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A NUMERICAL METHOD FOR A NONLOCAL ELLIPTIC BOUNDARY VALUE PROBLEM

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ABSTRACT. In 2005 Corrêa and Filho established existence and uniqueness results for the nonlinear PDE: $-\Delta u = \frac{g(x,u)^{\alpha}}{\left(\int_{\Omega} f(x,u)\right)^{\beta}}$, which arises in physical models of thermodynamical equilibrium via Coulomb potential, among others [3]. In this work we discuss a numerical method for a special case of this equation: $-\alpha\left(\int_0^1 u(t)dt\right)u'' = f(x)$, $0 < x < 1$, $u(0) = a$, $u(1) = b$. We first consider the existence and uniqueness of the analytic problem using a fixed point argument and the contraction mapping theorem. Next, we evaluate the solution of the numerical problem via a finite difference scheme. From there, the existence and convergence of the approximate solution will be addressed as well as a uniqueness argument, which requires some additional restrictions. Finally, we conclude the work with some numerical examples where an interval-halving technique was implemented.

1. Introduction. At the annual meeting of the American Mathematical Society in Baltimore in January 2003, the first named author above gave a talk at a special session organized by Zuhair Nashed. Part of the talk included an example of a boundary value problem which involved a coefficient that depended upon the integral of the solution over the domain within the differential equation. Namely,

$$u'' = \alpha\left(\int_0^{\infty} u(t)dt\right)u, \tag{1.1}$$

$$0 < x < \infty, \quad u(0) = 1, \quad \lim_{x\to\infty} u(x) = 0,$$

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where $\alpha = \alpha(q)$ is a positive function defined for $0 \leq q < \infty$. Integrating the solution's formula

$$u(x) = \exp\left\{-\sqrt{\alpha(q)}x\right\} \tag{1.2}$$

leads to the equation

$$q = \int_0^\infty u(x)dx = \beta(q) \equiv \frac{1}{\sqrt{\alpha(q)}}. \tag{1.3}$$

Clearly, it follows that depending upon $\alpha(q)$ there can exist a unique solution to (1.1), many solutions, or no solutions.

For example, $\beta(q) = \left(1+q^2\right)^{-1}, 0 \leq q < \infty$, implies the existence of a unique solution, $\beta(q) = q + \cos\frac{\pi q}{20}, 0 \leq q < \infty$, implies the existence of infinitely many solutions, and $\beta(q) = 1 + q^2$, $0 \leq q < \infty$, implies the nonexistence of solutions. Recently, the authors became aware of applications for elliptic partial differential equations involving coefficients depending on the integral of solution or the L^2 norm of the gradient of the solution over the domain of the solution. For physical applications, see [1]. For some existence and uniqueness results see [1, 2, 3]. The purpose of this paper is to consider a one-dimensional problem similar to that discussed in [3] and to analyze conditions on the coefficient and data, which lead to the existence and uniqueness of the solution, the existence and uniqueness of a numerical approximation, and the convergence of the numerical approximation to the solution.

We shall consider the problem of finding a solution $u = u(x)$ satisfying

$$-\alpha\left(\int_0^1 u(t)dt\right)u'' = f(x), \qquad 0 < x < 1, \;\; u(0) = a, \;\; u(1) = b \tag{1.4}$$

where $\alpha = \alpha(q)$ is a positive function of q defined over $-\infty < q < \infty$, $f(x)$ is defined over $0 \leq x \leq 1$, and a and b are real constants. In Section 2, we shall demonstrate the existence of a solution via the fixed point of a nonlinear mapping under various conditions on the data a, b, α, and f. For f sufficiently small we show that the mapping is a contraction yielding unicity of the solution. Section 3 deals with a

Fourier series approach to uniqueness, which serves as a motivation for the existence of the numerical approximation of the nonlinear finite difference scheme derived in Section 4. The existence of the numerical approximation is demonstrated in Section 5 via a fixed point of a nonlinear mapping derived from the finite Fourier representation of the solution of a linear auxillary finite difference scheme of the linear auxillary problem in Section 2. The estimates involved in the existence of the solution to the nonlinear algebraic problem in Section 5 carry over to the analysis of convergence, which is demonstrated in Section 6. Basically, convergence can be guaranteed if f is sufficiently small. In Section 7, we conclude the paper with some examples of the numerical process.

2. Existence. We shall start with the assumption that $\alpha = \alpha(q)$ is a continuous function bounded below by the positive constant α_0. If $f \in L^2([0,1])$ which is the Hilbert space of square integrable functions with inner product

$$(\phi, \psi) = \int_0^1 \phi(x)\psi(x)dx \tag{2.1}$$

and norm

$$\| \phi \|_0^2 = (\phi, \phi), \tag{2.2}$$

then for each q in $-\infty < q < \infty$, the problem

$$-\alpha(q)u'' = f(x), \quad 0 < x < 1, \ \ u(0) = a, \ \ u(1) = b \tag{2.3}$$

has a unique solution $u = u(x;q)$ belonging to the Sobolev space $H^1([0,1])$ with inner product

$$(\phi, \psi)_1 = (\phi, \psi) + (\phi', \psi'), \tag{2.4}$$

which is the closure in the norm

$$\| \phi \|_1^2 = (\phi, \phi) + (\phi', \phi') \tag{2.5}$$

of the restrictions of the continuously differentiable functions to $0 \le x \le 1$. The function $u = u(x;q)$ has the form

$$u = v + \varsigma, \tag{2.6}$$

where $\varsigma = a(1-x) + bx$ and $v = v(x;q) \in H_0^1([0,1])$ is the solution of the weak formulation

$$\int_0^1 v'\phi' dx = \frac{1}{\alpha(q)}(f,\phi), \quad \forall \phi \in H_0^1([0,1]), \tag{2.7}$$

where $H_0^1([0,1])$ is the Sobolev space with the inner product and norm of $H^1([0,1])$ that results in the closure in the norm of $H^1([0,1])$ of the space of continuously differentiable functions with compact support in $0 \le x \le 1$. The following inequality is easy to obtain from setting $\phi = v$ in (2.7) and employing Schwarz's lemma:

$$\int_0^1 (v')^2 dx \le \frac{1}{\alpha(q)} \|f\|_0 \|v\|_0 . \tag{2.8}$$

Since π^2 is the smallest eigenvalue for the problem $u'' + \lambda u = 0,\ u(0) = u(1) = 0$, we have

$$\| v \|_0^2 \le \frac{1}{\pi^2} \int_0^1 (v')^2 dx, \tag{2.9}$$

whence it follows

$$\| v \|_0 \le \frac{1}{\alpha(q)\pi^2} \| f \|_0, \tag{2.10}$$

and

$$\int_0^1 (v')^2 dx \le \frac{1}{[\alpha(q)\pi]^2} \| f \|_0^2 . \tag{2.11}$$

Utilizing the Green's function for the operator $-\frac{d^2}{dx^2}$ in $[0,1]$ with zero boundary conditions, we have

$$v(x) = \frac{1}{\alpha(q)} \int_0^1 G(x,t) f(t) dt, \tag{2.12}$$

where $G(x,t) = \begin{cases} x(1-t), & x \leq t \\ (1-x)t, & x \geq t \end{cases}$ and $0 \leq x,\ t \leq 1$, it follows that

$$\max_{0\leq x\leq 1} |v(x;q)| \leq \frac{1}{\alpha(q)} \parallel f \parallel_0 . \tag{2.13}$$

In a similar manner, we obtain

$$\max_{0\leq x\leq 1} |v(x;q_1) - v(x;q_2)| \leq \parallel f \parallel_0 \left| \frac{1}{\alpha(q_1)} - \frac{1}{\alpha(q_2)} \right| . \tag{2.14}$$

We now define the mapping

$$T(q) := \int_0^1 u(x;q)dx = \frac{a+b}{2} + \int_0^1 v(x;q)dx. \tag{2.15}$$

From (2.12) and $\alpha(q) \geq \alpha_0 > 0$, we obtain

$$|T(q)| \leq \frac{1}{2}|a+b| + \frac{1}{\alpha_0} \parallel f \parallel_0 . \tag{2.16}$$

Next, we see from (2.13) and (2.14) that

$$|T(q_1) - T(q_2)| \leq \frac{1}{\alpha_0^2} \parallel f \parallel_0 |\alpha(q_2) - \alpha(q_1)|. \tag{2.17}$$

Since $\alpha(q)$ is uniformly continuous on the $-C \leq q \leq C$, where

$$C = \frac{1}{2}|a+b| + \frac{1}{\alpha_0}\|f\|_o \tag{2.18}$$

it follows that $T(q)$ is uniformly continuous on $-C \leq q \leq C$. Consider the square $-C \leq q,\ y \leq C$ in the Cartesian plane. Since the graph of $y = T(q)$ is contained in the square and continuously traverses it from $q = -C$ to $q = +C$, it must intersect the diagonal $y = q$ in at least one point $q*$. Hence, there is at least one fixed point $T(q*) = q*$ and at least one solution for (1.4). We summarize the analysis above with the following statement.

Theorem 2.1. *If $\alpha(q)$ is a continuous real valued function defined on $-\infty < q < \infty$, which is bounded below by α_0 and $f(x)$ is square integrable on $0 \le x \le 1$, then there exists at least one weak solution $u = u(x) \in H^1([0,1])$ that satisfies*

$$(2.19)\qquad -\alpha\left(\int_0^1 u(t)\,dt\right)\, u'' = f(x), \\ 0 < x < 1, \;\; u(0) = a, \;\; u(1) = b,$$

where a and b are real numbers and $H^1([0,1])$ is the Sobolev space of square integrable functions with square integrable derivatives defined over $0 \le x \le 1$.

Proof: See the analysis above.

As a corollary of the argument above, we have the following result.

Theorem 2.2. *If $\alpha(q)$ is continuous and continuously differentiable on $-\infty < q < \infty$ such that $|\alpha'(q)| < M$, where M is a positive real number, the remaining assumptions of Theorem 2.1 hold, and if*

$$(2.20)\qquad \frac{M}{\alpha_0^2}\,\|f\|_0 \;<\; 1,$$

then the weak solution $u = u(x)$ of

$$(2.21)\qquad -\alpha\left(\int_0^1 u(t)\,dt\right)\, u'' = f(x), \\ 0 < x < 1, \;\; u(0) = a, \;\; u(1) = b,$$

is unique.

Proof: From (2.16) and (2.18), if follows that the mapping $T(q)$ is a contraction and thus possesses a unique fixed point.

Now we consider the case that $\alpha\,(q)$ and continuous on $0 < q < \infty$, $\alpha(0) = 0$, and $\alpha(q)$ is monotone increasing which requires some additional assumptions on the data $f(x)$, a and b in order to obtain a lower bound on $\alpha(q)$. Namely, we assume that $f(x)$ is continuous on $0 < x < 1$, $f(x) > 0$, and f is square integrable over $0 \leq x \leq 1$. Also we assume that a and b are nonnegative and at least one of them is positive. Under the above assumptions for $q > 0$, we have a unique classical solution for

$$-\alpha(q)u'' = f(x), \quad 0 < x < 1, \;\; u(0) = a, \;\; u(1) = b. \tag{2.22}$$

Recalling (2.6) with $u = v + \varsigma$, $\varsigma = a(1-x) + bx$, we see that v is a classical solution of

$$-\alpha(q)v'' = f(x), \quad 0 < x < 1, \;\; v(0) = v(1) = 0.$$

Thus, $v(x) \geq 0$ via the maximum principle. Otherwise v has a negative minimum at say, x_0, $0 < x_0 < 1$, at which $v''(x_0) \geq 0$. However, from the differential equation at x_0, $v''(x_0) = -f(x_0)/\alpha(q) < 0$, which is a contradiction. Thus, $u - \varsigma = v(x) \geq 0$ and

$$T(q) = \int_0^1 u(x,q)dx \geq \int_0^1 \varsigma(x)dx = \frac{a+b}{2}. \tag{2.23}$$

From the analysis for (2.8) through (2.14) we have

$$0 < \left(\frac{a+b}{2}\right) \leq T(q) \leq \left(\frac{a+b}{2}\right) + \frac{1}{\alpha_0}\|f\|_0 \tag{2.24,}$$

where here

$$\alpha_0 = \alpha\left(\frac{a+b}{2}\right). \tag{2.25}$$

Likewise (2.16) holds. As $\alpha\,(q)$ is uniformly continuous on $0 < \left(\frac{a+b}{2}\right) \leq q \leq C$, where C is defined by (2.17), it follows that T is uniformly continuous on $\left(\frac{a+b}{2}\right) \leq q \leq C$ and that from (2.23), $T\,(q)$ has a fixed point in that interval. We can summarize the above analysis in the following statement.

Theorem 2.3. *If $\alpha = \alpha(q)$ is continuous and monotone increasing on $0 \le q < \infty$ with $\alpha(0) \ge 0$, $f(x)$ is continuous, square integrable, and $f(x) > 0$ on $0 < x < 1$, and if a and b are nonnegative real numbers with at least one of them positive, then*

$$-\alpha\left(\int_0^1 u(t)dt\right) u'' = f(x), \quad 0 < x < 1, \quad u(0) = a, \quad u(1) = b \tag{2.26}$$

has at least one classical solution.

Proof: See the analysis preceding the statement of the theorem and the analysis preceding Theorem 2.1.

As a corollary we have the following result.

Theorem 2.4. *If the assumptions of Theorem 2.3 hold and if $\alpha(q)$ is continuously differentiable on $0 \le q < \infty$ with $|\alpha'(q)| < M$, where M is a positive real number, and if (2.20) holds, then the solution $u(x)$ is unique.*

Proof: As for Theorem 2.2, $T(q)$ is a contraction.

Remark: As an example of the contraction inequality (2.20), consider $\alpha = \alpha(q) = (q)^{\frac{1}{n}}$, then $\alpha'(q) = \frac{1}{n} q^{\frac{1}{n}-1}$ and (2.20) becomes

$$\frac{1}{n}\left(\frac{a+b}{2}\right)^{-\frac{n+1}{n}} \|f\|_0 < 1,$$

which may allow a larger f than $\alpha(q) = q^n$ for which (2.20) becomes

$$n\left(\frac{a+b}{2}\right)^{-2n}\left[\left(\frac{a+b}{2}\right) + \left(\frac{a+b}{2}\right)^{-2n}\|f\|_0\right]^{n-1}\|f\|_0 < 1.$$

3. Another analysis of uniqueness. We provide a Fourier analysis of uniqueness as a motivation of the analysis of the convergence of a numerical procedure for problem (1.4). Let $u_i = u_i(x)$ be two solutions of (1.4). Setting $z = u_1 - u_2$ and subtracting the equation for u_2 from u_1, we obtain

$$z'' = f(x)\left[\alpha\left(\int_0^1 u_1(t)\,dt\right)\alpha\left(\int_0^1 u_2(t)\,dt\right)\right]^{-1}\alpha'(\xi)\int_0^1 z(t)\,dt, \quad z(0) = z(1) = 0, \tag{3.1}$$

where the number ξ lies between the numbers $\int_0^1 u_1(t)\,dt$ and $\int_0^1 z(t)\,dt$. Let η denote the number

$$\left[\alpha\left(\int_0^1 u_1(t)\,dt\right)\alpha\left(\int_0^1 u_2(t)\,dt\right)\right]^{-1}\alpha'(\xi)\int_0^1 z(t)\,dt, \tag{3.2}$$

Expanding f in a Fourier sine series we see that

$$f(x) = \sum_{k=0}^{\infty} c_n \sin n\pi x, \quad 0 \le x \le 1, \tag{3.3}$$

where

$$c_n = 2\int_0^1 f(x)\sin n\pi x\,dx, \quad n = 1, 2, ... \tag{3.4}$$

So, it follows from the differential equation and boundary conditions for z that

$$z(x) = \eta\sum_{k=0}^{\infty} -\frac{c_n}{(n\pi)^2}\sin n\pi x \tag{3.5}$$

and

$$\int_0^1 z(x)dx = \eta\sum_{k=0}^{\infty} -\frac{2c_{2k+1}}{[(2k+1)\pi]^3}. \tag{3.6}$$

From (3.2) we see that

$$\left[\int_0^1 z(x)dx\right]\left[1 + \gamma\sum_{k=0}^{\infty}\frac{c_{2k+1}}{[(2k+1)\pi]^3}\right] = 0, \tag{3.7}$$

where

$$\gamma = 2\alpha'(\xi)\left[\alpha\left(\int_0^1 u_1(t)\,dt\right)\alpha\left(\int_0^1 u_2(t)\,dt\right)\right]^{-1}. \tag{3.8}$$

Since we have assumed above that $\alpha(q) \geq \alpha_0 > 0$, see Theorem 2.1 or (2.24), and that $|\alpha'(q)| \leq M$, it follows from elementary estimates that

$$\left|\gamma \sum_{k=0}^{\infty} \frac{c_{2k+1}}{[(2k+1)\pi]^3}\right| \leq \frac{6M\|f\|_0}{\alpha_0^2\pi^3}. \tag{3.9}$$

Hence, from

$$\frac{6M\|f\|_0}{\alpha_0^2\pi^3} < 1 \tag{3.10}$$

we see that

$$\int_0^1 z(x)dx = 0, \tag{3.11}$$

which implies that $z = 0$, $u_1 \equiv u_2$, and uniqueness.

We remark that estimate (2.19) yields a slightly better multiplier of $\|f\|_0$ than that of (3.9). However, as mentioned above the Fourier analysis will yield a viable approach for the numerical approximation estimates.

4. A Finite Difference Scheme. Let N denote a postive integer, $h = \frac{1}{N}$, and $x_i = \frac{i}{N}$, $i = 0, 1, 2, ..., N$. Denote $u(x_i)$ as u_i and note that it is well known [7] that

$$\Delta_h^2 u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = u''(x_i) + O(h^2) \tag{4.1}$$

for u sufficiently smooth and that

$$Q(\vec{u}) = \sum_{i=0}^{N-1} \frac{u_{i+1} + u_i}{2} h = \int_0^1 u(t)\,dt + O(h^2) \tag{4.2}$$

where $O(h^2)$ denotes a quantity bounded by a positive constant times h^2 and $\vec{u}$ denotes the vector $(u_0, u_2, ..., u_N)$. Consider now the problem (1.4). We have,

$$\left[\alpha\left(\int_0^1 u(t)\,dt\right)\right]^{-1} = [\alpha(Q(\vec{u}))]^{-1} + \left[\alpha\left(\int_0^1 u(t)\,dt\right)\right]^{-1} - [\alpha(Q(\vec{u}))]^{-1} \tag{4.3}$$

and

$$\left[\alpha\left(\int_0^1 u(t)\,dt\right)\right]^{-1} - [\alpha(Q(\vec{u}))]^{-1} = O(h^2), \tag{4.4}$$

where the constant in the $O(h^2)$ depends upon estimates of the term

$$\alpha'(\xi)\left[\alpha\left(\int_0^1 u(t)\,dt\right)\alpha(Q(\vec{u}))\right]^{-1} u''. \tag{4.5}$$

Consequently, at the points x_i, $i = 1, ..., N-1$, we have from the differential equation in (1.4)

$$\begin{gathered} -\Delta u_i = f(x_i)\,[\alpha(Q(\vec{u}))] + O(h^2), \\ i = 1, ..., N-1, \quad u_0 = a, \quad u_N = b. \end{gathered} \tag{4.6}$$

Setting $\vec{w} = (w_0, w_1, ..., w_N)$, deleting the $O(h^2)$ term and in (4.6), and substituting w_i and $\vec{w}$ for u_i and $\vec{u}$ in (4.6), we obtain the algebraic problem for the approximation $\vec{w}$ for $\vec{u}$. Namely, find $\vec{w}$ satisfying

$$\begin{gathered} -\Delta_h^2 w_i = f(x_1)\,[\alpha(Q(\vec{w}))]^{-1}, \\ i - 1, ..., N-1, \quad w_0 = a \text{ and } w_N = b. \end{gathered} \tag{4.7}$$

We turn now to the existence of a solution to the algebraic problem (4.7).

5. Existence of Approximate Solutions. As with the analytic case and under the same assumptions on the data, we consider the mapping

$$F(q) = Q(\vec{w}) \tag{5.1}$$

where $\vec{w} = (a, w_1, ..., w_{N-1}, b)$ is the solution of

$$(5.2) \qquad -\Delta_h^2 w_i = f(x_i)\,[\alpha(q)]\,, \quad i = 1, ..., N-1, \;\; w_0 = a, \;\; w_N = b.$$

For each q in the appropriate interval for q, there exists a unique $\vec{w} = \vec{w}(q)$. Hence the map is well-defined. In order to apply the appropriate fixed point theorem we need estimates of $F(q)$ and to obtain these estimates, it is necessary to write $\vec{w}$ in a form that can be estimated. Namely,

$$(5.3) \qquad w_i = a(1 - x_i) + bx_i + [\alpha(q)]^{-1} \sum_{n=1}^{N-1} \frac{c_n}{\lambda_n} \sin n\pi x_i,$$

where

$$(5.4) \qquad \lambda_n = \frac{4\sin^2 \frac{n\pi h}{2}}{h^2}.$$

Recall [6,7] that

$$(5.5) \qquad \Delta_h^2 \sin \alpha x = -\frac{4\sin^2 \frac{\alpha h}{2}}{h^2} \sin \alpha x,$$

and

$$(5.6) \qquad (\overrightarrow{\sin\, m\pi x}, \overrightarrow{\sin\, n\pi x}) = \frac{1}{2}\delta_{mn},$$

where

$$(5.7) \qquad \overrightarrow{\sin\, m\pi x} = (0, \sin\, m\pi x_1, \sin\, m\pi x_2, ..., \sin\, m\pi x_{N-1}, 0)$$

with

$$(5.8) \qquad (\vec{f}, \vec{\phi}) = \sum_{i=0}^{N} f_i \phi_i h,$$

for $\vec{f}$ and $\vec{\phi}$ $N+1$ dimensional vectors with the 1st and last components equal to zero,

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{5.9}$$

and m and n are positive integers that range from 1 to $N-1$. Also, we note that

$$c_n = 2(\vec{f}, \overrightarrow{\sin\, n\pi x}). \tag{5.10}$$

By Schwarz's lemma, we see that

$$|c_n| \leq 2 \left(\sum_{i=1}^{N-1} f_i^2 h \right)^{\frac{1}{2}} \left(\sum_{i=1}^{N-1} \sin^2\, n\pi x_i h \right)^{\frac{1}{2}} \leq \sqrt{2} \left(\vec{f}, \vec{f} \right)^{\frac{1}{2}}. \tag{5.11}$$

To finish the estimates for $F\,(q)$ we need the following result.

Lemma 5.1.

$$Q(\overrightarrow{\sin\, n\pi x}) = - \begin{cases} h \cot \dfrac{m\pi h}{2} & , \; m \textit{ odd} \\ 0 & , \; m \textit{ even} \end{cases} \tag{5.12}$$

Proof: From the identity $\sin\, x + \sin\, y = 2\, \sin\, \frac{x+y}{2} \cos\, \frac{x-y}{2}$, we obtain

$$Q(\overrightarrow{\sin\, m\pi x}) = \sum_{i=0}^{N-1} h\, \sin\, m\pi \left(\frac{x_{i+1} + x_i}{2} \right) \cos\, \frac{m\pi h}{2} \tag{5.13}$$

multiplying (5.13) by one in the form $\sin\, \frac{m\pi h}{2} / \sin\, \frac{m\pi h}{2}$ we see that

$$Q(\overrightarrow{\sin\, m\pi x}) = h\, \cot\, \frac{m\pi h}{2} \sum_{i=0}^{N-1} \left(\sin\, m\pi \left(i + \frac{1}{2} \right) h \right) \sin\, \frac{m\pi h}{2}. \tag{5.14}$$

Next, via the identity $\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$, it follows that

$$Q(\overrightarrow{\sin m\pi x}) = -\frac{h}{2}\left(\cot \frac{m\pi h}{2}\right) \cdot \sum_{i=0}^{N-1} [\cos m\pi x_{i+1} - \cos m\pi x_i] \tag{5.15}$$

$$= -\frac{h}{2} \cot \frac{m\pi h}{2} [\cos m\pi - \cos 0] \tag{5.16}$$

$$= \begin{cases} +h \cot \dfrac{m\pi h}{2} & , \ m \textit{ odd} \\ 0 & , \ m \textit{ even} \end{cases} \tag{5.17}$$

□

We can utilize the calculus inequality $\sin \theta < \theta < \tan \theta$ for $0 < \theta < \frac{\pi}{2}$ to obtain the estimate

$$\left| Q(\overrightarrow{\sin m\pi x}) \right| = \frac{2}{m\pi} \frac{m\pi h}{2} \cot \frac{m\pi h}{2} \leq \frac{2}{m\pi}, \quad m \text{ odd.} \tag{5.18}$$

From the linearity of $Q(\vec{w})$ we see that

$$F(q) = Q(\vec{w}) \tag{5.19}$$
$$= \frac{a+b}{2} + [\alpha(q)]^{-1} \sum_{n=1}^{N-1} \frac{c_n}{\lambda_n} Q(\overrightarrow{\sin n\pi x})$$
$$= \frac{a+b}{2} + [\alpha(q)]^{-1} \sum_{0 \leq k \leq \lceil \frac{N-1}{2} \rceil} \frac{c_{2k+1}}{\lambda_{2k+1}} \cdot h \cot \frac{(2k+1)\pi h}{2} \tag{5.20}$$

where $\lceil \frac{N-1}{2} \rceil$ is the largest integer in $\frac{N-1}{2}$. Consequently, from (5.15)

$$|F(q)| \leq \frac{1}{2} |a+b| + \frac{1}{\alpha_0} \sum_{0 \leq k \leq \lceil \frac{N-1}{2} \rceil} \frac{|c_{2k+1}|}{|\lambda_{2k+1}|} \cdot \frac{2}{(2k+1)\pi}. \tag{5.21}$$

Since $\sin x > \frac{2}{\pi}x$ for $0 < x < \frac{\pi}{2}$, we see that

$$\frac{1}{\lambda_n} = \frac{h^2}{4\sin^2 \frac{n\pi h}{2}} \le \frac{h^2}{4\left(\frac{2}{\pi}\frac{n\pi h}{2}\right)^2} = \frac{1}{4n^2} \tag{5.22}$$

and from (5.10) it follows that

$$\begin{aligned} |F(q)| &\le \frac{1}{2}|a+b| + \frac{\sqrt{2}}{\alpha_0}\left(\vec{f},\vec{f}\right) \sum_{0\le k\le \lceil \frac{N-1}{2} \rceil} \frac{1}{2\pi} \cdot \frac{1}{(2k+1)^3} \\ &\le \frac{1}{2}|a+b| + \frac{\sqrt{2}}{\pi\alpha_0}\left(\vec{f},\vec{f}\right). \end{aligned} \tag{5.23}$$

By a similar argument, we obtain

$$|F(q_1) - F(q_2)| \le \frac{\sqrt{2}}{\pi\alpha_0^3}\left(\vec{f},\vec{f}\right)|\alpha(q_1) - \alpha(q_2)| \tag{5.24}$$

From the uniform continuity of $\alpha(q)$ on $-R \le q \le R$, where

$$R = \frac{1}{2}|a+b| + \frac{\sqrt{2}}{\pi\alpha_0}\left(\vec{f},\vec{f}\right), \tag{5.25}$$

we see from (5.23) and (5.24) that $F(q)$ has at least one fixed point. As $\Delta_h^2 \vec{w_i}$ is positive at a minimum and negative at a maximum, we see that $F(q)$ has at least one fixed point under the conditions on the data in Theorem 2.3. Likewise to Theorem 2.2 and Theorem 2.4, if $\left(\vec{f},\vec{f}\right)^{\frac{1}{2}}$ is sufficiently small $F(q)$ is a contraction and the fixed point is unique. We summarize the analysis above in the following statement.

Theorem 5.3. *Under the assumptions on the data, a, b, $\alpha(q)$ and $f(x)$ given in Theorem 2.1 and Theorem 2.3, respectively, there exists a solution to the algebraic problem stated in (4.7) and if $\left(\vec{f},\vec{f}\right)^{\frac{1}{2}}$ is sufficiently small, the solution is unique.*

Proof: See the analysis preceding the statement of the theorem.

6. Convergence of the approximation to the analytic solution. Setting $z_i = u_i - w_i$, $i = 0, ..., N$ and subtracting (4.7) from (4.6), we obtain

$$-\Delta_h^2 z_i = f(x_i)\left\{[\alpha(Q(\vec{u}))]^{-1} - [\alpha(Q(\vec{w}))]^{-1}\right\} + O(h^2)\; z_0 = z_N = 0, \tag{6.1}$$

which can be written as

$$\Delta_h^2 z_i = s_i Q(\vec{z}) + r_i, \quad i = 1, ..., N-1, \;\; z_0 = z_N = 0, \tag{6.2}$$

where

$$s_i = f(x_i)\,[\alpha(Q(\vec{u}))]^{-1}\,[\alpha(Q(\vec{w}))]^{-1}\,\alpha'(\xi), \quad r_i = O(h^2), \tag{6.3}$$

and ξ is a number between $Q(\vec{u})$ and $Q(\vec{w})$. In a similar manner to (5.2), we obtain the representation for z_i as

$$z_i = \mu Q(\vec{z}) \sum_{n=1}^{N-1} \frac{c_n}{\lambda_n} \sin n\pi x_i - \sum_{n=1}^{N-1} \frac{d_n}{\lambda_n} \sin n\pi x_i \tag{6.4}$$

where c_n is defined by (5.9),

$$d_n = 2(\vec{r}, \overrightarrow{\sin n\pi x}), \tag{6.5}$$

and

$$\mu = -\,[\alpha(Q(\vec{u}))]^{-1}\,\alpha'(\xi). \tag{6.6}$$

From the linearity of $Q(\vec{w})$, we see that

$$Q(\vec{z}) = \mu Q(\vec{z}) \sum_{n=1}^{N-1} \frac{c_n}{\lambda_n} Q(\overrightarrow{\sin n\pi x}) - \sum_{n=1}^{N-1} \frac{d_n}{\lambda_n} Q(\overrightarrow{\sin n\pi x}) \tag{6.7}$$

and via Lemma 5.1 we have

$$Q(\vec{z})\left[1 - \mu \sum_{0 \le k \le \lceil \frac{N-1}{2} \rceil} \frac{c_{2k+1}}{\lambda_{2k+1}} \cdot h \cot \frac{(2k+1)\pi h}{2}\right] = \sum_{0 \le k \le \lceil \frac{N-1}{2} \rceil} \frac{d_{2k+1}}{\lambda_{2k+1}} \cdot h \cot \frac{(2k+1)\pi h}{2}. \tag{6.8}$$

From the estimates (5.11), (5.17) and (5.18) we see that

$$\left|\mu \sum_{0\le k\le\lceil\frac{N-1}{2}\rceil} \frac{c_{2k+1}}{\lambda_{2k+1}} h \cot\frac{(2k+1)\pi h}{2}\right| \le \frac{\sqrt{2}M}{2\pi\alpha_0^2}(\vec{f},\vec{f})^{\frac{1}{2}} \sum_{0\le k\le\lceil\frac{N-1}{2}\rceil} \frac{1}{(2k+1)^3} \tag{6.9}$$

$$\le \frac{\sqrt{2}M}{\pi\alpha_0^2}(\vec{f},\vec{f})^{\frac{1}{2}}. \tag{6.10}$$

Thus, the multiplier of $Q(\vec{z})$ on the left hand side of (6.9) is clearly not equal to zero if $(\vec{f},\vec{f})^{\frac{1}{2}}$ is sufficiently small. Since the

$$|d_{2k+1}| = O(h^2) \tag{6.11}$$

and the multiplier of $Q(\vec{z})$ in (6.9) can be bounded below in absolute value via (6.10) for $(\vec{f},\vec{f})^{\frac{1}{2}}$ sufficiently small, it follows that

$$Q(\vec{z}) = O(h^2) \tag{6.12}$$

Using (6.12) and the preceding estimates, it follows from (6.5) that

$$|z_i| = O(h^2), \tag{6.13}$$

where the constant in the $O(h^2)$ depends upon the size of $(\vec{f},\vec{f})^{\frac{1}{2}}$. Summarizing the analysis above we have the following statement.

Theorem 6.1. *For $f(x)$ sufficiently small, the approximate solutions $\vec{w}$ converge to the solution $u(x)$ at each x appearing in the grid at some h sufficiently small and remaining in the grid as h tends to zero.*

Proof: See the analysis preceding the statement of the theorem.

7. Numerical Examples. We present here the results of three examples. For example 1, we chose $u = x^3$. Integrating from zero to one we obtain

$$q = \int_0^1 u(x)dx = \frac{1}{4}. \tag{7.1}$$

Choosing

$$\alpha(q) = q^{\frac{1}{3}}, \tag{7.2}$$

generates from $u'' = 6x$ the source term

$$f(x) = 6\left(\frac{1}{4}\right)^{\frac{1}{3}} x. \tag{7.3}$$

The boundary conditions of $u(0) = 0$ and $u(1) = 1$ along with $\alpha(q)$ and $f(x)$ were employed in (5.1) for $h = .1, .01, .001$ and $.0001$. The solution of $q = Q(\vec{w}(q))$ for each h required a search of $q_k = \frac{k}{10}$ $k = 1, ..., 10$ until a change of sign in $q_k = Q(\vec{w}(q_k))$ was obtained followed by interval halving until $|Q(\vec{w}(q)) - q|$ diminished below a pre-set precision. In the Tables below the J-Value in the 2nd row denotes the number of interval halvings required for $s = |Q(\vec{w}(q_j)) - q_j|$ to fall below the pre-set Precision is recorded for each h in the third row. The actual precision $S = |Q - q|$ is recorded for each h in the fourth row. The max $|u(x_i) - w_i|$ for each h is given in the fifth row. The actual error $E = |Q - q|$ between the actual q and its approximation Q is found in sixth row. The values of h are found in the first row as labels for the columns of associated computed results listed under each value of h. The data u, q, $\alpha(q)$, and $f(x)$ for each example are summarized in the legend/title of each table.

Table 1				
	h=1/10	h=1/100	h=1/1000	h=1/10000
J-Value	49	46	47	47
Precision	10^{-16}	10^{-16}	10^{-16}	10^{-16}
$S-\|Q-q\|$	2.22040E-16	8.88180E-16	6.10620E-16	4.44090E-16
$\max\|u_i - w_i\|$	1.40000E-03	1.94000E-06	1.99400E-9	2.00050E-12
$E = \|Q-q\|$	3.70000E-03	3.74960E-05	3.75000E-07	3.75170E-09

TABLE 1. $u = x^3,\ q = \frac{1}{4},\ \alpha = q^{\frac{1}{3}},\ f = 6(\frac{1}{4})^{\frac{1}{3}}x.$

Table 2				
	h=1/10	h=1/100	h=1/1000	h=1/10000
J-Value	51	43	42	42
Precision	10^{-16}	10^{-14}	10^{-14}	10^{-14}
$S-\|Q-q\|$	5.55110E-17	5.38460E-15	1.77640E-15	8.88180E-16
$\max\|u_i - w_i\|$	5.88490E-04	4.10550E-07	3.87000E-10	3.91630E-13
$E = \|Q-q\|$	5.14140E-04	5.10340E-06	5.10300E-08	5.07910E-10

TABLE 2. $u = \cos\left(\frac{2\pi}{3}x\right),\ q = \frac{3\sqrt{3}}{4\pi},\ \alpha = q^2,\ f = -\frac{3}{4}\cos\left(\frac{2\pi}{3}x\right).$

Table 3				
	h=1/10	h=1/100	h=1/1000	h=1/10000
J-Value	43	39	42	42
Precision	10^{-14}	10^{-14}	10^{-14}	10^{-14}
$S-\|Q-q\|$	2.22040E-16	7.10540E-15	3.21960E-15	9.02060E-15
$\max\|u_i-w_i\|$	3.56810E-04	4.35570E-07	4.43560E-10	4.45430E-13
$E=\|Q-q\|$	1.30000E-03	1.29630E-05	1.29630E-07	1.29500E-09

TABLE 3. $u = x(1-x),\ q = \frac{1}{6},\ \alpha = (1+q)^2,\ f = -2\left(1+\frac{1}{6}\right)^2.$

Consideration of the results in the Tables above shows the error behaves as $O(h^2)$ or better. We note that a search followed by interval halfing was necessary since the various $f's$ were not small enough to cause a contraction or to satisfy the condition for uniqueness. A Newton's Method for solving $H(q) \equiv q - Q(\vec{w}(q)) = 0$ was not considered. Left open for consideration is the general question of uniqueness of the solution and the numerical approximation.

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