JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 22, Number 1, Spring 2010

STOCHASTIC VOLTERRA EQUATIONS IN WEIGHTED SPACES

JOHN A.D. APPLEBY AND MARKUS RIEDLE

Communicated by T. Herdman

ABSTRACT. In the following paper, we provide a stochastic analogue to work of Shea and Wainger by showing that when the measure and state-independent diffusion coefficient of a linear Itô-Volterra equation are in appropriate L^p-weighted spaces, the solution lies in a weighted L^p-space in both an almost sure and moment sense.

1. Introduction. This paper examines the asymptotic stability and decay rates, in various modes of stochastic convergence, of solutions of stochastically perturbed Volterra equations to the equilibrium solution of a related unperturbed deterministic Volterra equation. For deterministic equations the phenomenon of asymptotic stability has been shown to be distinct from that of exponential stability. These phenomena were shown to coincide in linear Volterra integrodifferential equations by Murakami [**30, 31**] if and only if the kernel lies in an exponentially weighted L^1-space. On the other hand, non-exponential rates of decay in spaces of integrable functions with general weights have been considered by Gelfand et al. [**12**], Shea and Wainger [**32**] and Jordan and Wheeler [**19**], with extensions of the last paper presented in Jordan, Staffans and Wheeler [**18**]. An account of this research is summarized in Gripenberg et al. [**13**].

Asymptotic stability results for stochastic functional and evolution equations without concern for the rate of decay appear in, e.g., Hauss-

---

2010 AMS *Mathematics subject classification.* Primary 60H10, 60H20, 34K20, Secondary 45D05.
Keywords and phrases. Volterra integral equations, Volterra integro-differential equations, stochastic functional differential equations, Itô-Volterra equations, stochastic Volterra equations, almost sure asymptotic stability, exponential asymptotic stability, weight function, submultiplicative weight function, weighted spaces, L^p-spaces.
The first author was partially supported by an Albert College Fellowship, awarded by the Dublin City University Research Advisory Panel.
Received by the editors on April 1, 2007, and in revised form on February 18, 2008.
DOI:10.1216/JIEA-2010-22-1-1

man [**15**], Ichikawa [**16**], and Zabczyk [**33**]. However, more recently a substantial proportion of researchers' efforts have centered on determining exponential stability of solutions of stochastic functional differential equations. To this end, it is principally equations with bounded delay and mild nonlinearity which have been studied cf., e.g., Mao [**23, 25**], Mohammed and Scheutzow [**29**] and Kolmanovskii and Myshkis [**22**]. In these works, the exponential decay exhibited by deterministic equations with bounded delay is recovered. In the cases when stochastic Volterra equations have been studied, exponentially fading memory has often been postulated (cf., e.g., Mao [**24**] and Appleby and Freeman [**3**]) in which case the rate of decay is exponential, just as in the deterministic case.

Fewer results for non-exponential decay in autonomous linear stochastic Volterra equations exist. Kadiev and Ponosov have determined conditions for the admissibility of solutions in certain weighted function spaces in [**20, 21**], so that general rates of decay can be computed. Other results give the so-called subexponential almost sure rate of convergence to an equilibrium e.g., Appleby [**1, 2**]. While this approach gives exact almost sure rates of convergence, and supersedes the L^p-weighted space results in determining exact L^∞-convergence rates, it suffers from some limitations. Firstly, in this approach conditions on the asymptotically stable deterministic resolvent are in general more restrictive than those needed to establish stochastic stability outright; for instance, only the L^1-stability of the underlying deterministic Volterra equation suffices for asymptotic stability of equations with state-independent diffusion coefficients, as shown in Appleby and Riedle [**9**]. Moreover, the class of weight functions employed in the subexponential approach appears somewhat more restrictive than the class of submultiplicative weight functions employed in Gelfand et al., Shea and Wainger, and Jordan and Wheeler. It is therefore highly desirable to obtain complementary and general results which estimate the rate of decay in weighted L^p-spaces, and which require only standard deterministic stability hypotheses. Furthermore, results on measures unify existing results on Volterra equations with point delays and continuous kernels. A first paper in this direction is Mao and Riedle [**26**].

The contribution of this work is to extend the results of Gelfand et al., Shea and Wainger, and Jordan and Wheeler to perturbed Volterra equations of the form

$$dX(t) = \left(\int_{[0,t]} \mu(ds)X(t-s) \right) dt + \Sigma(t)\, dW(t),$$

where μ is a finite measure lying in a weighted measure space, and Σ is a deterministic function lying in a weighted L^2-space. The underlying deterministic equation is presumed to be stable. Results are established which estimate the almost sure L^p convergence rates of X and which connect these convergence rates to weight functions describing the spaces in which Σ and μ lie. The convergence of the pth mean of the solution X in weighted spaces is also considered.

2. Supporting results. Let d, d' be some positive integers, and let $\mathbf{R}^{d\times d'}$ denote the space of all $d \times d'$ matrices with real entries. The identity matrix on $\mathbf{R}^{d\times d}$ is denoted by Id_d. We equip $\mathbf{R}^{d\times d'}$ with a norm $|\cdot|$ and write $\mathbf{R}^d$ if $d' = 1$ and $\mathbf{R}$ if $d = d' = 1$. We denote by $\mathbf{R}_+$ the half-line $[0, \infty)$. The complex plane is denoted by $\mathbf{C}$ and $\mathbf{C}_0 := \{z \in \mathbf{C} : \mathrm{Re}\, z \geq 0\}$.

Let $M(\mathbf{R}_+, \mathbf{R}^{d\times d'})$ be the space of finite Borel measures on $\mathbf{R}_+$ with values in $\mathbf{R}^{d\times d'}$. The total variation of a measure ν in $M(\mathbf{R}_+, \mathbf{R}^{d\times d'})$ on a Borel set $B \subseteq \mathbf{R}_+$ is defined by

$$|\nu|(B) := \sup \sum_{i=1}^{N} |\nu(E_i)|,$$

where $(E_i)_{i=1}^N$ is a partition of B and the supremum is taken over all partitions. The total variation defines a positive scalar measure $|\nu|$ in $M(\mathbf{R}_+, \mathbf{R})$. If one specifies temporarily the norm $|\cdot|$ as the l^1-norm on the space of real-valued sequences and identifies $\mathbf{R}^{d\times d'}$ by $\mathbf{R}^{dd'}$ one can easily establish for the measure $\nu = (\nu_{i,j})_{i,j=1}^d$ the inequality

$$|\nu|(B) \leq C \sum_{i=1}^{d} \sum_{j=1}^{d} |\nu_{i,j}|(B) \quad \text{for every Borel set } B \subseteq \mathbf{R}_+ \tag{2.1}$$

with $C = 1$. Then, by the equivalence of every norm on finite-dimensional spaces, the inequality (2.1) holds true for the arbitrary norms $|\cdot|$ and some constant $C > 0$. Moreover, as in the scalar case we have the fundamental estimate

$$\left| \int_{\mathbf{R}_+} \nu(ds)\, f(s) \right| \le \int_{\mathbf{R}_+} |f(s)|\, |\nu|(du)$$

for every function $f : \mathbf{R}_+ \to \mathbf{R}^{d' \times d''}$ which is $|\nu|$-integrable. The convolution of a function f and a measure ν is defined by

$$\nu * f : \mathbf{R}_+ \to \mathbf{R}^{d \times d''}, \qquad (\nu * f)(t) := \int_{[0,t]} \nu(ds)\, f(t-s).$$

The convolution of two functions is defined analogously.

We first turn our attention to the deterministic Volterra equation in $\mathbf{R}^d$:

$$x'(t) = \int_{[0,t]} \mu(ds)\, x(t-s) \quad \text{for } t \ge 0, \quad x(0) = x_0. \tag{2.2}$$

For any $x_0 \in \mathbf{R}^d$ there is a unique $\mathbf{R}^d$-valued function x which satisfies (2.2) on $[0,\infty)$. The function $x \equiv 0$ is a solution of (2.2) and is called the zero solution of (2.2). The definition of various standard notions of stability of the zero solution required for our analysis are detailed in Miller [**27**], to which the reader may refer.

The so-called *fundamental solution or resolvent of* (2.2) is the matrix-valued function $r : \mathbf{R}_+ \to \mathbf{R}^{d \times d}$, which is the unique solution of (2.2) with the initial condition $r(0) = \mathrm{Id}_d$, where Id_d denotes the identity matrix in $\mathbf{R}^{d \times d}$. Even such simple Volterra equations of the form (2.2) enjoy the property that not every solution decays exponentially if the zero solution is asymptotically stable. See Murakami [**30, 31**] and Appleby and Reynolds [**7**]. On the contrary, linear differential equations with bounded delay are asymptotically stable if and only if every solution decays exponentially, see Hale and Lunel [**14**]. Consequently, it is quite an open-ended task to determine the rate of convergence of the solutions of (2.2). Such results are covered by the work Shea and Wainger [**32**], Gelfand et al. [**12**], and Jordan and Wheeler [**19**], and are based on *weighted function spaces*, which we introduce in the sequel. Other results on *pointwise* and *subexponential* rates of decay of solutions of Volterra equations may be found in Appleby and Reynolds [**7**] and Appleby, Reynolds, and Győri [**5, 6**].

A function $\psi : \mathbf{R}_+ \to \mathbf{R}$ with $\psi(0) = 1$ is called a *weight function* if ψ is positive, Borel measurable, locally bounded and locally bounded away from zero. A weight function is called *submultiplicative*, if

$$\psi(s+t) \le \psi(s)\psi(t) \quad \text{for all } s, t \ge 0.$$

For submultiplicative weight functions $\psi : \mathbf{R}_+ \to \mathbf{R}$, the limit

$$\beta_\psi := -\lim_{t\to\infty} \frac{\ln \psi(t)}{t}$$

exists and is finite, see Gripenberg et al. [**13**, Lemma 4.4.1]. We present some examples of submultiplicative weight functions which occur frequently:

$$\begin{aligned}
&\psi(t) = e^{\alpha t} \quad \text{for } \alpha \in \mathbf{R} \text{ with } \beta_\psi = -\alpha,\\
&\psi(t) = (1+t)^\gamma \quad \text{for } \gamma \ge 0 \text{ with } \beta_\psi = 0\\
&\psi(t) = (1+\ln(1+t))^\gamma \quad \text{for } \gamma \ge 0 \text{ with } \beta_\psi = 0.
\end{aligned}$$

Let ψ be a weight function. We define the weighted space of integrable functions by

$$L^p(\mathbf{R}_+, \mathbf{R}^{d\times d'}; \psi) := \left\{ f : \mathbf{R}_+ \to \mathbf{R}^{d\times d'} : \int_0^\infty \psi(t)|f(t)|\,dt < \infty \right\}.$$

For the ordinary space of integrable functions with $\psi \equiv 1$ we use the notation $L^p(\mathbf{R}_+, \mathbf{R}^d)$. Similarly, we define the weighted space of measures

$$M(\mathbf{R}_p, \mathbf{R}^{d\times d'}; \psi) := \left\{ \nu \in M(\mathbf{R}_+, \mathbf{R}^{d\times d'}) : \int_{\mathbf{R}_+} \varphi(t)|\nu|(dt) < \infty \right\}.$$

Moreover, we use the Landau symbols for functions $f : \mathbf{R}_+ \to \mathbf{R}^{d\times d'}$ and $g : \mathbf{R}_+ \to \mathbf{R}$:

$$\begin{aligned}
f = o(g^{-1}) &\iff \lim_{t\to\infty} |g(t)||f(t)| = 0,\\
f = \mathcal{O}(g^{-1}) &\iff \limsup_{t\to\infty} |g(t)||f(t)| < \infty.
\end{aligned}$$

We denote by $\det A$ the determinant of a matrix $A \in \mathbf{C}^{d\times d}$.

The following result was first established in Shea and Wainger [**32**].

Theorem 2.1. *Let φ_1 be a submultiplicative weight function and μ a measure in $M(\mathbf{R}_+, \mathbf{R}^{d\times d}; \varphi_1)$. Then the resolvent r of (2.2) satisfies*

$$r \in L^1(\mathbf{R}_+, \mathbf{R}^{d\times d}; \varphi_1)$$

if and only if

$$\det\left[z\mathrm{Id}_d - \int_{\mathbf{R}_+} e^{-zu}\,\mu(du)\right] \neq 0 \quad \text{for every } z \in C \text{ with } \operatorname{Re} z \geq \beta_{\varphi_1}. \tag{2.3}$$

Moreover, in this case the resolvent r obeys $r = o(\varphi_1^{-1})$.

Since the function φ in Theorem 2.1 is not assumed to tend to infinity, one obtains a rate of convergence for the solutions of (2.2) even in the non-stable case. This remark will remain true in the sequel when we consider stochastic differential equations.

In general, the verification of (2.3) cannot be accomplished explicitly. But there are several analytical or numerical techniques to deal with this equation, see for instance Bellman and Cooke [**10**] or Hale and Lunel [**14**]. Moreover, in some specific examples equivalent conditions to (2.3) are known.

Example 2.2. Let μ be of the form $\mu(dt) = -a\delta_0(dt)+K(t)\,dt$ where δ_0 denotes the Dirac measure at 0 and K is a continuous and positive function in $L^1(\mathbf{R}_+, \mathbf{R})$. Then the resolvent of (2.2) is in $L^1(\mathbf{R}_+, \mathbf{R})$ if and only if

$$a > \int_0^\infty K(s)\,ds. \tag{2.4}$$

Consequently, if $K \in L^1(\mathbf{R}_+, \mathbf{R}; \varphi_1)$ for the weight function $\varphi_1(t) = (1+t)^\alpha$ for some $\alpha > 0$, then (2.4) implies $r \in L^1(\mathbf{R}_+, \mathbf{R}; \varphi_1)$ and $r = o(\varphi_1^{-1})$.

If $K \in L^1(\mathbf{R}_+, \mathbf{R}; \psi)$ for the weight function $\psi(t) = \exp(\beta t)$ for some $\beta > 0$ the function $z \mapsto z - \int_0^\infty e^{-zu}K(u)\,du$ is analytic for

$z \in \mathbf{C}$ with $\operatorname{Re} z > -\beta$. Therefore its zeros are isolated, and so it has only finitely many zeros in a compact region. Therefore, if (2.4) is satisfied, there exists a $\gamma \in (0, \beta)$ such that equation (2.3) has no roots in $\{z \in \mathbf{C} : \operatorname{Re} z \geq -\gamma\}$. For the weight function $\varphi_2(t) = \exp(\gamma t)$ one obtains $r \in L^1(\mathbf{R}_+, \mathbf{R}; \varphi_2)$ and $r = o(\varphi_2^{-1})$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$, and let $(W(t))_{t\geq 0}$ be a d'-dimensional Brownian motion on this probability space. We will consider the stochastic integro-differential equation with stochastic perturbations of the form

$$\begin{aligned} dX(t) &= \left(\int_{[0,t]} \mu(ds)\, X(t-s) \right) dt \\ &\qquad + \Sigma(t)\, dW(t) \quad \text{for } t \geq 0, \\ X(0) &= X_0, \end{aligned} \tag{2.5}$$

where μ is a measure in $M(\mathbf{R}_+, \mathbf{R}^{d\times d})$ and Σ is a continuous function from $\mathbf{R}_+$ to $\mathbf{R}^{d\times d'}$. The initial condition X_0 is an $\mathbf{R}^d$-valued, $\mathcal{F}_0$-measurable random variable with $\mathbf{E}|X_0|^2 < \infty$. The existence and uniqueness of a continuous solution X of (2.5) with $X(0) = X_0$ P-almost surely is covered in Berger and Mizel [**11**], for instance. Independently, the existence and uniqueness of solutions of stochastic functional equations was established in Itô and Nisio [**17**] and Mohammed [**28**].

3. Stochastic Shea-Wainger theorems.

In this section, we present some general results for the existence of solutions of (2.5) in weighted spaces, both in an almost sure and pth mean sense.

We start with a result which enables us in the rest of the paper to estimate the rate of decay of convolutions of functions and measures.

Lemma 3.1. *Let φ_1 and φ_2 be weight functions, and define φ by*

$$\varphi(t) := \min_{0\leq s\leq t} \varphi_1(t-s)\varphi_2(s).$$

Let $\nu \in M(\mathbf{R}_p, \mathbf{R}^{d\times d}; \varphi_1)$ and $f : \mathbf{R}_+ \to \mathbf{R}^{d\times d}$ be a measurable function.

(a) *If* $f = \mathcal{O}(\varphi_2^{-1})$, *then* $\nu * f = \mathcal{O}(\varphi^{-1})$.
(b) *If* $f = o(\varphi_2^{-1})$, *then* $\nu * f = o(\varphi^{-1})$.
(c) *If* $f \in L^1(\mathbf{R}_+, \mathbf{R}; \varphi_2)$, *then* $\nu * f \in L^1(\mathbf{R}_+, \mathbf{R}; \varphi)$.

Proof. By assumption there is a constant $c > 0$ such that $\varphi_2(t)|f(t)| \le c$ for every $t \ge 0$. Then, we obtain

$$\begin{aligned} \varphi(t)|(\nu * f)(t)| &\le \int_{[0,t]} \varphi(t)|f(t-s)|\,|\nu|(ds) \\ &\le c \int_{[0,t]} \frac{\varphi(t)}{\varphi_2(t-s)\varphi_1(s)} \varphi_1(s)\,|\nu|(ds) \\ &= \mathcal{O}(1), \end{aligned}$$

which proves the first assertion. The other assertions can be similarly established. □

We define for all $t \ge 0$ the random variable

$$Y(t) := \int_0^t \Sigma(s)\,dW(s). \tag{3.1}$$

If Σ is in $L^2(\mathbf{R}_+, \mathbf{R}^{d\times d'})$, then the process $(Y(t))_{t\ge 0}$ is a uniformly square-integrable martingale and converges therefore P-almost surely and in $L^2(\Omega, \mathbf{R}^d)$ to a random variable Y^*. We define for all $t \ge 0$ the random variable

$$\int_t^\infty \Sigma(s)\,dW(s) := Y^* - Y(t).$$

Note that $Y^* - Y(t)$ is not adapted but nevertheless a well-defined random variable on $(\Omega, \mathcal{F}, P)$. We call a function $\varphi : \mathbf{R}_+ \to \mathbf{R}$ eventually non-decreasing, if there is a t_0 such that φ is non-decreasing on $[t_0, \infty)$. In what follows we define the function $\log\log$ by $(\log\log)(x) = 1$ for $0 < x < e^e$ and $(\log\log)(x) = \log\log x$ for $x \ge e^e$.

Lemma 3.2. *Let* φ *be an eventually non-decreasing weight function which tends to infinity. If* Σ *is in* $L^2(\mathbf{R}_+, \mathbf{R}^{d\times d'}; \varphi^2)$, *then there exists a non-random constant* $c > 0$, *such that*

$$\limsup_{t\to\infty} \frac{\varphi(t)}{\sqrt{\log\log\varphi(t)}} \left| \int_t^\infty \Sigma(s)\,dW(s) \right| \le c \quad \text{P-}a.s.$$

Proof. If $\|\cdot\|_1$ is the l^1 norm in $\mathbf{R}^d$ and $\Sigma(t) = (\sigma(t))_{i,j}$ and $W = (W_1, \dots, W_{d'})^T$, then we have

$$\left\| \int_t^\infty \Sigma(s)\, dW(s) \right\|_1 \le \sum_{i=1}^{d} \sum_{j=1}^{d'} \left\| \int_t^\infty \sigma_{i,j}(s)\, dW_j(s) \right\|_{l^1}.$$

Therefore, we can assume $d = d' = 1$. An argument in Appleby, Gleeson and Rodkina [**2**] guarantees

$$\limsup_{t\to\infty} \frac{|\int_t^\infty \Sigma(s)\, dW(s)|}{(2\int_t^\infty \Sigma(s)^2\, ds \log\log(\int_t^\infty \Sigma(s)^2\, ds)^{-1})^{1/2}} = 1 \text{ P-a.s.}$$

By the monotonicity of φ we obtain for sufficiently large t that

$$\begin{aligned} \varphi^2(t) \int_t^\infty \Sigma(s)^2\, ds &\le \int_t^\infty \varphi^2(s)\Sigma(s)^2\, ds \\ &\le \int_0^\infty \varphi^2(s)\Sigma(s)^2\, ds \\ &=: c_0 < \infty. \end{aligned}$$

Since the function $x \mapsto x \log\log x^{-1}$ is increasing for sufficiently small x and $\varphi^{-2}(t)$ tends to 0 for $t \to \infty$ we obtain for t sufficiently large

$$\begin{aligned} \int_t^\infty \Sigma(s)^2\, ds \log\log \left(\int_t^\infty \Sigma(s)^2\, ds \right)^{-1} &\le c_0\varphi^{-2}(t) \log\log \frac{\varphi^2(t)}{c_0} \\ &\le c\, \varphi^{-2}(t) \log\log \varphi^2(t) \end{aligned}$$

for a constant $c > 0$. □

Theorem 3.3. *Let φ_1 be a submultiplicative weight function and φ_2 an eventually non-decreasing weight function which tends to infinity. If μ is in $M(\mathbf{R}_+, \mathbf{R}^{d\times d}; \varphi_1)$ and Σ is in $L^2(\mathbf{R}_+, \mathbf{R}^{d\times d'}; \varphi_2^2)$, then*

$$\det \left[z\mathrm{Id}_d - \int_{\mathbf{R}_+} e^{-zu}\, \mu(du) \right] \ne 0$$

for every $z \in \mathbf{C}$ with $\operatorname{Re} z \ge \beta_{\varphi_1}$

implies for every solution X *of* (2.5):

$$\limsup_{t\to\infty} \varphi(t)|X(t)| < \infty \quad P\text{-a.s.},$$

where $\varphi(t) := \varphi_1(t)\min_{0\le s\le t}(\varphi_2(s))/\varphi_1(s)\sqrt{\log\log\varphi_2(s)}$.

Proof. Let $Y(t)$ denote the random variables defined in (3.1). The stochastic random variables $Z(t) := X(t) - Y(t) + Y^*$, $t \ge 0$, are differentiable and obey

$$\begin{aligned} Z'(t) &= \int_{[0,t]} \mu(ds)\, X(t-s) \\ &= \int_{[0,t]} \mu(ds)\, Z(t-s) \\ &\quad + \int_{[0,t]} \mu(ds)\, (Y(t-s) - Y^*). \end{aligned}$$

Hence, the variation of constants formula implies for $t \ge 0$

$$Z(t) = r(t)Z(0) + \int_0^t r(t-s)F(s)\, ds, \quad P\text{-a.s.} \tag{3.2}$$

where r is the fundamental solution and $F = \mu * (Y(\cdot) - Y^*)$. Let $\psi(t) := (\log\log\varphi_2(t))^{-1/2}\varphi_2(t)$. Then, by Lemma 3.2 we have

$$\limsup_{t\to\infty}\ \psi(t)|Y(t) - Y^*| \le C_1\ P\text{-a.s.} \tag{3.3}$$

Since $\mu \in M(\mathbf{R}_+, \mathbf{R}^{d\times d}; \varphi_1)$, part (a) of Lemma 3.1 results in

$$F = \mathcal{O}(\varphi_3^{-1})\ \text{P-a.s., where } \varphi_3(t) := \min_{0\le s\le t} \psi(t-s)\varphi_1(s). \tag{3.4}$$

Theorem 2.1 guarantees $r \in L^1(\mathbf{R}_+, \mathbf{R}^{d\times d}; \varphi_1)$ for the fundamental solution r of (2.2) which, in combination with (3.4) and by applying part (a) of Lemma 3.1, yields:

$$\begin{gathered} r * F = \mathcal{O}(\varphi_4^{-1})\ \text{P-a.s.,} \\ \text{where } \varphi_4(t) := \min_{0\le s\le t} \varphi_3(t-s)\varphi_1(s). \end{gathered}$$

Since also $r = o(\varphi_1^{-1})$ by Theorem 2.1 and obviously $\varphi_4 \le \varphi_1$ the variation of constants formula (3.2) implies $Z = \mathcal{O}(\varphi_4^{-1})$ P-almost surely. Since $\varphi_4 \le \psi$, and by using (3.3), we arrive at

$$\varphi_4(t)|X(t)| \le \varphi_4(t)|Z(t)|_{\mathbf{R}^d} + \varphi_4(t)|Y(t) - Y^*|_{\mathbf{R}^d} = \mathcal{O}(1) \text{ P-a.s.}$$

By means of the submultiplicative property of φ_1, we obtain

$$\begin{aligned}\varphi_4(t) &= \min_{0\le s\le t}\left(\min_{0\le u\le s}\left(\psi(s-u)\varphi_1(u)\right)\varphi_1(t-s)\right)\\ &\ge \min_{0\le s\le t}\left(\min_{0\le u\le s}\left(\psi(u)\frac{\varphi_1(s)}{\varphi_1(u)}\right)\frac{\varphi_1(t)}{\varphi_1(s)}\right)\\ &= \varphi_1(t)\min_{0\le s\le t}\frac{\varphi_2(s)}{\varphi_1(s)\sqrt{\log\log\varphi_2(s)}},\end{aligned}$$

which completes the proof. □

The effect of the stochastic perturbation on the resulting convergence rate in Theorem 3.3 is weakened by the logarithmic term. This arises from the excursion associated with the law of the iterated logarithm.

Remark 3.4. Assume the conditions and notations of Theorem 3.3. If

$$\lim_{t\to\infty}\frac{\psi(t)}{\varphi_1(t)} = \infty, \text{ where } \psi(t) := \varphi_2(t)(\log\log\varphi_2(s))^{-1/2}, \tag{3.5}$$

one can even obtain $X = \mathcal{O}(\varphi_1^{-1})$ P-almost surely. This is a consequence of Theorem 3.3, because (3.5) forces the choice $\varphi = \varphi_1$ in that result.

Condition (3.5) can be interpreted to mean that the stochastic perturbation decays more rapidly than the solution of the unperturbed equation, because (3.5) states that the weight function for $|\Sigma|^2$ tends to infinity more rapidly than the weight of μ. Then Remark 3.4 states that the perturbed system inherits the decay rate of the deterministic system. If, on the other hand, we assume the relation

$$\lim_{t\to\infty}\frac{\psi(t)}{\varphi_1(t)} = 0,$$

and in addition $t \mapsto \psi(t)\varphi_1^{-1}(t)$ is decreasing for $t \geq T$, it follows that

$$\varphi(t) = \varphi_1(t) \min_{0 \leq s \leq t} \frac{\psi(s)}{\varphi_1(s)} = \psi(t)$$

for t large enough. In this situation, the stochastic perturbation fades out more slowly than the solution of the deterministic system, so the solution of the stochastic equation is dominated by the stochastic perturbation.

Finally, if one assumes

$$\lim_{t \to \infty} \frac{\psi(t)}{\varphi_1(t)} = c$$

for a constant $c \in (0, \infty)$ then $\varphi = \mathcal{O}(\varphi_1) = \mathcal{O}(\psi)$, and the effects of the noise and the underlying deterministic system are in balance; and this is also reflected in the decay rate.

We proceed to consider pathwise stability properties of the solution.

Theorem 3.5. *Let φ_1 be a submultiplicative weight function, and let φ_2 an eventually non-decreasing weight function which tends to infinity. If μ is in $M(\mathbf{R}_+, \mathbf{R}^{d\times d}; \varphi_1)$ and Σ is in $L^2(\mathbf{R}_+, \mathbf{R}^{d\times d'}; \varphi_2^2)$, then*

$$\det\left[z\mathrm{Id}_d - \int_{\mathbf{R}_+} e^{-zu}\, \mu(du)\right] \neq 0, \text{ for every } z \in \mathcal{C} \text{ with } \operatorname{Re} z \geq \beta_{\varphi_1}$$

implies for every solution X of (2.5):

$$X \in L^1(\mathbf{R}_+, \mathbf{R}^d; \varphi) \ P\text{-a.s.},$$

where $\varphi(t) := \varphi_1(t) \min_{0\leq s\leq t} \varphi_2(s)\zeta(s)\varphi_1(s)^{-1}$, and ζ is such that $\zeta\sqrt{\log\log\varphi_2}$ is in $L^1(\mathbf{R}_+, \mathbf{R})$.

Proof. By means of Lemma 3.2 one establishes that $Y(\cdot) - Y^*$ is in $L^1(\mathbf{R}_+, \mathbf{R}^d; \varphi_2 \cdot \zeta)$. Lemma 3.1 part (c) implies that $F = \mu * (Y - Y^*)$ is in $L^1(\mathbf{R}_+, \mathbf{R}^d; \varphi_3)$ where

$$\varphi_3(t) := \min_{0 \leq s \leq t} \varphi_1(t-s)\varphi_2(s)\zeta(s).$$

One can proceed as in the proof of Theorem 3.3, but this time employing part (c) of Lemma 3.1. □

Now we turn our attention to the stability properties of the p-th mean of the solution.

Theorem 3.6. *Let φ_1 be a submultiplicative weight function, and let φ_2 be a weight function. If $\mu \in M(\mathbf{R}_+, \mathbf{R}^{d\times d}; \varphi_1)$ and Σ is in $L^2(\mathbf{R}_+, \mathbf{R}^{d\times d'}; \varphi_2^2)$, then*

$$\det\left[z\mathrm{Id}_d - \int_{\mathbf{R}_+} e^{-zu}\,\mu(du)\right] \neq 0 \text{ for every } z \in \mathcal{C} \text{ with } \operatorname{Re} z \geq \beta_{\varphi_1}$$

implies for every solution X of (2.5):

$$\mathbf{E}|X(t)|^2 = o(\varphi^{-1}) \text{ and } \int_0^\infty \varphi(s)\mathbf{E}|X(s)|^2\,ds < \infty,$$

where $\varphi(t) := \min_{0\leq s\leq t} \varphi_1^2(t-s)\varphi_2^2(s)$.

Proof. The solution X obeys the following variation of constants formula for $t \geq 0$:

$$X(t) = r(t)X_0 + \int_0^t r(t-s)\Sigma(s)\,dW(s) \text{ P-a.s.,} \tag{3.6}$$

where the integral can be understood as an Itô integral or as a Riemann-Stieltjes integral defined pathwise. Itô's isometry yields

$$\mathbf{E}|X(t)|^2 \leq c\left(|r(t)|^2\mathbf{E}|X_0|^2 + \int_0^t |r(t-s)|^2|\Sigma(s)|^2\,ds\right), \tag{3.7}$$

with a constant $c > 0$ depending on the norm. Since $|r|^2 = o(\varphi_1^{-2})$ and $\Sigma^2 \in L^1(\mathbf{R}_+, \mathbf{R}^{d\times d'}; \varphi_2^2)$, part (b) of Lemma 3.1 implies $|r|^2 * |\Sigma|^2 = o(\varphi^{-2})$ which establishes the first assertion of the theorem by (3.7) and the fact that $\varphi \leq \varphi_1^2$.

Since $\varphi_1(t)|r(t)| \to 0$ for $t \to \infty$ and $r \in L^1(\mathbf{R}_+, \mathbf{R}^{d\times d}; \varphi_1)$ we obtain $r^2 \in L^1(\mathbf{R}_+, \mathbf{R}^{d\times d}; \varphi_1^2)$. Therefore, Lemma 3.1 part (c) implies $|r|^2 * |\Sigma|^2 \in L^1(\mathbf{R}_+, \mathbf{R}; \varphi)$. Since $\varphi \leq \varphi_1^2$ the result follows by (3.7). □

We conclude the paper with a corollary on p-th mean stability for general p in weighted spaces.

Corollary 3.7. *Let φ_1 be a submultiplicative weight function and let φ_2 be a weight function. If $\mu \in M(\mathbf{R}_+, \mathbf{R}^{d\times d}; \varphi_1)$, Σ is in $L^2(\mathbf{R}_+, \mathbf{R}^{d\times d'}; \varphi_2^2)$ and $\mathbf{E}|X_0|^p < \infty$ for some $p > 0$, then*

$$\det\left[z\mathrm{Id}_d - \int_{\mathbf{R}_+} e^{-zu}\,\mu(du)\right] \neq 0$$

$$\text{for every } z \in \mathcal{C} \text{ with } \operatorname{Re} z \geq \beta_{\varphi_1}$$

implies for every solution X of (2.5):

$$\mathbf{E}|X(t)|^p = o(\varphi^{-1})$$

and, if $p \geq 2$, we have

$$\int_0^\infty \varphi(s)\mathbf{E}|X(s)|^p\,ds < \infty,$$

where $\varphi(t) := \min_{0\leq s\leq t} \varphi_1^p(t-s)\varphi_2^p(s)$.

Proof. For $0 < p < 2$, Hölder's inequality implies

$$\mathbf{E}|X(t)|^p \leq \left(\mathbf{E}|X(t)|^2\right)^{p/2}$$

and the first result follows by Theorem 3.6.

Now, assume $p \geq 2$. Using (3.6), we obtain

$$\mathbf{E}|X(t)|^p \leq 2^{p-1}\left(c_1|r(t)|^p\mathbf{E}|X_0|^p + \mathbf{E}\left|\int_0^t r(t-s)\Sigma(s)\,dW(s)\right|^p\right), \tag{3.8}$$

with a constant $c_1 > 0$ depending on the norm. Let us assume for a moment that $d = d' = 1$. Then the last expectation on the right hand side is the p-th moment of a normally distributed random variable with expectation 0 and variance $\int_0^t r^2(t-s)\Sigma(s)^2\,ds$. Hence we can estimate from above the p-th moment of this variable in terms of its variance and a constant c depending only on p and the norm; this gives

$$\begin{aligned} \mathbf{E}\left|\int_0^t r(t-s)\Sigma(s)\,dW(s)\right|^p &\leq c\left(\int_0^t |r(t-s)|^2|\Sigma(s)|^2\,ds\right)^{p/2} \\ &= c\left(\left(|r|^2 * |\Sigma|^2\right)(t)\right)^{p/2}. \end{aligned} \tag{3.9}$$

By norm equivalence, we also obtain the estimate (3.9) generalized to arbitrary dimensions. To conclude the proof of the first assertion, we note from Theorem 2.1 that $r^2 = o(\varphi_1^2)$. Now, by part (b) of Lemma 3.1, it follows that $|r|^2 * |\Sigma|^2 = o(\varphi^{2/p})$. Using (3.8) and (3.9) we arrive at the first assertion of the theorem.

To establish the second assertion, we define the function

$$f(t) := (\varphi(t))^{2/p} \left(|r|^2 * |\Sigma|^2\right)(t), \quad t \geq 0.$$

Since $f = o(1)$ and $f \in L^1(\mathbf{R}_+, \mathbf{R})$ we have that $f^{p/2}$ is in $L^1(\mathbf{R}_+, \mathbf{R})$ for $p \geq 2$. The same argument results in $|r|^p \in L^1(\mathbf{R}, \mathbf{R}^{d\times d}; \varphi_1^p)$ which, using (3.8), completes the proof. □

Acknowledgments. The authors are pleased to thank the referees for their careful scrutiny of the manuscript, and for informing them of some additional relevant literature concerning the asymptotic behavior of solutions of deterministic Volterra equations in weighted spaces. JA gratefully acknowledges Science Foundation Ireland for the support of this research under the Mathematics Initiative 2007 grant 07/MI/008 "Edgeworth Centre for Financial Mathematics."

REFERENCES

1. J.A.D. Appleby, *Subexponential solutions of scalar linear Itô-Volterra equations with damped stochastic perturbations*, Functional Differ. Equations **11** (2004), 5–10.

2. ———, *Almost sure subexponential decay rates of scalar Itô-Volterra equations*, Electron. J. Qual. Theory Differ. Equations, Proc. 7th Coll. QTDE, (2004), 1–32.

3. J.A.D. Appleby and A. Freeman, *Exponential asymptotic stability of linear Itô-Volterra equations with damped stochastic perturbations*, Electron. J. Probab. **8** (2003), 22 pp.

4. J.A.D. Appleby, J.G. Gleeson and A. Rodkina, *Asymptotic constancy and stability in nonautonomous stochastic differential equations*, Cubo J. **10** (2008), 145–160.

5. J.A.D. Appleby, I. Győri and D.W. Reynolds, *On exact rates of decay of solutions of linear systems of Volterra equations with delay*, J. Math. Anal. Appl. **320** (2006), 56–77.

6. ———, *Subexponential solutions of scalar linear integro-differential equations with delay*, Functional Differ. Equations **11** (2004), 11–18.

7. J.A.D. Appleby and D.W. Reynolds, *Subexponential solutions of linear Volterra integro-differential equations and transient renewal equations*, Proc. Roy. Soc. Edinburgh. **132** (2002), 521–543.

8. ———, *On necessary and sufficient conditions for exponential stability in linear Volterra integro-differential equations*, J. Integral Equations Appl. **16** (2004), 221–240.

9. J.A.D. Appleby and M. Riedle, *Almost sure asymptotic stability of stochastic Volterra integro-differential equations with fading perturbations*, Stoch. Anal. Appl. **24**, (2006), 813–826.

10. R. Bellman and K.L. Cooke, *Differential-difference equations*, Academic Press, New York, 1963.

11. M.A. Berger and V.J. Mizel, *Volterra equations with Itô integrals* I, J. Integral Equations **2** (1980), 187–245.

12. I.M. Gelfand, D.A. Raikov and G.E. Shilov, *Commutative normed rings*, Chelsea, New York, 1964.

13. G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra integral and functional equations*, Cambridge University Press, Cambridge, 1990.

14. J.K. Hale and S.M. Verduyn Lunel, *Introduction to functional differential equations*, Springer-Verlag, New York, 1993.

15. U.G. Haussmann, *Asymptotic stability of the linear Itô equation in infinite dimensions*, J. Math. Anal. Appl. **65** (1978), 219–235.

16. A. Ichikawa, *Stability of semilinear stochastic evolution equations*, J. Math. Anal. Appl. **90** (1982), 12–44.

17. K. Itô and M. Nisio, *On stationary solutions of a stochastic differential equation*, J. Math. Kyoto Univ. **4** (1964), 1–75.

18. G.S. Jordan, O. Staffans and R.L. Wheeler, *Local analyticity in weighted L^1-spaces and applications to stability problems for Volterra equations*, Trans. Amer. Math. Soc. **274** (1982), 749–782.

19. G.S. Jordan and R.L. Wheeler, *Weighted L^1-remainder theorems for resolvents of Volterra equations*, SIAM J. Math. Anal. **11** (1980), 885–900.

20. R. Kadiev and A. Ponosov, *Relations between stability and admissibility for stochasatic linear functional differential equations*, Functional Differ. Equations **12** (2005), 209–244.

21. R. Kadiev and A. Ponosov, *Stability of stochastic functional differential equations and the W-transform*, Electron. J. Differential Equations **92** (2004), 36 pp.

22. V. Kolmanovskii and A. Myshkis, *Introduction to the theory and applications of functional differential equations*, in *Mathematics and its applications*, Kluwer, Dordrecht, 1999.

23. X. Mao, *Exponential stability of stochastic differential equations*, Pure Applied Mathematics, Marcel Dekker, New York, 1994.

24. ———, *Stability of stochastic integro-differential equations*, Stochastic Anal. Appl. **18** (2000), 1005–1017.

25. ———, *Almost sure exponential stability of delay equations with damped stochastic perturbation*, Stochastic Anal. Appl. **19** (2001), 67–84.

26. X. Mao and M. Riedle, *Mean square stability of stochastic Volterra integrodifferential equations*, Systems Control Lett. **55** (2006), 459–465.

27. R.K. Miller, *Asymptotic stability properties of linear Volterra integrodifferential equations*, J. Differential Equations **10** (1971), 485–506.

28. S.E.A. Mohammed, *Stochastic functional differential equations*, Pitman, Boston, Mass., 1984.

29. S-E.A. Mohammed and M.K.R. Scheutzow, *The stable manifold theorem for non-linear stochastic systems with memory.* II: *The local stable manifold theorem*, J. Functional Anal. **206** (2004), 253–306.

30. S. Murakami, *Exponential stability for fundamental solutions of some linear functional differential equations*, Proc. International Symposium: Functional differential equations (Singapore), T. Yoshizawa and J.Kato, eds., World Scientific, Singapore, 1990.

31. ———, *Exponential asymptotic stability of scalar linear Volterra equations*, J. Differential Integral Equations **4** (1991), 519–525.

32. D.F. Shea and S. Wainger, *Variants of the Wiener-Levy theorem, with applications to stability problems for some Volterra integral equations*, Amer. J. Math. **97** (1975), 312–343.

33. J. Zabczyk, *On the stability of infinite-dimensional linear stochastic systems*, in *Probability theory* (Papers, VIIth Semester, Stefan Banach Internat. Math. Center, Warsaw, 1976), Z. Ciesislski, ed., Banach Center Publications, PWN, Warsaw, 1979.

Edgeworth Centre for Financial Mathematics, School of Mathematical Sciences, Dublin City University, Dublin 9, Ireland
Email address: john.appleby@dcu.ie

School of Mathematics, The University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom
Email address: markus.riedle@manchester.ac.uk