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AN ALTERNATIVE APPROACH TO ILL-POSED PROBLEMS

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ABSTRACT. An approach to ill-posed problems is presented in which the domain of the operator is enlarged rather than the range restricted. A topology is then introduced which makes the inverse operator continuous. This leads to a regularization procedure based on analytic representations. A number of examples are presented as well.

1. Introduction. In one standard kind of ill-posed problem a linear operator T with non-closed range must be inverted in order to solve an equation of the form

$$Tf = g. \tag{1.1}$$

The difficulty arises when g belongs to the closure of the range but not to the range itself. Then (1.1) has no exact solution but has at best only an approximate solution. Even this approximate solution may not be adequate in that it may not be close to an exact solution because of the lack of continuity of the operator T^{-1}.

In a number of problems important in applications, T is an integral operator and the problem (1.1) is one of solving an integral equation of the first kind.

Many procedures have been proposed and used in resolving such problems. They are concisely summarized in the recent article by Nashed [**3**]; a more detailed exposition may be found in the book by Tikhonov and Arsenin [**8**].

We shall not review the current literature on the subject but merely remark that many of the standard approaches involve restricting the range of the operator. In particular, in Tikhonov's regularization method, the range is restricted to the image of a compact set. In

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the reproducing kernel Hilbert space (RKHS) approach of Nashed and Wahba [**4**], the range is restricted to this RKHS.

In this work we shall take a different approach. Rather than restricting the range, we shall enlarge the domain. We shall introduce a topology on both the range and the domain such that the inverse exists and is continuous. This converts the ill-posed problem into a well-posed one but at the expense of difficulty in interpreting the "solution." We shall assume that the equation is an integral equation in $L^2(a,b)$ whose kernel is Hilbert-Schmidt, i.e., that (1.1) has the form
(1.2)
$$g(x) = \int_a^b K(x,t)f(t)dt, \quad x \in (a,b), \; f,g \in L^2(a,b), \; K \in L^2(a,b)^2.$$

We shall also assume that the operator is 1-1 and self-adjoint.

The extension to the larger space is accomplished in two stages. We first restrict the problem to a subspace A of L^2 on which it is well-posed (§2) and then extend the problem to the conjugate space (§3). In §4, we present a regularization procedure based on analytic representations which avoids calculation of the eigenfunctions and apply this to some examples.

2. Construction of A, a linear topological space. We shall construct a sequence $\{A_n\}$ of Hilbert spaces associated with the operator T. and then take their intersection to obtain a linear topological space on which T^{-1} is continuous.

DEFINITION 2.1. Let $A_0 = L^2(a,b)$; then A_n, $n = 1,2,\ldots$, is given by the set
$$A_n = \{T^n f | f \in L^2(a,b)\}$$

and its topology determined by the inner product
$$\langle \phi, \psi \rangle_n = \int_a^b T^{-n}\phi\overline{T^{-n}\psi}.$$

We denote by $|| \; ||_n$ the corresponding norm.

REMARK 2.1. A_n is merely $T^n(L^2)$ with the induced topology. Clearly we have $A_0 \supset A_1 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots$, and since T is a compact operator, the unit ball in A_{n+1} is compact in the topology of each of $A_0, A_1, \ldots, A_n$.

DEFINITION 2.2. Let $A = \cap_{n=0}^{\infty} A_n$ with the natural topology; i.e., a neighborhood of 0 in A is a finite intersection of sets $U_{\varepsilon,n} = \{\phi \in A | \, \|\phi\|_i < \varepsilon, i = 0, 1, \ldots, n\}$.

PROPOSITION 2.1. *A is a complete countably normed space and hence a Fréchet space. Its metric may be given by*

$$d(\phi, \psi) = \sum_{n=0}^{\infty} 2^{-n-1} ||\phi - \psi||_n / (1 + ||\phi - \psi||_n).$$

PROOF. Each A_n is complete with respect to $\| \; \|_n$. Since the norms are in concordance because T is 1-1, A is a countably normed space. See Friedman [**1**, p. 7] for the remaining statements. □0

We now turn to the operator T and its spectrum. Since T is self adjoint and compact, it has a discrete real spectrum. We assume the eigenvalues $\{\lambda_k\}$ are arranged in order of decreasing magnitude. Since T is 1-1 it has no zero eigenvalues, but of course $\sum_{k=0}^{\infty} \lambda_k^2 < \infty$ since T was a $H - S$ operator on L^2.

REMARK 2.2. The expansion of an element $\phi \in A_n$ with respect to the eigenfunction $\{\phi_k\}$ is given by

$$\begin{aligned} \phi &= \sum_{k=0}^{\infty} \langle \phi, \phi_k \rangle_0 \phi_k = \sum \langle T^n f, \phi_k \rangle_0 \phi_k \\ &= \sum \langle f, T^n \phi_k \rangle_0 \phi_k = \sum \lambda_k^n \langle f, \phi_k \rangle_0 \phi_k. \end{aligned} \tag{2.1}$$

PROPOSITION 2.2. *Let $\phi \in A_n$ (resp. A); then the expansion coefficients satisfy*

$$\sum (\langle \phi, \phi_k \rangle_0 \lambda_k^{-n})^2 < \infty \quad (\textit{resp. } \langle \phi, \phi_k \rangle_0 = O(\lambda_k^n), \; n = 1, 2, \ldots) \tag{2.2}$$

and the expansion (2.1) *converges to* ϕ *in the sense of* A_n (*resp.* A). *Conversely, if* $\{a_k\}$ *satisfies* (2.2), *then* $\sum a_k\phi_k$ *converges to some* $\phi \in A_n$ (resp. A).

PROOF. The statement about the coefficients is clearly true. The convergence follows from (2.1) since

$$f = T^{-n}\phi = \sum \langle f, \phi_k\rangle_0 \phi_k = \sum T^{-n}\lambda_k^n \langle f, \phi_k\rangle_0 \phi_k.$$

The expansion of f converges in $L^2 (= A_0)$. Hence the convergence of the expansion of ϕ to ϕ also occurs.

Convergence in A follows from convergence in the sense of each A_n. □0

PROPOSITION 2.3. *The problem* $Tf = g$ *has a unique solution for* $g \in A$ *which depends continuously on* g.

REMARK 2.3. *On* A *the problem is well-posed.*

PROOF. If $g = \sum b_k\phi_k$, then $f = \sum (b_k/\lambda_k)\phi_k$ is clearly a solution, at least formally. By Proposition 2.2, $b_k = O(\lambda_k^n)$ for each integer n. Since the same is true of b_k/λ_k, and since the series must therefore converge in A, $f \in A$, and is unique since T is 1-1.

In order to show continuity of the inverse we assume $g_m \to g$ in the sense of A and show that $f_m \to f$ in A where $Tf_m = g_m$. If $g_m \to g$, then $\|g_m - g\|_n \to 0$ for each $n \geq 1$; i.e., $\|T^{-n}g_m - T^{-n}g\|_0 \to 0$. Hence $\|T^{-n+1}f_m - T^{-n+1}f\|_0 = \|f_m - f\|_{n-1}$ for $n - 1 \geq 0$, and hence $f_m \to f$ in A. □0

3. Dual spaces. The dual space of a Hilbert space is of course isomorphic to itself when linear functionals are taken to have values given by the inner product. This is true for our spaces A_n. However, each $A_n \subset L^2$, and hence has an L^2 inner product as well. For each pair $f, \phi \in A_n$, $\langle f, \phi\rangle_0$ determines a linear functional f on A_n. In fact, even if $f \in L^2$, this will be a linear functional on A_n. More generally still, any expression of the form

$$F_n(\phi) = \langle f, T^{-n}\phi\rangle_0, \tag{3.1}$$

where $f \in L^2$, will be a continuous linear functional on A_n. We denote F_n symbolically as $F_n = T^{-n} f$.

REMARK 3.1. It is easy to show that any continuous linear functional F on A_n has a representation given by (3.1). Indeed, since A_n is dense in $A_0 = L^2$, and is first countable, F has a continuous extension to L^2 and hence $F(g) = \langle f, g\rangle_0$ for some f and all $g \in L^2$. Since each $g \in L^2$ has the form $g = T^n \phi$ we obtain (3.1).

DEFINITION 3.1. Let A_{-n} denote A_n' with inner product given by

$$\langle F, G\rangle_{-n} = \langle T^{-n} f, T^{-n} g\rangle_{-n} = \langle f, g\rangle_0$$

PROPOSITION 3.1. *A_{-n} is a Hilbert space and each $F \in A_{-n}$ has an eigenfunction expansion*

$$F = \sum_{k=0}^{\infty} a_k \phi_k, \quad a_k = \langle f, T^{-n} \phi_k\rangle_0, \quad f = T^n f$$

satisfying

$$\sum a_k^2 \lambda_k^{2n} < \infty.$$

REMARK 3.2. The dual space A' of the countably normed space A contains each of the spaces A_{-n}. It itself is not a Hilbert space but is complete with respect to the weak topology. Moreover, since A is a perfect space (see [**1**, p. 15]), bounded sets in A' are (sequentially) compact in both the weak and strong topologies. Each element $F \in A'$ has an expansion coefficient given by

$$d_k = F(\phi_k) = O(\lambda_k^{-p}) \text{ for some } p \geq 0.$$

REMARK 3.3. Under the additional assumption that the kernel $K(x, y)$ is a continuous function, the space A_{-1} contains the point measures δ_x for each $x \in (a, b)$. Hence, the space A' is a space of generalized functions. (see [**1**, p. 28] or [**2**].)

We now return to our ill-posed problem $Tf = g$ for $g \in L^2$. Since $L^2 \subset A'$ we can interpret this as a problem in A'. In this case it becomes well-posed since, for each $G \in A'$, there is an $F \in A'$ which satisfied the equation and depends continuously on G.

Indeed $F(\phi) = G(T^{-1}\phi)$ satisfied the equation formally. But T has an inverse on A, and hence $T^{-1}\phi$ exists. Also, if $G_n \to 0$ in A', then $G_n(T^{-1}\phi) = F_n(\phi) \to 0$, and hence $F_n \to 0$ in A'.

The solution to the problem with $g \in L^2$ is of course an element of A_{-1} but may not itself be a function. However it may be approximated by finite partial sums of its eigenfunction expansion. We have

$$F(\phi) = \sum_{k=0}^{\infty} \frac{b_k \alpha_k}{\lambda_k} \tag{3.2}$$

where $\alpha_k = \langle \phi, \phi_k \rangle$. This series converges very rapidly since α_k and hence $b_k \alpha_k = O(\lambda_k^n)$ for each integer n. For the same reason there is no difficulty in dividing by λ_k.

However, from a practical point of view, we are often interested in a solution which is itself a function. One approach is to choose the value in the unit ball of L^2 closest to F.

PROPOSITION 3.2. *Let $G \in A_{-n}$ for some $n \geq 1$, then there exists a unique $f \in L^2$ such that $||f|| \leq 1$ and*

$$||Tf - G||_{-n}$$

is minimized.

The proof is clear when we observe that the unit ball in L^2 is compact in the norm of A_{-n}. This corresponds to a quasi-solution [**6**, p. 35].

This proposition may be applied when G is a point mass. For each δ_x there exists an f_x in the unit ball of L^2 such that Tf is the best approximation to δ_x in A_{-1}. However, the best approximation to $\sum \alpha_i \delta_{x_i}$ is not necessarily given by $T \sum \alpha_i f_{x_i}$, since the best approximation operator is not linear. Nevertheless, this will still be a fairly good approximation since

$$||\sum_i \alpha_i \delta_{x_i} - T \sum_i \alpha_i f_{x_i}||_{-1} \leq \sum_i |\alpha_i| \, ||\delta_{x_i} - T f_{x_i}||_{-1}. \tag{3.3}$$

Such linear combinations are used to represent the outcomes of experiments in which a quantity α_i is measured at time (or location) x_i.

The function Tf_x may be considered to be an approximation in the sense of A_{-1} to δ_x. We can get a closer approximation by using a larger ball instead of the unit ball in L^2, and in fact a sequence $[\delta_{n,x}]$ of functions in A_1 which converge to δ_x in the sense of A_{-1}. This is one example that satisfies

DEFINITION 3.2. A sequence (family) $\{\delta_n(x,y)\}_{n=0}^{\infty}(\{\delta_\alpha(x,y)\}, \alpha \in \Lambda)$ of functions in $L^2(a,b) \times (a,b)$ is a *delta-sequence* (*delta-family*) of level $m, m = 0, 1, \ldots,$ if

(3.4)

(i) $\int_a^b \delta_n(x,y)\phi(y)dy \in A_m$ for each $\phi \in A_m$,

(ii) $\int_a^b \delta_n(x,y)\phi(y)dy \to \phi(x)$ in the sense of A_m, as $n \to \infty$,

(iii) $\delta_n(x,y) \to \delta(x-y) = \delta_x(y)$ for x fixed in the sense of A_{-1}, as $n \to \infty$, (and similarly for a family as $\alpha \to \alpha_0$).

Some examples of delta-sequences are:

(i) The partial sums of the expansion of $\delta(x-y)$ constitute a delta sequence given by

$$\delta_n(x,y) = \sum_{k=1}^{n} \phi_k(x)\phi_k(y) \tag{3.5}$$

which does belong to A_m for each $m \geq 0$ and hence belongs to A. Here, $\{\phi_k\}$ are the eigenfunctions of the operator T.

(ii) A delta family may be obtained by minimizing

$$||\delta - Tf||_{-1}^2 + \alpha||f||_m^2$$

and then operating with T. This gives us

$$\delta_\alpha(x,y) = \sum_{n=0}^{\infty} \frac{\lambda_n^{2m+4}\phi_n(x)}{\lambda_n^{2m+4} + \alpha}\phi_n(y). \tag{3.6}$$

(iii) Still another delta family may be obtained from the formula

$$\delta_\varepsilon(x,y) = \frac{1}{\pi}\frac{\varepsilon}{(x-y)^2+\varepsilon^2}, \quad \varepsilon > 0. \tag{3.7}$$

This δ_ε is not necessarily in A_m and may have to be modified to meet this requirement. It is related to the analytic representation of functions which we shall explore in the next section. The following indicates how the delta-sequences may be used.

PROPOSITION 3.2. *Let $\{\delta_n\}$ be a delta sequence of level $m \geq 1$. Then the problem*

$$Th = g,$$

where $g = \sum \alpha_i \delta(x - x_i)$ has an approximate solution given by

$$h_n = \sum \alpha_i T^{-1}\delta_n(x_i, x)$$

and $T\ h_n \to \sum \alpha_i \delta(x - x_i)$ in the sense of A_{-1}.

4. Analytic representations. In the previous sections the techniques involved the eigenfunctions $\{\phi_k\}$ of the operator T. However, finding the ϕ_k is often as difficult as solving the problem. Hence, in this section, we consider an approach to the problem which avoids eigenfunctions but does require some additional assumptions about the kernel $K(x, y)$. The approach uses an "analytic representation" of g, i.e., a pair of functions defined respectively in the upper and lower complex half plane whose "jump" across the real axis gives g. For simplicity we also assume our interval $[a, b]$ to be $[0, 1]$.

ASSUMPTION 4.1. Associated with the continuous kernel $K(x, y)$ is a degenerate kernel $K_m(x, z)$ continuous on $[0, 1] \times X$ where S is a region of $\mathbf{C}$ containing $(0, 1)$ such that

a) $(x - z)^{-1} - K_m(x, z) \in A_1, \ \operatorname{Im} z \neq 0.$

b) $T(K_m(x, \cdot + i\varepsilon) - K_m(s, \cdot - i\varepsilon))(x) \to 0$ in L^2 as $\varepsilon \to 0.$

c) $(x - z)^{-1} - K_m(x, z) = \int_0^1 K(x, y)G(y - z)dy, \ \operatorname{Im} z \neq 0.$

where $G(x \pm i\varepsilon)$ is bounded in x for $\varepsilon > 0$.

DEFINITION 4.1. Let $g \in L^2(0,1)$; then an analytic representation $\hat{g}(z)$ is given by

$$\hat{g}(z) = \frac{1}{2\pi i}\int_0^1 \frac{g(x)}{x-z}dx, \quad \mathrm{Im}\, z \neq 0,$$

and a modified representation $\hat{g}_m(z)$ by

$$\hat{g}_m(z) = \frac{1}{2\pi i}\int_0^1 g(x)\Big(\frac{1}{x-z} - K_m(\mathrm{Re}\, z; x - i\mathrm{Im}\, z)\Big)dx, \ \mathrm{Im}\ z \neq 0.$$

THEOREM 4.1. *Let $K(x,y)$ be an $H-S$ kernel which satisfies Assumption* 4.1 *and let $g \in L^2$: then, for each $\varepsilon > 0$, there is an $f_\varepsilon \in L^2$ such that*

(i) $\hat{g}(x+i\varepsilon) - \hat{g}(x-i\varepsilon) = (Tf_\varepsilon(x)$
and an $e_t \in L^2$ such that

(ii) *if $g \in A_1$, then $\hat{g}(x+i\varepsilon) - \hat{g}(x-i\varepsilon) - (Te_\varepsilon)(x) \to g(x)$ as $\varepsilon \to 0$ in the sense of A_1 and $Te_\varepsilon \to 0$ in L^2.*

Before we present the proof we consider an example which contains all the essential ideas.

EXAMPLE 1. Let

$$K(x,y) = \begin{cases} x(1-y), \ x < y \\ \\ y(1-x), \ y < x \end{cases} \quad x, y \in (0,1),$$

Then we may calculate that

$$\frac{1}{x-z} = \int_0^1 K(x,y)\frac{2}{(z-y)^3}dy + \frac{x-1}{z} + \frac{x}{1-z} \tag{4.1}$$

by using the fact that $K(x,y)$ is a Green's functions of $-D^2$. Hence

$$K_2(x,z) = \frac{x-1}{z} + \frac{x}{1-z} = \psi_1(x)\phi_1(z) + \psi_2(x)\phi_2(z),$$

and

$$\hat{g}_2(z) = \frac{1}{2\pi i}\int_0^1 g(x)\Big(\frac{1}{x-z} + \frac{\operatorname{Re} z - 1}{x - i\operatorname{Im} z} + \frac{\operatorname{Re} z}{1 - x + i\operatorname{Im} z}\Big)dy.$$

We may interchange the role of x and $\operatorname{Re} z = s$ in (4.1) to obtain ($\operatorname{Im} z = t$)

(4.2)

$$\begin{aligned}\frac{1}{x-s-it} &= \frac{-1}{s-x+it} \\ &= -\int_0^1 K(x,y)\frac{2}{(x-it-y)^3}dy \; - \frac{s-1}{x-it} - \frac{s}{1-x+it}\end{aligned}$$

which we then substitute in the expression for $\hat{g}(z)$ to obtain

$$\begin{aligned}\hat{g}_2(z) &= \frac{-1}{2\pi i}\int_0^1 g(x)\int_0^1 K(s,y)\frac{2}{(x-it-y)^3}dy\,dx \\ &= -\frac{1}{2\pi i}\int_0^1 K(s,y)\int_0^1 \frac{2g(x)}{(x-it-y)^3}dx\,dy \qquad (4.3)\\ &= T\Big\{\frac{-1}{\pi i}\int_0^1 \frac{g(x)}{(x-it-y)^3}dx\Big\}(s), \quad (z = s + (s+it)).\end{aligned}$$

Hence, by defining f_ε by

$$f_\varepsilon(y) = -\frac{1}{\pi i}\int_0^1 g(x)\Big(\frac{1}{(x-i\varepsilon-y)^3} - \frac{1}{(x+i\varepsilon-y)^3}\Big)dx,$$

we see that

$$\hat{g}_2(x+i\varepsilon) - \hat{g}_2(x-i\varepsilon) = (Tf_\varepsilon)(x),$$

thus illustrating part (i) of the theorem.

To see that (ii) holds in this case, we note that, for $g \in A_1$, we may obtain another expression for f_ε.

(4.4)

$$\begin{aligned}\hat{g}_2(z) &= T\Big(\frac{-1}{\pi i}\int_0^1 \frac{\int_0^1 K(x,r)f(r)dr}{(x-it-y)^3}dx\Big)(s) \\ &= T\Big(\frac{1}{2\pi i}\int_0^1 f(r)\int_0^1 \frac{K(r,x)}{(y-s+it)^3}dx\,dr\Big)(s). \\ &= T\Big(\frac{1}{2\pi i}\int_0^1 f(r)\Big(\frac{1}{r-y-it} - \frac{r-1}{y+it} - \frac{r}{1-y-it}\Big)dr\Big)(s), \\ &\qquad (z = s+it).\end{aligned}$$

It remains to be shown that f_ε converges to f in the sense of L^2 as $\varepsilon \to 0$. Rather than work out this special case, we shall show that the conclusion holds in general and hence in this case.

PROOF OF THEOREM. The proof of part (i) is the same as in the example. It requires only that we obtain an expression similar to (4.3). Since the only operations needed involved the symmetry of $x - z$ and interchange of order of integration, we may conclude that (4.3) in the form

$$\hat{g}_m(z) = \frac{-1}{2\pi i} \int_0^1 K(s,y) \int_0^1 g(x)G(x - it - y)dx\, dy \tag{4.5}$$

holds in general.

LEMMA. *Let $g \in L^2(0,1)$, and $P_\varepsilon(x) = \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}$. Then*

(i) $(g^*P_\varepsilon) \to g^*$ in $L^2(-\infty,\infty)$ where $g^*(x) = \begin{cases} g(x), 0 \le x \le 1, \\ 0, o.w. \end{cases}$ and $(g^*P_\varepsilon)(x) = \int_0^1 g(t)P_\varepsilon(x-t)dt$;

(ii) $(g^*P_\varepsilon)(x) \to g^*(x)$ pointwise as each point of continuity of g^*.

This lemma is well known since $P_\varepsilon(x)$ is the Poisson kernel on $(-\infty,\infty)$. A proof may be found in [**5**].

Returning to $\hat{g}_m$ we see that, for $g \in A$, we have, by calculations similar to (4.4),

$$\hat{g}_m(s+i\varepsilon) = \int_0^1 K(s,y)\Big(\frac{1}{2\pi i}\int_0^1 f(r)\Big(\frac{i}{r-y-i\varepsilon} - K_m(r,y+i\varepsilon\Big)dr\Big)dy. \tag{4.6}$$

Hence we have

$$\begin{aligned} \hat{g}_m(s+i\varepsilon) - \hat{g}_m(s-i\varepsilon) = \int_0^1 K(s,y)\Big(&(f^*P_\varepsilon)(y) \\ &- \frac{1}{2\pi i}\int_0^1 f(r)(K_m(r,y+i\varepsilon) \\ &- K_m(r,y-i\varepsilon))dr\Big)dr \\ = T(f^*P_\varepsilon)(s) &+ T(e_\varepsilon)(s). \end{aligned} \tag{4.7}$$

By the lemma, $f^* P_\varepsilon \to f$ in L^2 as $\varepsilon \to 0$, while, by the assumption b, $Te_\varepsilon \to 0$ on L^2. Hence conclusion (ii) follows. □0

EXAMPLE 2. Let $K(x, y)$ be the Green's function of an m-th order linear differential operator $P(D)$ on $(0, 1)$ with boundary operator $B_i(f) = 0$, $i = 1, 2, \ldots, m$. Then the procedure of Example 1 may be followed by setting

$$\frac{1}{x - z} = \int_0^1 K(x, y) P(D) \frac{1}{y - z} dy + K_m(x, z), \tag{4.8}$$

where $K_m(x, z)$ is chosen such that

$$B_i\Big(\frac{1}{x - z}\Big) = B_i(K_m(x, z)), \quad i = 1, 2, \ldots, m. \tag{4.9}$$

This may be accomplished by setting $\psi_i(z)$ to

$$\psi_i(z) = B_i\Big(\frac{1}{x - z}\Big)$$

and then choosing $\phi_i(x)$ to be polynomials of degree $< m$ such that

$$B_j(\phi_i) = \delta_{ij}.$$

The degenerate kernel $K_m(x, z)$ is then taken to be

$$K_m(x, z) = \sum_{i=1}^{m} \phi_i(x) \psi_i(z).$$

A modification of this method may be used in other cases as well. One such is the following example.

EXAMPLE 3. A problem arising in antenna theory leads to the equation (see [**3**, p. 221])

$$g(u) = \int_{-1}^{1} e^{icxu} f(x) dx, \quad -1 \le u \le 1, \quad c \ne 0, \tag{4.10}$$

whose kernel is not self-adjoint but is in $L^2(-1,1)^2$. We can find an analytic representation by first assuming that g has been extended to the entire real axis. Indeed, if g is in the range of this operator it can be extended to an entire function which is in L^2 on the real axis. The one sided Fourier transform can then be applied to both sides of (4.10) to obtain

$$\begin{aligned}\tilde{g}(z) &= \int_0^\infty e^{iuz} g(u)du = \int_{-1}^1 \int_0^\infty e^{iuz} e^{icxu} f(x)dudx \\ &= \int_{-1}^1 \frac{(-1)}{i(z+cx)} f(x)dx = \frac{2\pi}{c}\hat{f}\Big(-\frac{z}{c}\Big), \quad \operatorname{Im} z > 0.\end{aligned} \tag{4.11}$$

The interchange of integration implicitly done is valid since, for $\operatorname{Im} z > 0$, the double integral is absolutely integrable. Here $\hat{f}$ is the analytic representation of f. For $\operatorname{Im} z < 0$ we use, in place of (4.11),

$$\tilde{g}(z) = \int_{-\infty}^0 e^{iuz} g(u)du = -\frac{2\pi}{c}\hat{f}(-\frac{z}{c}), \quad \operatorname{Im} z < 0. \tag{4.12}$$

These two equations give us an analytic representation of the solution when g is in the range of T. Even if $g \notin$ range (T), then procedures may be followed formally to obtain an analytic representation of an element of one of the spaces A_{-m}. For example, if g is the point mass at $a > 0$, we have

$$\tilde{g}(z) = \begin{cases} \int_0^\infty e^{iuz}\delta(u-a)du = e^{iaz}, & \operatorname{Im} z > 0, \\ 0, & \operatorname{Im} z < 0, \end{cases}$$

which is clearly not the analytic representation of a function with support on $[-1,1]$.

The technique may be further modified for kernels not in L^2.

EXAMPLE 4. In sterology one encounters the integral equation (see Wahba [**8**])

$$g(t) = t\int_t^\infty \frac{f(s)}{\sqrt{s^2-t^2}}ds, \quad t > 0. \tag{4.13}$$

The kernel $K(t,s) = (t/\sqrt{s^2-t^2})H(s-t)$ is neither symmetric nor in L^2. However, by means of a few simple transformations, it can be changed to a more tractable form and a technique involving analytic representations used.

We first replace t^2 by u and s^2 by v to obtain

$$g(\sqrt{u}) = \sqrt{u}\int_u^\infty \frac{1}{(v-u)^{1/2}}\frac{f(\sqrt{v})}{2\sqrt{v}}dv.$$

Subsequently, we replace u by x^{-1} and v by y^{-1}, whence we obtain a form of Abel's equation:

$$g(x^{-1/2}) = \int_0^x (x-y)^{-1/2}\frac{1}{2}y^{-1}f(y^{-1/2})dy. \tag{4.14}$$

One can then use a fractional derivative and the fact that

$$D^{1/2}\frac{x_+^{-1/2}}{\Gamma(1/2)} = \delta(x)$$

(See [**2**; Vol I, p. 117]) to simplify this some more. We therefore operate on both sides of (4.14) by $D^{1/2}$ and then take the analytic representation of both sides to obtain

$$\frac{1}{2\pi i}\int_0^\infty \frac{D^{1/2}g(x^{-1/2})}{x-z}dx = \Gamma(3/2)\hat{f}_1(z), \tag{4.15}$$

where $\hat{f}_1$ is the analytic representation of the function $f_1(y) = f(y^{-1/2})$ y^{-1}. But the fractional derivative $D^{1/2}$ of a function h is given by ([**2**; Vol I, p. 115])

$$D^{1/2}h(x) = \frac{x_+^{-3/2}}{\Gamma(-1/2)} * h(x).$$

Hence we have

$$\hat{f}_1(z) = \frac{1}{2\pi i}\int_0^\infty \frac{\{\int_0^x (x-t)^{-3/2}g(t^{-1/2})dt\}}{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2})(x-z)}dx \tag{4.16}$$

which, after integration by parts, becomes

$$\hat{f}_1(z) = \frac{1}{2\pi i\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}\int_0^\infty \frac{\{\int_0^x (x-t)^{-1/2}g(t^{-1/2})dt\}}{(x-z)^2}dx \tag{4.17}$$

which converges for $g \in L^\infty$.

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