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ERROR BOUNDS FOR INTEGRAL EQUATIONS ON THE HALF LINE

Dedicated to Professor Günther Hämmerlin
to commemorate his 60th birthday

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ABSTRACT. We compare solutions of integral equations

$$x(s) - \int_0^\infty k(s,t)x(t)dt = y(s),$$

$$x_\beta(s) - \int_0^\beta k(s,t)x_\beta(t)dt = y(s).$$

The setting for the analysis is the space of bounded, continuous functions on $[0,\infty)$. Under reasonable hypotheses, there are unique solutions x and x_β such that $x_\beta \to x$ as $\beta \to \infty$, uniformly on any finite interval. The main purpose of this paper is to obtain computable bounds for the error $|x_\beta - x|$, particularly for certain classes of Wiener-Hopf operators and for compact perturbations of Wiener-Hopf operators.

1. Background and objectives. We are concerned with integral equations of the form

$$x(s) - \int_0^\infty k(s,t)x(t)dt = y(s), \quad 0 \le s < \infty, \tag{1.1}$$

and the corresponding finite-section equations

$$x_\beta(s) - \int_0^\beta k(s,t)x_\beta(t)dt = y(s), \quad 0 \le s < \infty. \tag{1.2}$$

Such equations arise in probability theory, wave propagation, and radiative transfer, amongst other fields.

The functions x, x_β and y are assumed to be bounded and continuous. The hypotheses on the kernel $k(s,t)$ are

H1
$$\sup_{s\ge 0} \int_0^\infty |k(s,t)|dt < 1,$$

H2 $$\int_0^\infty |k(s',t) - k(s,t)|dt \to 0 \quad \text{as } s' \to s, \quad \text{for } 0 \le s < \infty.$$

These conditions imply that the equations (1.1) and (1.2) have unique solutions $x(s)$ and $x_\beta(s)$. Moreover, $x_\beta(s) \to x(s)$ as $\beta \to \infty$, uniformly for s in any finite interval. See Anselone and Sloan [**2**] for a proof under more general hypotheses.

Our main objective is to obtain computable bounds for the error $|x_\beta(s) - x(s)|$. Our analysis is influenced by that of Atkinson [**5**], who worked in a somewhat different context. He derived realistic error bounds which however are not very easy to compute. We shall relate our bounds to those of Atkinson in §8.

An important special case is that of a Wiener-Hopf kernel $k(s,t) = \kappa(s-t)$, where $\kappa \in L^1(-\infty,\infty)$ and $||\kappa||_1 < 1$. A particular example is the Picard kernel

$$k(s,t) = a\, e^{-|s-t|}, \quad 0 < a < \frac{1}{2}.$$

Another important case is $k(s,t) = \ell(s,t)$, where $\ell(s,t)$ satisfies H1, H2 and, moreover,

$$\sup_{s\ge 0} \int_\beta^\infty |\ell(s,t)|dt \to 0 \text{ as } \beta \to \infty.$$

For example,

$$\ell(s,t) = \frac{1}{\pi} \frac{1}{s^2 + t^2 + 1}.$$

We also consider sums, $\kappa(s-t) + \ell(s,t)$, of the two types of kernels. In [**7**], de Hoog and Sloan dealt with kernels of this form. They relaxed H1 by requiring the left member only to be finite, but imposed other conditions in order to ensure that (1.1) and (1.2) are uniquely solvable. They obtained theoretical bounds for $|x_\beta(s) - x(s)|$ which give qualitative information about the order of convergence, but do not seem to be suitable for numerical calculation.

The equation (1.2) reduces to an integral equation with $0 \le s \le \beta$. Numerical integration can be used in order to obtain approximate solutions $x_{\beta n}(s)$ of (1.2) which converge to $x_\beta(s)$ as $n \to \infty$. Bounds for $|x_{\beta n} - x_\beta|$ are available. A triangle inequality gives

$$|x_{\beta n} - x| \leq |x_{\beta n} - x_\beta| + |x_\beta - x|.$$

See Anselone and Sloan [**3, 4**] for further details.

A number of investigators have studied the convergence of finite-section approximations in various settings. However, the literature on error bounds, especially computable error bounds, is rather sparse. Some pertinent references on both aspects of the problem are Chandler and Graham [**6**], Gähler and Gähler [**8**], Gohberg and Feldman [**9**], Krein [**10**], Silbermann [**11**], Sloan [**12**], and Sloan and Spence [**13, 14**]. See also the references cited in [**2**].

2. Notation and basic relations. Let X^+ be the Banach space of bounded, continuous, real or complex functions f on $\mathbf{R}^+ = [0, \infty)$ with the norm $||f|| = \sup_{s \geq 0} |f(s)|$. Thus, convergence in norm is uniform convergence on $\mathbf{R}^+$.

The integral equations (1.1) and (1.2) are expressed in operator forms on X^+ by

$$(I - K)x = y, \quad (I - K_\beta)x_\beta = y, \tag{2.1}$$

where K and K_β, $\beta \in \mathbf{R}^+$, are the integral operators defined by

$$Kf(s) = \int_0^\infty k(s,t)f(t)dt, \quad K_\beta f(s) = \int_0^\beta k(s,t)f(t)dt. \tag{2.2}$$

The hypotheses H1 and H2 on $k(s,t)$ imply that K and K_β are bounded linear operators on X^+ into X^+ and

$$||K_\beta|| \leq ||K|| = \sup_{s \geq 0} \int_0^\infty |k(s,t)|dt < 1. \tag{2.3}$$

It follows that the equations $(I - K)x = y$ and $(I - K_\beta)x_\beta = y$ have the unique solutions

$$x = (I - K)^{-1}y, \quad x_\beta = (I - K_\beta)^{-1}y, \tag{2.4}$$

where

$$(2.5)\qquad (I-K)^{-1}=\sum_{n=0}^{\infty}K^n,\quad (I-K_\beta)^{-1}=\sum_{n=0}^{\infty}K_{\beta'}^n$$

$$(2.6)\qquad ||x||\le\frac{||y||}{1-||K||},\quad ||x_\beta||\le\frac{||y||}{1-||K_\beta||}\le\frac{||y||}{1-||K||}.$$

The solutions x and x_β in (2.1) satisfy

$$(2.7)\qquad x-x_\beta=(I-K_\beta)^{-1}(K-K_\beta)x.$$

This basic relation will yield several bounds for $|x_\beta-x|$.

To facilitate the derivation of computable error bounds, we introduce nonnegative kernels $\hat{k}(s,t)$ such that

$$(2.8)\qquad |k(s,t)|\le\hat{k}(s,t)$$

and $\hat{k}$ satisfies H1 and H2. Define operators $\hat{K}$ and $\hat{K}_\beta$ by

$$(2.9)\qquad \hat{K}f(s)=\int_0^\infty\hat{k}(s,t)f(t)dt,\quad \hat{K}_\beta f(s)=\int_0^\beta\hat{k}(s,t)f(t)dt.$$

Then $||K||\le||\hat{K}||<1$. If $\hat{k}(s,t)=|k(s,t)|$ then $||\hat{K}||=||K||$. Let

$$(2.10)\qquad \hat{K}1(s)=\int_0^\infty\hat{k}(s,t)dt.$$

Then

$$(2.11)\qquad ||\hat{K}1||=||\hat{K}||=\sup_{s\ge0}\int_0^\infty\hat{k}(s,t)dt.$$

From (2.2) and (2.8)-(2.10),

$$(2.12)\qquad |Kf(s)|\le||f||\hat{K}1(s).$$

3. Norm estimates for solutions. The solutions of $(I - K)x = y$ and $(I - K_\beta)x_\beta = y$ are related by (2.7), in which $(K - K_\beta)x$ is given by

$$(K - K_\beta)x(s) = \int_\beta^\infty k(s,t)x(t)dt. \tag{3.1}$$

This involves $x(t)$ only for $t \geq \beta$. Let

$$||x||_{[\beta,\infty)} = \sup_{t\geq\beta} |x(t)|. \tag{3.2}$$

LEMMA 3.1. *Bounds for $||x||_{[\beta,\infty)}$ are given by the inequalities*

$$||x||_{[\beta,\infty)} \leq ||x|| \leq \frac{||y||}{1 - ||K||} \leq \frac{||y||}{1 - ||\hat{K}||}, \tag{3.3}$$

$$||x||_{[\beta,\infty)} \leq ||y||_{[\beta,\infty)} + \frac{||y||\,||\hat{K}1||_{[\beta,\infty)}}{1 - ||K||} \leq \frac{||y||}{1 - ||\hat{K}||}. \tag{3.4}$$

PROOF. The inequalities in (3.3) are elementary. Since $x = y + Kx$,

$$||x||_{[\beta,\infty)} \leq ||y||_{[\beta,\infty)} + ||Kx||_{[\beta,\infty)}. \tag{3.5}$$

By (3.5) and (2.12),

$$||x||_{[\beta,\infty)} \leq ||y||_{[\beta,\infty)} + ||x||\,||\hat{K}1||_{[\beta,\infty)}. \tag{3.6}$$

Now (3.6) and (2.6) give the first inequality in (3.4). Since

$$||y||_{[\beta,\infty)} + \frac{||y||\,||\hat{K}1||_{[\beta,\infty)}}{1 - ||K||} \leq ||y|| + \frac{||y||\,||\hat{K}||}{1 - ||\hat{K}||} = \frac{||y||}{1 - ||\hat{K}||}, \tag{3.7}$$

the second inequality in (3.4) is established. □

In Lemma 3.1, suppose that $\hat{k}(s,t) = k(s,t)$ and hence $||\hat{K}|| = ||K||$. Then it follows from (3.7) that the first bound for $||x||_{[\beta,\infty]}$ in (3.4) is sharper than the second bound if and only if

$$||y||_{[\beta,\infty)} < ||y|| \quad \text{or} \quad ||\hat{K}1||_{[\beta,\infty)} < ||\hat{K}||.$$

The two bounds coincide if and only if

$$||y||_{[\beta,\infty)} = ||y|| \quad \text{and} \quad ||\hat{K}1||_{[\beta,\infty)} = ||\hat{K}||.$$

Clearly,

$$||y||_{[\beta,\infty)} < ||y|| \quad \text{if} \quad |y(s)| \text{ is decreasing,}$$
$$||y||_{[\beta,\infty)} = ||y|| \quad \text{if} \quad |y(s)| \text{ is nondecreasing.}$$

Simple examples of the two cases are furnished by $y(s) = e^{-s}$ and $y(s) = 1 - e^{-s}$. In view of (2.10) and (2.11),

$$||\hat{K}1||_{[\beta,\infty)} < ||\hat{K}|| \text{ if } \hat{k}(s,t) \text{ is decreasing in } s,$$
$$||\hat{K}1||_{[\beta,\infty)} = ||\hat{K}|| \text{ if } \hat{k}(s,t) \text{ is nondecreasing in } s.$$

Examples will be given later.

Uniform convergence on finite intervals plays a central role in our analysis. Note that $f_\beta(t) \to f(t)$ as $\beta \to \infty$, uniformly on each finite interval, if and only if $||f_\beta - f||_{[0,\alpha]} \to 0$ as $\beta \to \infty$ for $\alpha \in \mathbf{R}^+$.

LEMMA 3.2. *Assume*

$$(I - K_\beta)f_\beta = g_\beta, \quad ||g_\beta|| \le b < \infty, \tag{3.8}$$

$$||g_\beta||_{[0,\alpha]} \to 0 \text{ as } \beta \to \infty \text{ for } \alpha \in \mathbf{R}^+. \tag{3.9}$$

Then

$$||f_\beta|| \le \frac{||g_\beta||}{1 - ||K_\beta||} \le \frac{b}{1 - ||K||}, \tag{3.10}$$

$$||f_\beta||_{[0,\alpha]} \to 0 \quad \text{as } \beta \to \infty \text{ for } \alpha \in \mathbf{R}^+. \tag{3.11}$$

PROOF. Since more general results were proved by Anselone and Sloan [**2**], we merely outline the main steps in the argument. First, (3.10) is clear. Now H1 and H2 imply that $\{K_\beta f_\beta : \beta \in \mathbf{R}^+\}$ is bounded and equicontinuous. Repeated applications of the Arzelá-Ascoli lemma on successive intervals $[0,m]$, $m = 1,2,\ldots$, yield a sequence $\beta_i \to \infty$ and $f \in X^+$ such that

$$||K_{\beta_i} f_{\beta_i} - f||_{[0,\alpha]} \to 0 \text{ as } \beta_i \to \infty \text{ for } \alpha \in \mathbf{R}^+.$$

By (3.8), (3.9) and (2.2),

$$||f_{\beta_i} - f||_{[0,\alpha]} \to 0 \text{ as } \beta_i \to \infty \text{ for } \alpha \in \mathbf{R}^+,$$

$$||K_{\beta_i} f_{\beta_i} - Kf||_{[0,\alpha]} \to 0 \text{ as } \beta_i \to \infty \text{ for } \alpha \in \mathbf{R}^+,$$

It follows that $Kf = f, (I-K)f = 0, f = 0$, and

$$||f_{\beta_i}||_{[0,\alpha]} \to 0 \text{ as } \beta_i \to \infty \text{ for } \alpha \in \mathbf{R}^+.$$

Similarly, every sequence $\{f_{\beta_i}\}$ has a subsequence which converges to zero uniformly on finite intervals. A contrapositive argument yields (3.11). □

For the case of a Wiener-Hopf kernel, we shall give a constructive proof of (3.11) which produces error bounds.

LEMMA 3.3. *Let $k(s,t) = \kappa(s-t)$, where $\kappa \in L^1(\mathbf{R})$ and $||\kappa||_1 < 1$. Then $||K_\beta|| \leq ||K|| = ||\kappa||_1 < 1$. Assume*

$$(I - K_\beta) f_\beta = g_\beta. \tag{3.12}$$

Fix $\varepsilon > 0$. Choose r and n such that

$$\int_{-\infty}^{-r} |\kappa(u)| du < (1 - ||K||)\frac{\varepsilon}{2}, \quad ||K||^{n+1} < \frac{\varepsilon}{2}. \tag{3.13}$$

Then

$$||f_\beta||_{[0,\alpha]} \leq \frac{||g_\beta||_{[0,\alpha+nr]} + ||g_\beta||\varepsilon}{1 - ||K||}. \tag{3.14}$$

PROOF. From $(I - K_\beta) f_\beta = g_\beta$,

$$f_\beta = g_\beta + K_\beta g_\beta + K_\beta^2 g_\beta + \cdots + K_\beta^n g_\beta + K_\beta^{n+1} f_\beta. \tag{3.15}$$

By (3.13),

$$|K_\beta g_\beta(s)| \le \Big(\int_0^{\alpha+r} + \int_{\alpha+r}^{\infty}\Big)|\kappa(s-t)g_\beta(t)|dt,$$

$$|K_\beta g_\beta(s)| \le ||K||\,||g_\beta||_{[0,\alpha+r]} + ||g_\beta|| \int_{-\infty}^{s-\alpha-r} |\kappa(u)|du,$$

$$||K_\beta g_\beta||_{[0,\alpha]} \le ||K||\,||g_\beta||_{[0,\alpha+r]} + (1-|K||)||g_\beta||\frac{\varepsilon}{2}.$$

Repetition of this argument yields

$$||K_\beta^2 g_\beta||_{[0,\alpha]} \le ||K||^2||g_\beta||_{[0,\alpha+2r]} + 2||K||(1-||K||)||g_\beta||\frac{\varepsilon}{2},$$

$$||K_\beta^3 g_\beta||_{[0,\alpha]} \le ||K||^3||g_\beta||_{[0,\alpha+3r]} + 3||K||^2(1-||K||)||g_\beta||\frac{\varepsilon}{2},$$

and, in general,

$$||K_\beta^m g_\beta||_{[0,\alpha]} \le ||K||^m||g_\beta||_{[0,\alpha+mr]} + m||K||^{m-1}(1-||K||)||g_\beta||\frac{\varepsilon}{2}.$$

Hence (3.15) and (3.13) yield

$$\begin{aligned} ||f_\beta||_{[0,\alpha]} \le {} & (1+||K||+||K||^2+\cdots+||K||^n)||g_\beta||_{[0,\alpha+nr]} \\ & + (1+2||K||+3||K||^2+\cdots+n||K||^{n-1})(1-||K||)||g_\beta||\frac{\varepsilon}{2} \\ & + ||f_\beta||\frac{\varepsilon}{2}, \end{aligned}$$

which implies (3.14). □

For the special case of a Wiener-Hopf kernel, (3.8), (3.9) and (3.14) imply (3.11). In principle, the bound in (3.14) is computable.

4. Nonnegative kernels. Now let $\hat{k}(s,t)$ be any kernel such that

$$\hat{k}(s,t) \ge 0 \tag{4.1}$$

and $\hat{k}(s,t)$ satisfies H1 and H2. Then the corresponding integral operators $\hat{K}$ and $\hat{K}_\beta$ are given by (2.9), and $||\hat{K}_\beta|| \le ||\hat{K}|| < 1$.

We shall compare solutions of the equations

$$(I - \hat{K})\hat{x} = \hat{y}, \quad (I - \hat{K}_\beta)x_\beta = \hat{y}, \quad \hat{y} \geq 0. \tag{4.2}$$

Thus $\hat{y}(s) \geq 0$ for $s \in \mathbf{R}^+$.

All of the results of §2 and §3 are valid with $k(s,t) = \hat{k}(s,t), K = \hat{K}$, and $K_\beta = \hat{K}_\beta$.

Since $\hat{k}(s,t) \geq 0, \hat{K}$ and $\hat{K}_\beta$ are positive operators. By (2.9),

$$f \geq 0 \Rightarrow \hat{K}f \geq \hat{K}_\beta f \geq 0. \tag{4.3}$$

By (2.12) and (2.5),

$$f \geq 0 \Rightarrow \hat{K}f \leq ||f||\hat{K}1, \tag{4.4}$$

$$f \geq 0 \Rightarrow (I - \hat{K})^{-1}f \geq (I - \hat{K}_\beta)^{-1}f \geq 0. \tag{4.5}$$

Thus, $\hat{x} \geq \hat{x}_\beta \geq 0$ in (4.2).

Bounds for $|\hat{x} - \hat{x}_\beta|$ will involve the functions

$$\hat{v}_\beta(s) = (\hat{K} - \hat{K}_\beta)1(s) = \int_\beta^\infty \hat{k}(s,t)dt, \tag{4.6}$$

$$\hat{u}_\beta = (I - \hat{K}_\beta)^{-1}\hat{v}_\beta = (I - K_\beta)^{-1}(\hat{K} - \hat{K}_\beta)1. \tag{4.7}$$

Among other properties,

$$||\hat{v}_\beta|| = ||\hat{K} - \hat{K}_\beta|| \leq ||\hat{K}||, \tag{4.8}$$

$$||\hat{u}_\beta|| \leq \frac{||\hat{K} - \hat{K}_\beta||}{1 - ||\hat{K}_\beta||} \leq \frac{||\hat{K}||}{1 - ||\hat{K}||}. \tag{4.9}$$

Thus, $\{\hat{v}_\beta\}$ and $\{\hat{u}_\beta\}$ are bounded uniformly on $\mathbf{R}^+$.

LEMMA 4.1. *The functions $\hat{v}_\beta$ and $\hat{u}_\beta$ satisfy*

$$||\hat{v}_\beta||_{[0,\alpha]} \to 0 \ \text{ as } \beta \to \infty \text{ for } \alpha \in \mathbf{R}^+, \tag{4.10}$$

$$(4.11)\qquad ||\hat{u}_\beta||_{[0,\alpha]} \to 0 \quad \textit{as } \beta \to \infty \textit{ for } \alpha \in \mathbf{R}^+.$$

PROOF. By H1, H2 and (4.6),

$$\hat{v}_\beta(s) \to 0 \quad \text{as } \beta \to \infty \text{ for } s \in \mathbf{R}^+,$$

$$\{\hat{v}_\beta : \beta \in \mathbf{R}^+\} \text{ is equicontinuous,}$$

which imply (4.10). Lemma 3.2 yields (4.11). □

THEOREM 4.2. *The solutions* $\hat{x}$ *and* $\hat{x}_\beta$ *in* (4.2) *satisfy*

$$(4.12)\qquad \hat{x} - \hat{x}_\beta = (I - \hat{K}_\beta)^{-1}(\hat{K} - \hat{K}_\beta)\hat{x},$$

$$(4.13)\qquad 0 \le \hat{x} - \hat{x}_\beta \le ||x||_{[\beta,\infty)}\hat{u}_\beta,$$

$$(4.14)\qquad ||\hat{x}||_{[\beta,\infty)} \le ||\hat{y}||_{[\beta,\infty)} + \frac{||\hat{y}||\,||\hat{K}1||_{[\beta,\infty)}}{1 - ||\hat{K}||} \le \frac{||\hat{y}||}{1 - ||\hat{K}||},$$

$$(4.15)\qquad ||\hat{x} - \hat{x}_\beta||_{[0,\alpha]} \le ||x||_{[\beta,\infty)}||\hat{u}_\beta||_{[0,\alpha]} \quad \textit{for } \alpha \in \mathbf{R}^+,$$

$$(4.16)\qquad ||\hat{x} - \hat{x}_\beta||_{[0,\alpha]} \to 0 \quad \textit{as } \beta \to \infty \quad \textit{for } \alpha \in \mathbf{R}^+.$$

PROOF First, (4.12) and (4.14) are special cases of (2.7) and (3.4). By (2.9) and (4.6),

$$0 \le (\hat{K} - \hat{K}_\beta)\hat{x} \le ||\hat{x}||_{[\beta,\infty)}\hat{v}_\beta.$$

Now (4.12), (4.5) and (4.7) yield (4.13) and (4.15). Finally, (4.16) follows from (4.15) and (4.11). □

The quantities $||\hat{y}||$, $||\hat{y}||_{[\beta,\infty)}$, $||\hat{K}||$ and $||\hat{K}1||_{[\beta,\infty)}$ in (4.14) could be estimated numerically, since only suprema and integrals of functions are involved. But the function $\hat{u}_\beta$ in (4.13) is not generally available. Fortunately, $\hat{u}_\beta$ can be evaluated in the case of a Picard kernel, and $\hat{u}_\beta$ can be estimated in the case of a Wiener-Hopf kernel.

EXAMPLE 4.1. Let $\hat{k}(s,t)$ be the Picard kernel

$$\hat{k}(s,t) = a\, e^{-|s-t|}, \quad 0 < a < \frac{1}{2}. \tag{4.17}$$

Then $||\hat{K}|| = 2a < 1$. Let $b = \sqrt{1-2a}$ and $c = \frac{1-b}{1+b}$. Then

$$\hat{v}_\beta(s) = a\, e^{s-\beta}, \quad 0 \le s \le \beta, \tag{4.18}$$

$$\hat{u}_\beta(s) = \frac{2a}{1+c}\left[\frac{e^{cs} - be^{-cs}}{e^{c\beta} - b^2 e^{-c\beta}}\right], \quad 0 \le s \le \beta. \tag{4.19}$$

There are other formulas for $s \ge \beta$; see Atkinson [**5**]. For the Picard kernel, (4.18) and (4.19) imply (4.10) and (4.11). Note that $||\hat{u}_\beta||_{[0,\alpha]}$ is small only if β is significantly larger than α.

EXAMPLE 4.2. Let $\hat{k}(s,t)$ be a non-negative Wiener-Hopf kernel,

$$\hat{k}(s,t) = \hat{\kappa}(s-t),\ \hat{\kappa} \in L^1(\mathbf{R}),\ ||\hat{\kappa}||_1 < 1,\ \hat{\kappa} \ge 0. \tag{4.20}$$

Then

$$\hat{K}1(s) = \int_0^\infty \hat{\kappa}(s-t)dt = \int_{-\infty}^s \hat{\kappa}(u)du, \tag{4.21}$$

$$||\hat{K}1||_{[\beta,\infty)} = ||\hat{\kappa}||_1 = ||\hat{K}||. \tag{4.22}$$

Now the first bound for $||\hat{x}||_{[\beta,\infty)}$ in (4.14) simplifies, so the first bound for $||x||_{[\beta,\infty)}$ is sharper than the second bound if and only if $||\hat{y}||_{[\beta,\infty]} < ||\hat{y}||$, e.g., when $\hat{y}(s)$ is decreasing in s. An estimate for $||\hat{u}_\beta||_{[0,\alpha]}$ to use in (4.15) is available from Lemma 3.3 applied to $(I - \hat{K}_\beta)\hat{u}_\beta = \hat{v}_\beta$. Fix $\varepsilon > 0$. Choose r and n such that

$$\int_{-\infty}^{-r} |\hat{\kappa}(u)|du < (1 - ||\hat{K}||)\frac{\varepsilon}{2}, \quad ||\hat{K}||^{n+1} < \frac{\varepsilon}{2}. \tag{4.23}$$

Then

$$||\hat{u}_\beta||_{[0,\alpha]} \leq \frac{||\hat{v}_\beta||_{[0,\alpha+nr]} + ||\hat{v}_\beta||\varepsilon}{1 - ||\hat{K}||}. \tag{4.24}$$

The right member involves only suprema and integrals of functions. So the bound for $||\hat{x} - \hat{x}_\beta||$ in (4.15) could be estimated numerically.

5. Kernels of mixed sign Again consider the integral equations

$$(I - K)x = y, \quad (I - K_\beta)x_\beta = y, \tag{5.1}$$

where K and K_β are given by (2.2) with a kernel which satsifies H1 and H2. As before,

$$x - x_\beta = (I - K_\beta)^{-1}(K - K_\beta)x. \tag{5.2}$$

Again let

$$|k(s,t)| \leq \hat{k}(s,t), \tag{5.3}$$

where $\hat{k}(s,t)$ also satisfies H1 and H2. For example, $\hat{k}(s,t) = |k(s,t)|$ or $k(s,t) = \hat{k}(s,t) \geq 0$.

Define $\hat{K}$ and $\hat{K}_\beta$ by (2.9). Then

$$||K_\beta|| \leq ||K|| \leq ||\hat{K}|| < 1, \quad ||K_\beta|| \leq ||\hat{K}_\beta|| \leq ||\hat{K}|| < 1. \tag{5.4}$$

For $f \in X^+$ let $|f|(s) = |f(s)|$. Then

$$|Kf| \leq \hat{K}|f|, \quad |K_\beta f| \leq \hat{K}_\beta|f|, \quad |(K - K_\beta)f| \leq (\hat{K} - \hat{K}_\beta)|f|. \tag{5.5}$$

By (2.5),

$$|(I - K)^{-1}f| \leq (I - \hat{K})^{-1}|f|, |(I - K_\beta)^{-1}f| \leq (I - \hat{K}_\beta)^{-1}|f|. \tag{5.6}$$

THEOREM 5.1. *The solutions x and x_β in* (5.1) *satisfy*

$$|x - x_\beta| \leq (I - \hat{K}_\beta)^{-1}(\hat{K} - \hat{K}_\beta)|x|, \tag{5.7}$$

$$(5.8) \qquad |x - x_\beta| \leq ||x||_{[\beta,\infty)} \hat{u}_\beta,$$

$$(5.9) \qquad ||x||_{[\beta,\infty)} \leq ||y||_{[\beta,\infty)} + \frac{||y||\,||\hat{K}1||_{[\beta,\infty)}}{1 - ||K||} \leq \frac{||y||}{1 - ||\hat{K}||},$$

$$(5.10) \qquad ||x - x_\beta||_{[0,\alpha]} \leq ||x||_{[\beta,\infty]} ||\hat{u}_\beta||_{[0,\alpha]} \quad \textit{for } \alpha \in \mathbf{R}^+,$$

$$(5.11) \qquad ||x - x_\beta||_{[0,\alpha]} \to 0 \quad \textit{as } \beta \to \infty \textit{ for } \alpha \in \mathbf{R}^+.$$

PROOF. The arguments parallel those for Theorem 4.2, with the use of (5.4)-(5.6). □

EXAMPLE 5.1. For $0 < a < \frac{1}{2}$, let

$$k(s,t) = a\ \sin(s-t)e^{-|s-t|}, \quad \hat{k}(s,t) = a\ e^{-|s-t|}.$$

Then $\hat{u}_\beta$ is given by (4.19) and $||x - x_\beta||_{[0,\alpha]}$ can be estimated numerically.

For a general Wiener-Hopf kernel $k(s,t) = \kappa(s-t)$, we can define $\hat{k}(s,t) = |\kappa(s-t)|$ and obtain numerical estimates for $||x - x_\beta||_{[0,\alpha]}$ with the aid of the bound for $||\hat{u}_\beta||_{[0,\alpha]}$ given by (4.24).

6. Uniform convergence on $\mathbf{R}^+$. As we have seen, the hypotheses H1 and H2 on the kernel $k(s,t)$ imply that the solutions of $(I-K)x = y$ and $(I - K_\beta)x_\beta = y$ satisfy $x_\beta(s) \to x(s)$ as $\beta \to \infty$, uniformly on finite intervals. More restrictive conditions on $k(s,t)$ will give uniform convergence on $\mathbf{R}^+$. Different notation is used in this case.

Define integral operators L and L_β on X^+ by

$$(6.1) \qquad Lf(s) = \int_0^\infty \ell(s,t)f(t)dt, \quad L_\beta f(s) = \int_0^\beta \ell(s,t)f(t)dt,$$

where $\ell(s,t)$ satisfies

$$(6.2) \qquad ||L - L_\beta|| = \sup_{s \geq 0} \int_\beta^\infty |\ell(s,t)|dt \to 0 \text{ as } \beta \to \infty$$

in addition to H1 and H2. Then $||L_\beta|| \leq ||L|| < 1$.

The conditions H1, H2 and (6.2) do not imply that L or L_β is compact. However, if $\ell(s,t)$ is also bounded and uniformly continuous (the latter can be weakened), then L and L_β are compact. See Anselone and Sloan [**3**] for details.

Assume that

$$(6.3) \qquad |\ell(s,t)| \leq \hat{\ell}(s,t),$$

where $\hat{\ell}(s,t)$ satisfies H1 and H2. Define $\hat{L}$ and $\hat{L}_\beta$ by

$$(6.4) \qquad \hat{L}f(s) = \int_0^\infty \hat{\ell}(s,t)f(t)dt, \quad \hat{L}_\beta f(s) = \int_0^\beta \hat{\ell}(s,t)f(t)dt.$$

Consider the equations

$$(6.5) \qquad (I - L)x = y, \quad (I - L_\beta)x_\beta = y.$$

THEOREM 6.1. *The solutions x and x_β in* (6.5) *satisfy*

$$(6.6) \qquad x - x_\beta = (I - L_\beta)^{-1}(L - L_\beta)x,$$

$$(6.7) \qquad ||x - x_\beta|| \leq \frac{||L - L_\beta||\, ||x||_{[\beta,\infty)}}{1 - ||L_\beta||} \to 0 \quad \text{as } \beta \to \infty,$$

$$(6.8) \qquad ||x||_{[\beta,\infty)} \leq ||y||_{[\beta,\infty)} + \frac{||y||\, ||\hat{L}1||_{[\beta,\infty)}}{1 - ||L||} \leq \frac{||y||}{1 - ||\hat{L}||}.$$

PROOF. First, (6.6) is a special case of (2.7). Then (6.7) follows easily. Finally, (5.9) implies (6.8). □

Since the bounds for $||x - x_\beta||$ involve only suprema and integrals of given functions, they are suitable for numerical calculation. For better accuracy, the choice $\hat{\ell}(s,t) = |\ell(s,t)|$ is recommended unless it makes the calculations difficult.

EXAMPLE 6.1. Let

$$\ell(s,t) = \frac{1}{\pi}\frac{\sin st}{s^2+t^2+1}, \qquad \hat{\ell}(s,t) = \frac{1}{\pi}\frac{1}{s^2+t^2+1}.$$

Then $||L|| < ||\hat{L}|| = \frac{1}{2}$ and

$$||L - L_\beta|| \leq \frac{1}{2} - \frac{1}{\pi}\text{arc tan } \beta \to 0 \text{ as } \beta \to \infty.$$

Since $\hat{\ell}(s,t)$ is decreasing in s, the first bound for $||x||_{[\beta,\infty)}$ in (6.8) is sharper than the second. This also follows from

$$\hat{L}1(s) = \frac{1}{2\sqrt{s^2+1}}, \qquad ||\hat{L}1||_{[\beta,\infty)} = \frac{1}{2\sqrt{\beta^2+1}} < ||\hat{L}||.$$

By (6.8), if $y(s) \to 0$ as $s \to \infty$, then $x(s) \to 0$ as $s \to \infty$. The operator L is compact and maps X_0^+ and into X_0^+, the subspace of X^+ consisting of the functions which vanish at infinity. See Sloan [**12**].

EXAMPLE 6.2. Let

$$\ell(s,t) = \frac{1}{2}e^{-t}\frac{s}{s+t+1}, \qquad \hat{\ell}(s,t) = \frac{1}{2}e^{-t}\frac{s}{s+1}.$$

Then $||L|| < ||\hat{L}|| = \frac{1}{2}$ and

$$||L - L_\beta|| \leq \frac{1}{2}e^{-\beta} \to 0 \text{ as } \beta \to \infty.$$

Since $\hat{\ell}(s,t)$ is increasing in s, $||\hat{L}1||_{[\beta,\infty)} = ||\hat{L}||$. This also follows from an easy calculation. If $||y||_{[\beta,\infty)} = ||y||$, e.g., if $y(s)$ is nondecreasing, then the two bounds for $||x||_{[\beta,\infty)}$ in (6.8) coincide. The operator L is compact and maps X_ℓ^+ into X_ℓ^+, the subspace of X^+ consisting of the functions with any finite limits at infinity. See Sloan [**12**].

7. Operator sums. Consider an operator sum $K+L$ with K as in §5 and L as in §6. Define $\hat{K}$ and $\hat{L}$ as before. We impose the additional condition

$$||\hat{K}+\hat{L}|| < 1. \tag{7.1}$$

Then $||K_\beta + L_\beta|| \le ||K+L|| \le ||\hat{K}+\hat{L}|| < 1$.

We shall compare solutions of the equations

$$(I-K-L)x = y, \quad (I-K_\beta-L_\beta)x_\beta = y. \tag{7.2}$$

The following theorem generalizes Theorems 4.2, 5.1 and 6.1.

THEOREM 7.1. *The solutions x and x_β in* (7.2) *satisfy*

$$x - x_\beta = (I-K_\beta-L_\beta)^{-1}(K-K_\beta+L-L_\beta)x, \tag{7.3}$$

$$|x-x_\beta| \le ||x||_{[\beta,\infty)}(\hat{u}_\beta + \Delta_\beta), \tag{7.4}$$

$$||x||_{[\beta,\infty)} \le ||y||_{[\beta,\infty)} + \frac{||y||\,||(\hat{K}+\hat{L})1||_{[\beta,\infty)}}{1-||K+L||} \le \frac{||y||}{1-||\hat{K}+\hat{L}||}, \tag{7.5}$$

$$\Delta_\beta = \frac{||L_\beta \hat{u}_\beta|| + ||L-L_\beta||}{1-||K_\beta+L_\beta||} \to 0 \quad as\ \beta\to\infty, \tag{7.6}$$

$$||x-x_\beta||_{[0,\alpha]} \le ||x||_{[\beta,\infty)}(||\hat{u}_\beta||_{[0,\alpha]} + \Delta_\beta), \tag{7.7}$$

$$||x-x_\beta||_{[0,\alpha]} \to 0 \quad as\ \beta\to\infty \quad for\ \alpha\in\mathbf{R}^+. \tag{7.8}$$

PROOF. First, (7.3) and (7.5) are special cases of (2.7) and (3.4). By (7.3),

$$\begin{aligned} x - x_\beta &= (I-K_\beta-L_\beta)^{-1}(K-K_\beta)x \\ &\quad + (I-K_\beta-L_\beta)^{-1}(L-L_\beta)x. \end{aligned}$$

The standard identity $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ yields

$$(I - K_\beta - L_\beta)^{-1} = (I - K_\beta)^{-1} + (I - K_\beta - L_\beta)^{-1} L_\beta (I - K_\beta)^{-1}.$$

Therefore,

$$\begin{aligned}&(I - K_\beta - L_\beta)^{-1}(K - K_\beta)\\ &= (I - K_\beta)^{-1}(K - K_\beta) + (I - K_\beta - L_\beta)^{-1} L_\beta (I - K_\beta)^{-1}(K - K_\beta).\end{aligned}$$

It follows that

$$\begin{aligned}x - x_\beta = {}& (I - K_\beta)^{-1}(K - K_\beta)x\\ &+ (I - K_\beta - L_\beta)^{-1} L_\beta (I - K_\beta)^{-1}(K - K_\beta)x\\ &+ (I - K_\beta - L_\beta)^{-1}(L - L_\beta)x.\end{aligned}$$

An adaptation of (5.8) gives

$$|(I - K_\beta)^{-1}(K - K_\beta)x| \leq ||x||_{[\beta,\infty)} \hat{u}_\beta.$$

Hence,

$$||(I - K_\beta - L_\beta)^{-1} L_\beta (I - K_\beta)^{-1}(K - K_\beta)x|| \leq \frac{||x||_{[\beta,\infty)} ||L_\beta \hat{u}_\beta||}{1 - ||K_\beta + L_\beta||}.$$

By similar reasoning,

$$||(I - K_\beta - L_\beta)^{-1}(L - L_\beta)x|| \leq \frac{||x||_{[\beta,\infty)} ||L - L_\beta||}{1 - ||K_\beta + L_\beta||}.$$

These results imply (7.4) and (7.7) with Δ_β defined by (7.6). We prove next that $\Delta_\beta \to 0$ as $\beta \to \infty$. By the triangle inequality,

$$\begin{aligned}&||L_\beta \hat{u}_\beta|| \leq ||L_\alpha \hat{u}_\beta|| + ||(L_\beta - L_\alpha)\hat{u}_\beta||,\\ &||L_\beta \hat{u}_\beta|| \leq ||L||\,||\hat{u}_\beta||_{[0,\alpha]} + ||L_\beta - L_\alpha||\,||\hat{u}_\beta||.\end{aligned} \tag{7.9}$$

By (4.9), (4.11) and (6.2),

$$\begin{gathered}||\hat{u}_\beta|| \leq \frac{||\hat{K}||}{1 - ||\hat{K}||},\\ ||\hat{u}_\beta||_{[0,\alpha]} \to 0 \text{ as } \beta \to \infty \text{ for } \alpha \in \mathbf{R}^+,\end{gathered}$$

$$||L_\beta - L_\alpha|| \le ||L - L_\alpha|| \to 0 \quad \text{as } \alpha \to \infty.$$

Fix $\varepsilon > 0$. Choose $\alpha = \alpha(\varepsilon)$ such that

$$||L|| \, ||\hat{u}_\beta|| < \frac{\varepsilon}{2} \quad \text{for } \beta \ge \beta(\varepsilon).$$

There exists $\beta(\varepsilon) \ge \alpha(\varepsilon)$ such that

$$||L|| \quad ||\hat{u}_\beta|| < \frac{\varepsilon}{2} \quad \text{for } \beta \ge \beta(\varepsilon).$$

Now (7.9) yields

$$\begin{aligned} &||L_\beta \hat{u}_\beta|| < \varepsilon \quad \text{for } \beta \le \beta(\varepsilon), \\ &||L_\beta \hat{u}_\beta|| \to 0 \quad \text{as } \beta \to \infty. \end{aligned} \tag{7.10}$$

By (6.2),

$$||L - L_\beta|| \to 0 \quad \text{as } \beta \to \infty. \tag{7.11}$$

Since $||K_\beta + L_\beta|| \le ||\hat{K} + \hat{L}|| < 1$,

$$\frac{1}{1 - ||K_\beta + L_\beta||} \le \frac{1}{1 - ||\hat{K} + \hat{L}||}. \tag{7.12}$$

Now (7.10)-(7.12) imply that $\Delta_\beta \to 0$ in (7.6). Finally, (4.11), (7.6) and (7.7) yield (7.8). □

In the bounds for $|x - x_\beta|$, the quantities $\hat{u}_\beta$ and $||L_\beta \hat{u}_\beta||$ could present numerical difficulties. As for $\hat{u}_\beta$ the remarks in §4 apply here as well. If $\hat{k}(s,t)$ is a Picard kernel, then $\hat{u}_\beta$ is given by (4.19). If $\hat{k}(s,t)$ is any Wiener-Hopf kernel, then (4.24) estimates $\hat{u}_\beta$. As for $||L_\beta \hat{u}_\beta||$, it might be possible to estimate it directly in the Picard case. In the general Wiener-Hopf case, (7.9) and (4.24) could be exploited.

8. Related bounds. Return to the setting of §5. Thus, K and K_β are integral operators with a kernel $k(s,t)$ which satisfies H1 and H2. Again consider the equations

$$(I - K)x = y, \quad (I - K_\beta)x_\beta = y. \tag{8.1}$$

Bounds for $|x - x_\beta|$ are given in Theorem 5.1.

Another bound for $|x - x_\beta|$ comes from $x = y + Kx$ and

$$x - x_\beta = (I - K_\beta)^{-1}(K - K_\beta)x = (I - K_\beta)^{-1}(K - K_\beta)y + (I - K_\beta)^{-1}(K - K_\beta)Kx.$$

It follows, by now familiar arguments, that

$$(8.2) \qquad |x - x_\beta| \le ||y||_{[\beta,\infty)}\hat{u}_\beta + \frac{||y||(I - \hat{K}_\beta)^{-1}(\hat{K} - \hat{K}_\beta)\hat{K}1}{1 - ||K||}.$$

The bound for $|x - x_\beta|$ from Theorem 5.1 can be expressed in a similar form. By (5.8), (5.9) and (4.7),

$$(8.3) \qquad |x - x_\beta| \le ||y||_{[\beta,\infty)}\hat{u}_\beta + \frac{||y||\,||\hat{K}1||_{[\beta,\infty)}}{1 - ||K||}\hat{u}_\beta,$$

$$(8.4) \quad |x - x_\beta| \le ||y||_{[\beta,\infty)}\hat{u}_\beta + \frac{||y||(I - \hat{K}_\beta)^{-1}||\hat{K}1||_{[\beta,\infty)}(\hat{K} - \hat{K}_\beta)1}{1 - ||K||}.$$

The right members of (8.3) and (8.4) are equal.

Since

$$(8.5) \qquad (\hat{K} - \hat{K}_\beta)\hat{K}1 \le ||\hat{K}1||_{[\beta,\infty)}(\hat{K} - \hat{K}_\beta)1,$$

(8.2) implies (8.3). However it is likely to be more difficult to compute the bound in (8.2) because $(\hat{K} - \hat{K}_\beta)\hat{K}1$ is a double integral.

Moreover, as we shall show, (8.2) gives only a marginal improvement over (8.3) in the important case of a Wiener-Hopf kernel. Let

$$\hat{k}(s,t) = \hat{\kappa}(s - t), \quad \hat{\kappa} \in L^1(\mathbf{R}), \quad ||\hat{\kappa}||_1 < 1, \quad \hat{\kappa} \ge 0.$$

Then

$$\hat{K}f(s) = \int_0^\infty \hat{\kappa}(s - t)f(t)dt, \quad \hat{K}_\beta f(s) = \int_0^\beta \hat{\kappa}(s - t)f(t)dt,$$

and $||\hat{K}_\beta|| = ||\hat{K}|| = ||\hat{\kappa}||_1 < 1$. To compare (8.2) and (8.3), it suffices to compare the two members of (8.5). By (4.22), $||\hat{K}1||_{[\beta,\infty)} = ||\hat{K}||$. Therefore, (8.5) is now equivalent to

$$(8.6) \qquad (\hat{K} - \hat{K}_\beta)\hat{K}1 \le ||\hat{K}||(\hat{K} - \hat{K}_\beta)1.$$

The right member of (8.6) is given by

$$(8.7)\quad ||\hat{K}||(\hat{K}-\hat{K}_\beta)1(s) = ||\hat{K}||\int_\beta^\infty \hat{\kappa}(s-t)dt = ||\hat{K}||\int_{-\infty}^{s-\beta} \kappa(u)du,$$

which goes to zero as $\beta \to \infty$. Since

$$(8.8)\quad \hat{K}1(s) = \int_0^\infty \hat{\kappa}(s-t)dt = \int_{-\infty}^{s} \hat{\kappa}(u)du = ||\hat{K}|| - \int_s^\infty \hat{\kappa}(u)du,$$

the left member of (8.6) can be expressed by

$$(8.9)\quad \begin{aligned} &(\hat{K}-\hat{K}_\beta)\hat{K}1(s) \\ &= ||\hat{K}||(\hat{K}-\hat{K}_\beta)1(s) - \int_\beta^\infty \hat{\kappa}(s-t)\int_t^\infty \hat{\kappa}(u)du\,dt. \end{aligned}$$

Therefore the difference of the two members of (8.6) satisfies

$$(8.10)\quad \begin{aligned} &||\hat{K}||(\hat{K}-\hat{K}_\beta)1(s) - (\hat{K}-\hat{K}_\beta)\hat{K}1(s) \\ &= \int_\beta^\infty \hat{\kappa}(s-t)\int_t^\infty \hat{\kappa}(u)du\,dt = \int_{-\infty}^{s-\beta} \hat{\kappa}(r)\int_{s-r}^\infty \hat{\kappa}(u)du\,dr \\ &\le \int_{-\infty}^{s-\beta} \hat{\kappa}(r)dr \int_\beta^\infty \hat{\kappa}(u)du, \end{aligned}$$

where *both* of the last two integrals go to zero as $\beta \to \infty$. Thus, for large β, the difference of the two members of (8.6) is small compared with either member. As a consequence, the bounds for $|x - x_\beta|$ given by (8.2) and (8.3) are comparably accurate for large β. Since (8.3) is easier to implement, it is preferable, at least in the Wiener-Hopf case.

For example, let $\hat{k}(s,t)$ be the Picard kernel

$$\hat{k}(s,t) = a\,e^{-|s-t|}, \quad 0 < a < \frac{1}{2}.$$

Then the right member of (8.6) is

$$||\hat{K}||(\hat{K}-\hat{K}_\beta)1(s) = 2a^2 e^{s-\beta} \quad \text{for } 0 \le s \le \beta.$$

The left member is

$$(\hat{K} - \hat{K}_\beta)\hat{K}1(s) = 2a^2e^{s-\beta} - \frac{1}{2}a^2e^{s-2\beta} \quad \text{for } 0 \le s \le \beta.$$

Therefore, the difference of the two sides of (8.6) is

$$||\hat{K}||(\hat{K} - \hat{K}_\beta)1(s) - (\hat{K} - \hat{K}_\beta)\hat{K}1(s) = \frac{1}{2}a^2e^{s-2\beta} \quad \text{for } 0 \le s \le \beta,$$

which goes to zero faster than either member of (8.6) as $\beta \to \infty$ with s fixed or restricted to a finite interval.

Atkinson [**5**] derived an analogue of (8.2) in a wider context. He obtained a bound for $|x - x_{\alpha\beta}|$, where

$$x_{\alpha\beta}(s) - \int_\alpha^\beta k(s,t)x_{\alpha\beta}(t)dt = y(s).$$

His bound reduces to (8.2) when $\alpha = 0$.

Our analysis can be extended to the more general setting studies by Atkinson.

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