

and such that system (2.1) has a unique solution. α, β, σ and γ are real constants independent of $t, v, x, y, z, \widetilde{x}, \widetilde{y}, \widetilde{z}$. Problems of type (2.1) with (2.2)–(2.3) will be called *the class* $\mathbf{GRI}\,(\alpha,\beta,\sigma,\gamma)$. Some examples of the class $\mathbf{GRI}\,(\alpha,\beta,\sigma,\gamma)$ can be found in reference [**10**].

The system (2.1) can be discretized by the extended general linear methods (cf. [**10**])

(2.4)

$$\begin{cases} Y_i^{(n)}=h\sum_{j=1}^s c_{ij}^{(11)}f(t_j^{(n)},Y_j^{(n)},Y_j^{(n-m)},Z_j^{(n)})+\sum_{j=1}^r c_{ij}^{(12)}y_j^{(n-1)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1,2,\dots,s,\\ y_i^{(n)}=h\sum_{j=1}^s c_{ij}^{(21)}f(t_j^{(n)},Y_j^{(n)},Y_j^{(n-m)},Z_j^{(n)})+\sum_{j=1}^r c_{ij}^{(22)}y_j^{(n-1)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1,2,\dots,r,\end{cases}$$

where $c_{ij}^{(IJ)}$ $(I,J=1,2)$ are some real coefficients, stepsize $h=\tau/m$, $t_n=t_0+nh$, $t_j^{(n)}=t_n+c_jh$, $Y_i^{(n)}$ and $y_i^{(n)}$ are approximations to $y(t_i^{(n)})$ and $H_i(t_n+\nu_ih)$, respectively, in which each $H_i(t_n+\nu_ih)$ denotes a piece of information about true solution $y(t)$, and $Z_j^{(n)}$ is an approximation to

$$Z(t_j^{(n)}):=\int_{t_j^{(n-m)}}^{t_j^{(n)}} g(t_j^{(n)},v,y(v))\,dv$$

and computed by a uniform repeated rule (cf. [**10**])

$$(2.5)\qquad Z_j^{(n)}=h\sum_{q=0}^m \nu_q g(t_j^{(n)},t_j^{(n-q)},Y_j^{(n-q)}),\quad j=1,2,\dots,s.$$

The method $\{(2.4),\ (2.5)\}$, for the class $\mathbf{GRI}\,(\alpha,\beta,\sigma,\gamma)$, has been testified as being quite effective in Zhang and Stefan's paper [**10**]. In particular, we have the extended Runge-Kutta methods

(2.6)

$$\begin{cases} Y_i^{(n)}=y_n+h\sum_{j=1}^s a_{ij}f(t_j^{(n)},Y_j^{(n)},Y_j^{(n-m)},Z_j^{(n)})\quad i=1,2,\dots,s,\\ y_{n+1}=y_n+h\sum_{j=1}^s b_jf(t_j^{(n)},Y_j^{(n)},Y_j^{(n-m)},Z_j^{(n)})\end{cases}$$

and the extended linear multistep methods

$$\sum_{j=1}^{k} \alpha_j y_{n+j} = h \sum_{j=1}^{k} \beta_j f(t_{n+j}, y_{n+j}, y_{n+j-m}, z_j^{(n)}), \tag{2.7}$$

where y_n is an approximation to $y(t_n)$ and

$$z_j^{(n)} = h \sum_{q=0}^{m} \nu_q g(t_{n+j}, t_{n+j-q}, y_{n+j-q}), \quad j = 1, 2, \dots, k. \tag{2.8}$$

3. Some lemmas. In this section, we will distill some conclusions from reference [**10**] as lemmas. These conclusions have been used successfully to deal with the unique solvability of the implicit multivalue multiderivative methods for stiff ODEs (cf. [**6**]). Here, again, we will use them to study the unique solvability of the implicit numerical methods induced in Section 2.

For convenience of the statement, we define some notations. A linear mapping $\mathcal{L}: X^\iota \to X^\kappa$ corresponding to matrix $L := (l_{ij}) \in \mathbf{R}^{\kappa\times\iota}$ is given by

$$\mathcal{L}U = \left(\sum_{j=1}^{\iota} l_{1j} u_j, \sum_{j=1}^{\iota} l_{2j} u_j, \dots, \sum_{j=1}^{\iota} l_{\kappa j} u_j \right), \\ \forall U := (u_1, u_2, \dots, u_\iota) \in X^\iota.$$

On space X^ι, an inner product and the induced norm are defined as follows:

$$\langle U,\ V\rangle = \sum_{i=1}^{\iota} \langle u_i,\ v_i\rangle, \qquad \|U\| = \sqrt{\langle U,\ U\rangle},$$

where $U := (u_1, u_2, \dots, u_\iota)$, $V := (v_1, v_2, \dots, v_\iota) \in X^\iota$. If a matrix D is real symmetric positive definite (respectively semi-positive definite), it will be written as $D > 0$ (respectively $D \geq 0$). Moreover, the symbols $\lambda_{\min}^D$, $\lambda_{\max}^D$ denote the minimum and the maximum eigenvalues of matrices D, respectively.

Consider the nonlinear equation

$$\mathcal{Q}(x) := x - \mathcal{E}\psi(x) - \omega = 0, \quad x \in X^s, \tag{3.1}$$

where $\omega \in X^s$ is a given vector, $\mathcal{E}$ is a linear mapping corresponding to a given nonzero matrix $E = (e_{ij}) \in \mathbf{R}^{s\times s}$, and the mapping $\psi(x)$ is continuous on X^s and subject to

$$\Re\langle \mathcal{M}[\psi(x) - \psi(\widetilde{x})],\ x - \widetilde{x}\rangle \leq \langle x - \widetilde{x}, \mathcal{N}[x - \widetilde{x}]\rangle, \quad x, \widetilde{x} \in X^s, \tag{3.2}$$

where $\mathcal{M}, \mathcal{N}$ are the linear mappings corresponding to the matrices $M,\ N \in \mathbf{R}^{s\times s}$ with $N^T = N$.

Let

$$P_0 = P - N, \quad P_1 = M^T E + E^T M - 2E^T P E,$$
$$\Lambda(P_0, P_1, E) = \lambda_{\min}^{P_0} + \lambda_{\min}^{P_1}/(2\|E\|_2^2),$$

where $P \in \mathbf{R}^{s\times s}$ is a given symmetric matrix and $\|E\|_2$ denotes the spectral norm of the nonzero matrix E, that is the square root of the largest eigenvalue of the matrix E^*E. For the unique solvability of equation (3.1) with (3.2), Li [**7**] derived the following conclusions.

Lemma 3.1 (cf. [**7**]). *Assume that there is a real symmetric matrix P such that one of the following conditions is fulfilled*:

(i) $P_0 > 0,\ P_1 \geq 0$;

(ii) $P_0 \geq 0,\ P_1 > 0$;

(iii) $P_0 \geq 0,\ P_1 \geq 0$ *and* (3.2) *holds strictly whenever* $x \neq \widetilde{x}$.

Then equation (3.1) *has at most one solution.*

Lemma 3.2 (cf. [**7**]). *Assume that the condition* (i) *or* (ii) (*in Lemma* 3.1) *holds and the matrices M, E are invertible. Then equation* (3.1) *has a unique solution $x_* \in X^s$ with*

$$\|x_* - x_0\| \leq \|\mathcal{M}\mathcal{E}^{-1}\mathcal{Q}(x_0)\|/\Lambda(P_0, P_1, E), \quad \forall x_0 \in X^s,$$

where $\mathcal{M}\mathcal{E}^{-1}$ denotes the linear mapping corresponding to matrix ME^{-1}.

Lemma 3.3 (cf. [**7**]). *Assume that X is a finite-dimensional space, the matrix $E \neq 0$, M is invertible, and there exist an invertible matrix $B \in \mathbf{R}^{s\times s}$, a symmetric matrix $P \in \mathbf{R}^{s\times s}$ and real numbers $\widehat{a}_i$, $\widehat{b}_i$* $(i = 1, 2, 3)$, *$\widehat{d}$ with*

$$\widehat{a}_2\widehat{b}_3 - \widehat{a}_3\widehat{b}_2 \neq 0, \quad 0 \leq \operatorname{sgn}(\widehat{a}_3) + \operatorname{sgn}(\widehat{b}_3) \leq 1 \tag{3.3}$$

such that

$$P_0 > 0, \quad P_1 \geq 0, \quad \sum_{j=1}^{3} \widehat{a}_j P_j \geq 0, \quad \sum_{j=1}^{3} \widehat{b}_j P_j \geq 0,$$

where

$$P_2 = M^T B + B^T M - 2(B^T PE + E^T PB - \widehat{d} E^T PE),$$
$$P_3 = \widehat{d}(M^T B + B^T M) - 2B^T PB.$$

Then equation (3.1) *has a unique solution* $x_* \in X^s$ *with*

$$\|x_* - \omega\| \leq \|\mathcal{M}\psi(\omega)\| / \Lambda(P_0, P_1, E),$$

where $\mathcal{M}$ *denotes the linear mapping corresponding to matrix* M.

4. Unique solvability of numerical methods. In this section, we will apply the previous lemmas to the extended general linear methods (2.4) and its special cases (2.6), (2.7). This will lead to the unique solvability of the methods.

With the defined linear mapping, the implicit equations in (2.4) can be rewritten in a more compact form:

$$\mathcal{Q}_h(Y^{(n)}) := Y^{(n)} - h\mathcal{C}_{11} F(Y^{(n)}) - \mathcal{C}_{12} y^{(n-1)} = 0, \tag{4.1}$$

where $\mathcal{C}_{11}$ and $\mathcal{C}_{12}$ are the linear mappings corresponding to the matrices $C_{11} = (c_{ij}^{(11)})$ and $C_{12} = (c_{ij}^{(12)})$, respectively, and

$$y^{(n-1)} = \left(y_1^{(n-1)}, y_2^{(n-1)}, \ldots, y_r^{(n-1)}\right),$$
$$Y^{(n)}) = \left(Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_s^{(n)}\right),$$
$$F(Y^{(n)}) = \left(f_{1,\ n}(Y_1^{(n)}), f_{2,\ n}(Y_2^{(n)}), \ldots, f_{s,\ n}(Y_s^{(n)})\right),$$

with

$$f_{i,\ n}(Y_i^{(n)}) = f\left(t_i^{(n)}, Y_i^{(n)}, Y_i^{(n-m)}, h \sum_{q=0}^{m} \nu_q g\left(t_i^{(n)}, t_i^{(n-q)}, Y_i^{(n-q)}\right)\right),$$
$$i = 1, 2, \ldots, s.$$

In particular, for the extended Runge-Kutta methods (2.6) we have

$$C_{11} = A := (a_{ij}), \quad C_{12} = e := (\underbrace{1, 1, \dots, 1}_{s})^T, \quad y^{(n-1)} = y_n;$$

and for the extended linear multistep methods (2.7) we have

$$\begin{aligned} C_{11} &= \frac{\beta_k}{\alpha_k}, \quad C_{12} = \frac{1}{\alpha_k}(-\alpha_0, -\alpha_1, \dots, -\alpha_{k-1}, \beta_0, \beta_1, \dots, \beta_{k-1}), \\ y^{(n-1)} &= (y_n, y_{n+1}, \dots, y_{n+k-1}, hf_n, hf_{n+1}, \dots, hf_{n+k-1}), \\ Y^{(n)} &= y_{n+k}, \\ F(Y^{(n)}) &= f\left(t_{n+k}, y_{n+k}, y_{n+k-m}, h\sum_{q=0}^{m} \nu_q g(t_{n+k}, t_{n+k-q}, y_{n+k-q})\right). \end{aligned}$$

It is clear that the extended general linear method (2.4) has a unique solution if and only if the equation (4.1) is uniquely solvable. Hence, in the following, we will say *the method* (2.4) *has a unique solution* whenever the equation (4.1) is uniquely solvable.

Let $\mathcal{D}$ be a linear mapping corresponding to the diagonal matrix $D = \operatorname{diag}(d_1, d_2, \dots, d_s) \geq 0$. Then, for any $x = (x_1, x_2, \dots, x_s)$, $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_s) \in X^s$, it follows from (2.2)–(2.3) that

$$\begin{aligned} (4.2) \quad & \Re\langle \mathcal{D}[F(x) - F(\widetilde{x})],\ x - \widetilde{x}\rangle \\ &= \sum_{i=1}^{s} d_i \Re\langle f_{i,\ n}(x_i) - f_{i,\ n}(\widetilde{x}_i),\ x_i - \widetilde{x}_i\rangle \\ &\leq \sum_{i=1}^{s} d_i[\alpha\|x_i - \widetilde{x}_i\|^2 + h^2\sigma\nu_0^2\|g(t_i^{(n)}, t_i^{(n)}, x_i) - g(t_i^{(n)}, t_i^{(n)}, \widetilde{x}_i)\|^2] \\ &\leq (\alpha + h^2\sigma\nu_0^2\gamma^2)\sum_{i=1}^{s} d_i\|x_i - \widetilde{x}_i\|^2 \\ &= \langle x - \widetilde{x}, (\alpha + h^2\sigma\nu_0^2\gamma^2)\mathcal{D}(x - \widetilde{x})\rangle. \end{aligned}$$

This shows that condition (3.2) holds with

$$\mathcal{M} = \mathcal{D}, \quad \mathcal{N} = h(\alpha + h^2\sigma\nu_0^2\gamma^2)\mathcal{D}, \quad \psi(x) = hF(x).$$

Taking advantage of Lemmas 3.1 and 3.2, the following theorem can be obtained.

Theorem 4.1. *Assume that there exist a real symmetric matrix $P \in \mathbf{R}^{s\times s}$ and a nonnegative diagonal matrix $D \in \mathbf{R}^{s\times s}$ such that one of the following conditions is fulfilled:*

$$\text{(i)} \quad \begin{cases} P_0 := P - h(\alpha + h^2\sigma\nu_0^2\gamma^2)D > 0, \\ P_1 := DC_{11} + C_{11}^T D - 2C_{11}^T P C_{11} \geq 0; \end{cases}$$

$$\text{(ii)} \quad P_0 \geq 0, \;\; P_1 > 0;$$

$$\text{(iii)} \quad \begin{cases} P_0 \geq 0, \;\; P_1 \geq 0, \;\; D > 0 \\ (4.2) \text{ holds strictly whenever } Y^{(n)} \neq \widetilde{Y}^{(n)}. \end{cases}$$

Then the extended general linear method (2.4) *for the class* $\mathbf{GRI}(\alpha, \beta, \sigma, \gamma)$ *has at most one solution. If the above condition* (i) *or* (ii) *holds, matrix $D > 0$ and C_{11} is invertible, the extended general linear method* (2.4) *for the class* $\mathbf{GRI}\,(\alpha, \beta, \sigma, \gamma)$ *has a unique solution and its implicit equation's solution $Y_*^{(n)} \in X^s$ satisfies*

$$(4.3) \qquad \|Y_*^{(n)} - Y_0\| \leq \|\mathcal{D}\mathcal{C}_{11}^{-1}\mathcal{Q}_h(Y_0)\| / \Lambda(P_0, P_1, C_{11}), \quad \forall Y_0 \in X^s,$$

where $\mathcal{D}\mathcal{C}_{11}^{-1}$ denotes the linear mapping corresponding to matrix DC_{11}^{-1}.

When we set $P = lD$ ($l \in \mathbf{R}$) in Theorem 4.1, an interesting conclusion can be derived as follows.

Corollary 4.2. *Assume that C_{11} is invertible, and there exist a positive diagonal matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_s)$ and a real number l such that one of the following conditions are fulfilled*:

$$\text{(i)} \quad h(\alpha + h^2\sigma\nu_0^2\gamma^2) < l, \;\; DC_{11} + C_{11}^T D - 2lC_{11}^T DC_{11} \geq 0;$$

$$\text{(ii)} \quad h(\alpha + h^2\sigma\nu_0^2\gamma^2) \leq l, \;\; DC_{11} + C_{11}^T D - 2lC_{11}^T DC_{11} > 0.$$

Then the extended general linear method (2.4) *for the class* $\mathbf{GRI}(\alpha, \beta, \sigma, \gamma)$ *has a unique solution and its implicit equation's solution $Y_*^{(n)} \in X^s$ satisfies* (4.3) *with*

$$P_0 = [l - h(\alpha + h^2\sigma\nu_0^2\gamma^2)]D, \quad P_1 = DC_{11} + C_{11}^T D - 2lC_{11}^T DC_{11}.$$

Applying Lemma 3.3 to equation (4.1) yields

Theorem 4.3. *Assume that X is a finite-dimensional space, matrix $C_{11} \neq 0$ and there exist a positive diagonal matrix $D \in \mathbf{R}^{s\times s}$, an invertible matrix $B \in \mathbf{R}^{s\times s}$, a symmetric matrix $P \in \mathbf{R}^{s\times s}$ and real numbers $\widehat{a}_i$, $\widehat{b}_i$ $(i = 1, 2, 3)$, $\widehat{d}$ with (3.3) such that*

$$P_0 > 0, \quad P_1 \geq 0, \quad \sum_{j=1}^{3} \widehat{a}_j P_j \geq 0, \quad \sum_{j=1}^{3} \widehat{b}_j P_j \geq 0,$$

where P_0, P_1 are indicated in Theorem 4.1 and

$$\begin{aligned} P_2 &= DB + B^T D - 2(B^T P C_{11} + C_{11}^T P B - \hat{d} C_{11}^T P C_{11}), \\ P_3 &= \widehat{d}(DB + B^T D) - 2B^T P B. \end{aligned}$$

Then the extended general linear method (2.4) for the class $\mathbf{GRI}(\alpha, \beta, \sigma, \gamma)$ has a unique solution and its implicit equation's solution $Y_*^{(n)} \in X^s$ satisfies

$$\|Y_*^{(n)} - \mathcal{C}_{12} y^{(n-1)}\| \leq h \|\mathcal{D} F(\mathcal{C}_{12} y^{(n-1)})\| / \Lambda(P_0, P_1, C_{11}), \tag{4.4}$$

where $\mathcal{D}$ denotes the linear mapping corresponding to matrix D.

Let

$$\begin{aligned} &\widehat{a}_1 = \widehat{d} = 2l, \quad \widehat{a}_2 = \widehat{b}_3 = 1, \quad \widehat{a}_3 = 0, \\ &\widehat{b}_1 = -2l^2, \quad \widehat{b}_2 = -l, \quad P = l\widehat{P}, \quad B = \widehat{P}^{-1} D, \end{aligned}$$

where $\widehat{P} \in \mathbf{R}^{s\times s}$ is a symmetric-positive-definite matrix and $D \in \mathbf{R}^{s\times s}$ is a positive diagonal matrix. Then, by Theorem 4.3 we have

Corollary 4.4. *Assume that X is a finite-dimensional space, matrix $C_{11} \neq 0$, and there exist a positive diagonal matrix $D \in \mathbf{R}^{s\times s}$, a symmetric-positive-definite matrix $\widehat{P} \in \mathbf{R}^{s\times s}$ and a real number l such that*

$$\begin{aligned} \widetilde{P}_0 &:= l\widehat{P} - h(\alpha + h^2 \sigma \nu_0^2 \gamma^2) D > 0, \\ \widetilde{P}_1 &:= DC_{11} + C_{11}^T D - 2l C_{11}^T \widehat{P} C_{11} \geq 0. \end{aligned}$$

Then the extended general linear method (2.4) *for the class* $\mathbf{GRI}(\alpha, \beta, \sigma, \gamma)$ *has a unique solution, and its implicit equation's solution* $Y_*^{(n)} \in X^s$ *satisfies* (4.4) *with* $P_i = \widetilde{P}_i$ $(i = 0, 1)$.

In particular, when set $\widehat{P} = D$, the above theorem can be reduced into a simple criterion.

Corollary 4.5. *Assume that* X *is a finite-dimensional space, matrix* $C_{11} \neq 0$, *and there exist a positive diagonal matrix* $D = \operatorname{diag}(d_1, d_2, \dots, d_s)$ *and a real number* l *such that*

$$h(\alpha + h^2\sigma\nu_0^2\gamma^2) < l, \qquad DC_{11} + C_{11}^T D - 2lC_{11}^T DC_{11} \geq 0.$$

Then the extended general linear method (2.4) *for the class* $\mathbf{GRI}(\alpha, \beta, \sigma, \gamma)$ *has a unique solution and its implicit equation's solution* $Y_*^{(n)} \in X^s$ *satisfies* (4.4) *with*

$$P_0 = [l - h(\alpha + h^2\sigma\nu_0^2\gamma^2)]D, \quad P_1 = DC_{11} + C_{11}^T D - 2lC_{11}^T DC_{11}.$$

Corollary 4.2 is quite similar to Corollary 4.5. The difference between the two is that the latter can cover the case where C_{11} is a nonzero singular matrix when X is a finite-dimensional space. Moreover, it is interesting that when we apply these two corollaries to the extended Runge-Kutta methods, respectively, the conclusions obtained can be viewed as extensions to those in the references [**2, 3, 5**], where the unique solvability of Runge-Kutta methods for stiff ODEs was concerned.

Corollary 4.6. *Assume that* A *is invertible, and there exist a positive diagonal matrix* $D = \operatorname{diag}(d_1, d_2, \dots, d_s)$ *and a real number* l *such that one of the following conditions is fulfilled*:

$$\text{(i)} \quad h(\alpha + h^2\sigma\nu_0^2\gamma^2) < l, \quad DA + A^T D - 2lA^T DA \geq 0;$$
$$\text{(ii)} \quad h(\alpha + h^2\sigma\nu_0^2\gamma^2) \leq l, \quad DA + A^T D - 2lA^T DA > 0.$$

Then the extended Runge-Kutta method (2.6) *for the class* $\mathbf{GRI}(\alpha, \beta, \sigma, \gamma)$ *has a unique solution and its implicit equation's solution* $Y_*^{(n)} \in X^s$

satisfies (4.3) *with*

$$\begin{cases} C_{12} = e, \ C_{11} = A, \ y^{(n-1)} = y_n, \\ P_0 = [l - h(\alpha + h^2\sigma\nu_0^2\gamma^2)]D, \ P_1 = DA + A^TD - 2lA^TDA. \end{cases}$$

Corollary 4.7. *Assume that X is a finite-dimensional space, matrix $A \neq 0$, and there exist a positive diagonal matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_s)$ and a real number l such that*

$$h(\alpha + h^2\sigma\nu_0^2\gamma^2) < l, \qquad DA + A^TD - 2lA^TDA \geq 0.$$

Then the extended Runge-Kutta method (2.6) *for the class* **GRI** $(\alpha, \beta, \sigma, \gamma)$ *has a unique solution and its implicit equation's solution $Y_*^{(n)} \in X^s$ satisfies* (4.4) *with* (4.5).

Furthermore, when applying Corollary 4.2 to the extended linear multistep methods (2.7), we have the following conclusion.

Corollary 4.8. *Assume that $\alpha_k\beta_k \neq 0$, and there exists a real number l such that one of the following conditions is fulfilled*:

$$\begin{aligned} &\text{(i)} \quad h(\alpha + h^2\sigma\nu_0^2\gamma^2) < l, \quad \beta_k(\alpha_k - \beta_k l) \geq 0; \\ &\text{(ii)} \quad h(\alpha + h^2\sigma\nu_0^2\gamma^2) \leq l, \quad \beta_k(\alpha_k - \beta_k l) > 0. \end{aligned}$$

Then the extended linear multistep methods (2.7) *for the class* **GRI**$(\alpha, \beta, \sigma, \gamma)$ *has a unique solution $y_{n+k} \in X$, which satisfies*

$$\|y_{n+k} - x_0\| \leq \left| \frac{\alpha_k}{\alpha_k - h\beta_k(\alpha_k + h^2\sigma\nu_0^2\gamma^2)} \right| \|\mathcal{Q}_h(x_0)\|, \quad \forall x_0 \in X, \tag{4.6}$$

where

$$\mathcal{Q}_h(x_0) = x_0 - h\frac{\beta_k}{\alpha_k}F(x_0) - \frac{1}{\alpha_k}\bigg(-\sum_{j=0}^{k-1}\alpha_j y_{n+j} + h\sum_{j=0}^{k-1}\beta_j f_{n+j}\bigg),$$

$$F(x_0) = f\Big(t_{n+k}, x_0, y_{n+k-m}, h\nu_0 g(t_{n+k}, t_{n+k}, x_0) + h\sum_{q=1}^{m} \nu_q g(t_{n+k}, t_{n+k-q}, y_{n+k-q})\Big).$$

When applying Corollary 4.5 to the extended linear multistep methods (2.7), the result obtained will be covered by Corollary 4.8, hence we omit it. In fact, for the extended Runge-Kutta methods and the extended linear multistep methods, besides the above-presented results we may also get more general results by application of Theorem 4.1, Theorem 4.3 and Corollary 4.4. Since the general results will involve only some changes in the symbols for those of the extended general linear methods, it is unnecessary to rewrite them.

5. An illustration with concrete examples. In order to give an illustration for the results obtained, we consider several concrete examples as follows.

Example 5.1. Consider an extended Runge-Kutta method composed by the two-stage four-order Gauss method

$$\begin{array}{c|cc} \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\ \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \tag{5.1}$$

with the four-order compound Simpson quadrature rule

$$\begin{aligned} Z_j^{(n)} = \frac{h}{3}[&g(t_n + c_j h, t_n + c_j h, Y_j^{(n)}) \\ &+ 4\sum_{q=1}^{m/2} g(t_n + c_j h, t_{n-2q+1} + c_j h, Y_j^{(n-2q+1)}) \\ &+ 2\sum_{q=1}^{(m-2)/2} g(t_n + c_j h, t_{n-2q} + c_j h, Y_j^{(n-2q)}) \\ &+ g(t_n + c_j h, t_{n-m} + c_j h, Y_j^{(n-m)})], \\ &\qquad j = 1, 2; \quad m \geq 4. \end{aligned} \tag{5.2}$$

For the above method, one easily verifies that matrix

$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \end{pmatrix}$$

is invertible. Moreover, when we set $D = \mathrm{diag}\,(1, 7 - 4\sqrt{3})$, Cooper [**6**] has shown that

$$DA + A^T D - 2lA^T DA \geq 0 \quad \text{if and only if } l \leq 3,$$

where $DA + A^T D - 2lA^T DA = 0$ if and only if $l = 3$. Hence, by Corollary 4.6, we know that the method (5.1)–(5.2) for the class $\mathbf{GRI}\,(\alpha, \beta, \sigma, \gamma)$ has a unique solution and its implicit equation's solution $Y_*^{(n)} \in X^2$ satisfies (4.3) with (4.5), whenever

$$h\Big(\alpha + \frac{1}{9}h^2\sigma\gamma^2\Big) < l \leq 3 \text{ or } h\Big(\alpha + \frac{1}{9}h^2\sigma\gamma^2\Big) \leq l < 3.$$

It is well known that the above-mentioned Gauss method is algebraically stable. In fact, our result is also applicable for the methods that are not algebraically stable. The following is just such an example.

Example 5.2. Consider the extended Runge-Kutta method composed by the two-stage third order method (cf. [**3**])

$$\text{(5.3)} \qquad \begin{array}{c|cc} \frac{1}{6} & \frac{5}{42} & \frac{1}{21} \\ \frac{3}{4} & \frac{15}{28} & \frac{3}{14} \\ \hline & \frac{3}{7} & \frac{4}{7} \end{array}$$

with the three-order compound Gregory rule

$$\begin{aligned} Z_j^{(n)} = \frac{h}{12}[&5g(t_n + c_j h, t_n + c_j h, Y_j^{(n)}) \\ &+ 13g(t_n + c_j h, t_{n-1} + c_j h, Y_j^{(n-1)}) \\ &+ 12\sum_{q=2}^{m-2} g(t_n + c_j h, t_{n-q} + c_j h, Y_j^{(n-q)}) \\ &+ 13g(t_n + c_j h, t_{n-m+1} + c_j h, Y_j^{(n-m+1)}) \\ &+ 5g(t_n + c_j h, t_{n-m} + c_j h, Y_j^{(n-m)})], \quad j = 1, 2. \end{aligned} \tag{5.4}$$

Although the underlying method (5.3) is not algebraically stable and $\det(A) = 0$, we still give the unique solvability criteria of the extended method on a finite-dimensional space X. Set $D = \operatorname{diag}(1, (4/45))$; it can be shown that

$$DA + A^T D - 2lA^T DA \geq 0 \quad \text{if and only if } l \leq 3.$$

Therefore, we conclude from Corollary 4.7 that the method (5.3)–(5.4) for the class $\mathbf{GRI}\,(\alpha, \beta, \sigma, \gamma)$ has a unique solution, and its implicit equation's solution $Y_*^{(n)} \in X^2$ satisfies (4.4) with (4.5), whenever

$$h\left(\alpha + \frac{25}{144}h^2\sigma\gamma^2\right) < l \leq 3.$$

In the end, we give an example for the extended linear multistep methods.

Example 5.3. Consider an extended linear multistep method which composed by the two-step second order BDF method and the compound trapezoidal rule:

$$y_{n+2} = \frac{2}{3}hf(t_{n+2}, y_{n+2}, y_{n+2-m}, z^{(n)}) + \frac{4}{3}y_{n+1} - \frac{1}{3}y_n, \tag{5.5}$$

where $z^{(n)}$ is computed by the compound trapezoidal rule:

$$\begin{aligned} z^{(n)} = \frac{h}{2}\Big[g(t_{n+2}, t_{n+2}, y_{n+2}) &+ 2\sum_{q=1}^{m-1} g(t_{n+2}, t_{n+2-q}, y_{n-q+2}) \\ &+ g(t_{n+2}, t_{n+2-m}, y_{n-m+2})\Big]. \end{aligned} \tag{5.6}$$

Since for the method (5.5) it holds that

$$\beta_2(\alpha_2 - l\beta_2) \geq 0 \quad \text{if and only if } l \leq \frac{3}{2},$$

where $\beta_2(\alpha_2 - l\beta_2) = 0$ if and only if $l = 3/2$, it follows from Corollary 4.8 that the method (5.5)–(5.6) for the class $\mathbf{GRI}\,(\alpha, \beta, \sigma, \gamma)$ has a unique solution $y_{n+k} \in X$ with (4.6), whenever

$$h\Big(\alpha + \frac{1}{4}h^2\sigma\gamma^2\Big) < l \leq \frac{3}{2} \text{ or } h\Big(\alpha + \frac{1}{4}h^2\sigma\gamma^2\Big) \leq l < \frac{3}{2}.$$

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