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A VARIATIONAL APPROACH FOR A NONLOCAL AND NONVARIATIONAL ELLIPTIC PROBLEM

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ABSTRACT. We prove results concerning the existence of solutions for the problem

$$-a(x, \int_\Omega u\,dx)\Delta u = f(x,u) \quad \text{in } \Omega,$$

where Ω is a bounded regular domain and $f : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a function having subcritical growth. Although we are facing a problem with lack of variational structure we will be able to apply variational technique (the Mountain pass theorem) by suitably using a device introduced in [**6**].

1. Introduction. In this paper we investigate questions of existence of solutions for the following problem

$$(P_1) \qquad \begin{cases} -a\left(x, \int_\Omega u\,dx\right)\ \Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{for all } x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, $N \geq 3$ and the functions a and f enjoy the following assumptions:

The function $a : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is continuous, and there are constants a_0, a_∞, R_1 and L_1 such that

$$(a_1) \qquad 0 < a_0 \leq a(x,t) \leq a_\infty \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbf{R},$$

and

$$(a_2) \qquad |a(x,s_1) - a(x,s_2)| \leq L_1|s_1 - s_2|,$$

for all $s_1, s_2 \in [0, R_1]$ and for all $x \in \overline{\Omega}$.

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The nonlinearity $f : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a continuous function satisfying

$$(f_1) \qquad f(x,s) = 0 \quad \text{for all } s < 0 \text{ and } x \in \overline{\Omega},$$

$$(f_2) \qquad \lim_{|s|\to 0} \frac{|f(x,s)|}{s} = 0, \quad \text{uniformly in } x \in \overline{\Omega}.$$

There exists a $2 < q < 2^* = 2N/(N-2)$ such that

$$(f_3) \qquad \lim_{s\to+\infty} \frac{|f(x,s)|}{|s|^{q-1}} = 0, \quad \text{uniformly in } x \in \overline{\Omega},$$

where $2 < q < 2^*$ and $2^* = 2N/(N-2)$.

From assumptions (f_2)–(f_3), given $\epsilon > 0$, there exist C_ϵ such that

$$f(x,s) \le \epsilon|s| + C_\epsilon |s|^{q-1}, \tag{1.1}$$

for $s \in \mathbf{R}$ and $x \in \overline{\Omega}$.

In this article, the classical Palais-Smale condition will play a key role. Related to this condition, we have the well known Ambrosetti-Rabinowitz superlinear condition, that is, there exists a $\theta \in \mathbf{R}$ with $2 < \theta < q$ such that

$$(f_4) \qquad \begin{gathered} 0 < \theta F(x,s) = \theta \int_0^s f(x,t)\,dt \le s f(x,s) \\ \text{for all } s > 0 \text{ and for all } x \in \overline{\Omega}. \end{gathered}$$

(f_5)

$$\text{The function } s \to \frac{f(x,s)}{s} \text{ is increasing in } (0,+\infty), \text{ for all } x \in \overline{\Omega}.$$

We also suppose that there exists a constant L_2 such that

$$(f_6) \qquad \begin{gathered} |f(x,t_1) - f(x,t_2)| \le L_2|t_1 - t_2| \\ \text{for all } t_1, t_2 \in [0,R_1] \text{ and for all } x \in \overline{\Omega}. \end{gathered}$$

We say that $u \in H_0^1(\Omega)$ is a weak solution of the problem (P_1) if

$$\int_\Omega \nabla u \nabla \phi\,dx = \int_\Omega \frac{f(x,u)\phi}{a\left(x, \int_\Omega u\,dx\right)}\,dx,$$

for all $\phi \in H_0^1(\Omega)$.

Problem (P_1) is a generalization of the equation

$$(P_2) \qquad \begin{cases} -a\left(\int_\Omega u\,dx\right)\ \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$f \in H^{-1}(\Omega)$, which is the steady-state counterpart of the parabolic problem

$$(P_2') \qquad \begin{cases} u_t - a\left(\int_\Omega u\,dx\right)\ \Delta u = f & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x). \end{cases}$$

Such an equation arises in various situations. For instance, u could describe the density of a population (bacteria, for instance) subject to spreading. The diffusion coefficient a is then supposed to depend on the entire population in the domain Ω, rather than on the local density, that is, moves are guided by considering the global state of the medium.

Furthermore, with respect to the stationary problem (P_2), it has the special feature of not being variational. It has been studied by several authors such as [**2**–**5**] by using techniques such as the fixed point theory, sub and supersolution, quasi-variational inequalities, Galerkin method and so on.

In problem (P_1), besides the lack of variational structure, the function a also depends on the variable $x \in \Omega$ situation that, at least to our knowledge, has not been studied in the existing literature.

However, inspired by ideas developed in [**6**], we use the mountain pass theorem to find a solution of (P_1).

We point out that the techniques we will use are valid, mutatis mutandis, for equations of the Kirchhoff-type like

$$(P_3) \qquad \begin{cases} -M\left(x, \int_\Omega |\nabla u|^2\,dx\right)\ \Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{for all } x \in \Omega, \end{cases}$$

where Ω is as before and $M : \overline{\Omega} \times \mathbf{R}^+ \to \mathbf{R}$ is a given function.

In this work, we denote by S_r is the best constant of the embedding of $H_0^1(\Omega)$ into $L^r(\Omega)$, that is, $S_r = \inf_{u \not\equiv 0} \|u\|/|u|_r$, where $\|u\| =$

$(\int_\Omega |\nabla u|^2\,dx)^{1/2}$ and $|u|_r = (\int_\Omega |u|^r\,dx)^{1/r}$ are, respectively, the usual norms in $H_0^1(\Omega)$ and $L^r(\Omega)$.

Note that if there exists a constant K_2 such that $|s| \le K_2$, then, from (1.1), there exists a constant C_1, depending on K_2, such that $\int_\Omega |f(x,s)|^{2dx \le C_1}$ for all $x \in \overline{\Omega}$.

Our main result is as follows:

Theorem 1.1. *Assume conditions* (a_1)–(a_2) *and* (f_1)–(f_6) *hold. If*

$$\frac{S_2 L_2 C_1^{1/2}}{S_1(a_0 S_2^2 - L_1 a_\infty)} < 1,$$

then problem (P_1) *has a positive solution.*

2. The variational framework. As in [**6**], the technique used in this paper consists of associating with problem (P_1) a family of local semilinear elliptic problems. Namely, for each $w \in H_0^1(\Omega)$ we consider the problem

$$(P_w) \qquad \begin{cases} -\Delta u = (f(x,u))/(a\left(x, \int_\Omega w\,dx\right)) & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega \text{ and } u > 0 \text{ in } \Omega. \end{cases}$$

Now, problem (P_w) is variational and we can treat it by Variational Methods.

As usual, a weak solution of a problem as in (P_w) is obtained as a critical point of the associated functional

$$I_w(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \frac{F(x,u)}{a\left(x, \int_\Omega w\,dx\right)}\,dx.$$

The proof of Theorem 1.1 is broken in several lemmas. We prove that the functional I_w has the geometry of the mountain pass theorem, that it satisfies the Palais-Smale condition and finally that the obtained solutions have the uniform bounds stated in the theorem.

Lemma 2.1. *Let* $w \in H_0^1(\Omega)$. *Then there exist positive numbers* ρ *and* α, *which are independent of* w, *such that*

$$I_w(u) \ge \alpha > 0, \quad \text{for all } u \in H_0^1(\Omega) : \|u\| = \rho.$$

Proof. From (a_1), (1.1) and using Sobolev embedding theorem, we conclude

$$I_w(u) \geq \left(\frac{1}{2} - \frac{\epsilon}{2a_0 S_2^2}\right)\|u\|^2 - \frac{C_\epsilon}{a_0 S_q^q}\|u\|^q.$$

Since $2 < q$, the result follows. □

Lemma 2.2. *Let $w \in H_0^1(\Omega)$. Fix $v_0 \in H_0^1(\Omega)$, with $v_0 > 0$ and $\|v_0\| = 1$. Then there is a $T > 0$, independent of w, such that*

$$I_w(tv_0) \leq 0, \quad \textit{for all } t \geq T.$$

Proof. It follows from (f_4) that there exist constants C_3 and C_4 such that

$$I_w(tv_0) \leq \frac{t^2}{2} - \frac{C_3 t^\theta}{a_\infty S_\theta^\theta} - \frac{C_4}{a_\infty}|\Omega|.$$

Since $\theta > 2$, we obtain T independent of v_0 and also of w, such that the result holds. □

Lemma 2.3. *Assume (f_1)–(f_4). Then problem (P_w) has at least one positive solution u_w for any $w \in H_0^1(\Omega)$.*

Proof. Lemmas 2.1 and 2.2 show that the functional I_w has the mountain pass geometry. From (1.1) we conclude that I_w satisfies the (PS) condition. So, by the mountain pass theorem, a weak solution u_w of (P_w) is obtained as a critical point of I_w at an inf max level. Namely,

$$I_w'(u_w) = 0$$

and

$$I_w(u_w) = c_w = \inf_{\gamma \in \Gamma_w} \max_{t \in [0,1]} I_w(\gamma(t)), \tag{2.1}$$

where $\Gamma_w = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = Tv_0\}$, for some v_0 and T as in Lemma 2.2. From now on we fix such a v_0 and such a T. Multiplying both sides of the equation in (P_w) by u_w^-, using (f_1) and integrating by parts, we conclude that $u_w^- \equiv 0$. So u_w is positive. □

Lemma 2.4. *Let $w \in H_0^1(\Omega)$. There exists a positive constant K_1 independent of w, such that $\|u_w\| \geq K_1$, for all solutions u_w obtained in Lemma* 2.3.

Proof. Using u_w as a test function in (P_w), we obtain

$$\|u_w\|^2 = \int_\Omega \frac{f(x,u_w)u_w}{a\left(x, \int_\Omega w\,dx\right)}\,dx.$$

From (a_1), (1.1) and using Sobolev embedding theorem, we conclude

$$\left(1 - \frac{\epsilon}{S_2^2 a_0}\right)\|u_w\|^2 \leq \frac{C_\epsilon}{S_q^q a_0}\|u_w\|^q.$$

So, the result follows. □

Lemma 2.5. *Let $w \in H_0^1(\Omega)$. There exists a positive constant K_2 independent of w, such that $\|u_w\| \leq K_2$, for all solutions u_w obtained in Lemma* 2.3.

Proof. Using (f_5), we obtain the inf max characterization of u_w in Lemma 2.3. So,

$$c_w \leq \max_{t\geq 0} I_w(tv_0)$$

with v_0 chosen in Lemma 2.3. We estimate c_w using (f_4):

$$c_w \leq \max_{t\geq 0} I_w(tv_0) \leq \max_{t\geq 0}\left\{\frac{t^2}{2} - \frac{C_3 t^\theta}{S_\theta^\theta} - C_4|\Omega|\right\} = \widetilde{K}.$$

Also from (f_4), we obtain

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_w\| \leq I_w(u_w) - \frac{1}{\theta}I_w'(u_w)u_w = c_w \leq \widetilde{K}.$$

The result follows by considering $K_2 = [\widetilde{K}((1/2) - (1/\theta))^{-1}]^{1/2}$. □

Remark 2.6 (On the regularity of the solution of (P_2)). In Lemma 2.3 we have obtained a weak solution u_w of (P_w) for each given $w \in H_0^1(\Omega)$.

Since $q < 2^*$, a standard bootstrap argument, using the L^p-regularity theory, shows that u_w is, in fact, in $C^{1,\beta}(\overline{\Omega})$. As a consequence of the Sobolev embedding theorems and Lemma 2.5, we conclude with the following:

Lemma 2.7. *Let $w \in H_0^1(\Omega) \cap C^{1,\beta}(\overline{\Omega})$. Then there exists a positive constant R_1, independent of w, such that the solution u_w obtained in Lemma 2.3 satisfies $\|u_w\|_{C^{0,\beta}} \le R_1$.*

3. Proof of Theorem 1.1. We construct a sequence (u_n), $n \in \mathbf{N}$, of solutions as

$$(P_n) \qquad \begin{cases} -\Delta u_n = (f(x,u_n))/(a(x, \int_\Omega u_{n-1}\,dx)) & \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega \text{ and } u_n > 0 \text{ for all } x \in \Omega. \end{cases}$$

obtained by the mountain pass theorem, starting with an arbitrary $u_0 \in H_0^1(\Omega) \cap C^{1,\beta}(\overline{\Omega})$. By Remark 2.6, we see that $\|u_n\|_{C^{0,\beta}(\Omega)} \le R_1$. On the other hand, using (P_n) and (P_{n+1}) we obtain

$$\int_\Omega \nabla u_{n+1}(\nabla u_{n+1} - \nabla u_n)\,dx = \int_\Omega \frac{f(x,u_{n+1})}{a\left(x, \int_\Omega u_n\,dx\right)}(u_{n+1} - u_n)\,dx$$

and

$$\int_\Omega \nabla u_n(\nabla u_{n+1} - \nabla u_n)\,dx = \int_\Omega \frac{f(x,u_n)}{a\left(x, \int_\Omega u_{n-1}\,dx\right)}(u_{n+1} - u_n)\,dx.$$

Note that from (1.1) and Lemma 2.5, we have that $(\int_\Omega |f(x,u_n)|^2\,dx)^{1/2} \le C_1^{1/2}$. Thus,

$$\begin{aligned} &\|u_{n+1} - u_n\|^2 \\ &\le \frac{1}{a_0^2}\left[\int_\Omega (f(x,u_{n+1}) - f(x,u_n))a\left(x, \int u_{n-1}\,dx\right)|u_{n+1} - u_n|\,dx\right] \\ &\quad + \frac{1}{a_0^2}\left[\int_\Omega f(x,u_n)\left(a\left(x, \int u_{n-1}\,dx\right)\right.\right. \\ &\qquad\qquad\qquad \left.\left. - a\left(x, \int u_n\,dx\right)\right)|u_{n+1} - u_n|\,dx\right]. \end{aligned}$$

Using (a_2) and (f_6),

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &\\ &\leq \frac{L_1 a_\infty}{a_0^2}\int_\Omega |u_{n+1}-u_n|^2\,dx\\ &\quad+\frac{L_2}{a_0^2}\int_\Omega |f(x,u_n)|\left[\int_\Omega |u_n-u_{n-1}|\right]|u_{n+1}-u_n|\,dx.\end{aligned}$$

Using the Sobolev embedding theorem

$$\begin{aligned}\|u_{n+1}-u_n\|^2 \leq \frac{L_1 a_\infty}{a_0^2 S_2^2}\|u_{n+1}-u_n\|^2+\frac{L_2}{a_0^2}\int_\Omega |u_n-u_{n-1}|\,dx&\\ \times\int_\Omega |f(x,u_n)||u_{n+1}-u_n|\,dx.&\end{aligned}$$

From the Hölder inequality

$$\left(\frac{a_0^2 S_2^2 - L_1 a_\infty}{a_0^2 S_2^2}\right)\|u_{n+1}-u_n\|^2 \leq \frac{L_2 C_1^{1/2}}{a_0^2 S_2 S_1}\|u_{n+1}-u_n\|\|u_n-u_{n-1}\|.$$

So,

$$\left(\frac{a_0^2 S_2^2 - L_1 a_\infty}{S_2}\right)\|u_{n+1}-u_n\| \leq \frac{L_2 C_1^{1/2}}{S_1}\|u_n-u_{n-1}\|.$$

Hence,

$$\|u_{n+1}-u_n\| \leq \frac{L_2 C_1^{1/2} S_2}{S_1\left(a_0^2 S_2^2 - L_1 a_\infty\right)}\|u_n-u_{n-1}\| =: k\|u_n-u_{n-1}\|.$$

Since the coefficient k is less than 1, it follows, by a straightforward argument, that the sequence (u_n) converges strongly in $H_0^1(\Omega)$ to some function $u \in H_0^1(\Omega)$. Since $K_1 \leq \|u_n\|$ for all n, we have $u > 0$ in Ω.

From (P_n), we obtain

$$\int_\Omega \nabla u_n \nabla \phi\,dx = \int_\Omega \frac{f(x,u_n)\phi}{a\left(x,\int_\Omega u_{n-1}\,dx\right)}\,dx.$$

Since $u_n \to u$ in $H_0^1(\Omega)$, we conclude that

$$\int_\Omega \nabla u \nabla \phi \, dx = \int_\Omega \frac{f(x,u)\phi}{a\left(x, \int_\Omega u dx\right)} \, dx, \text{ for all } \phi \in H_0^1(\Omega),$$

and the proof of the theorem is over. □

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