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THE CLASSICAL SOLUTIONS FOR NONLINEAR PARABOLIC INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we consider the solvability in the classical sense of a class of nonlinear one-dimensional integrodifferential equations of parabolic type. The motivation for studying this problem comes from the many physical models in such fields as heat transfer, nuclear reactor dynamics and thermoelasticity. One of the characteristics of this kind of equation is that the maximum principle is no longer valid in general. We combine the integral estimate method and Schauder estimate theory for a linear parabolic equation to derive an *a priori* bound for the solution of our nonlinear problem in the norm of the Banach space $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$. The method of continuity then allows us to establish the global existence of the solution. For completeness, we also demonstrate the uniqueness and continuous dependence of the solution.

1. Introduction. Let $\overline{Q}_T = [0,1] \times [0,T]$ with $T > 0$ arbitrary. In this paper we consider a nonlinear integrodifferential initial-boundary value problem of finding a function $u(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$ which satisfies:

$$(1.1)\quad u_t = a(x,t,u,u_x)u_{xx} + b(x,t,u,u_x) + \int_0^t c(x,\tau,u,u_x)d\tau \text{ in } Q_T,$$

$$(1.2)\qquad u(0,t) = f_1(t), \quad 0 \le t \le T,$$

$$(1.3)\qquad u(1,t) = f_2(t), \quad 0 \le t \le T,$$

$$(1.4)\qquad u(x,0) = u_0(x), \quad 0 \le x \le 1.$$

The motivation for studying (1.1)-(1.4) arises from a variety of physical and engineering problems (see [**13, 20, 21**], etc.). Considerable

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research on the wellposedness for special kinds of integrodifferential equations has been previously carried out (cf. [**1 - 11, 14 - 15, 17, 18 - 25**] and their references). Various approaches such as abstract semi-group theory, perturbation method, compactness arguments, etc. have been applied to this kind of equation. When the principal part of such an equation is nonlinear, one needs certain strong assumptions to obtain the global solution ([**1, 5, 15, 19,**] and [**24**]). In this paper, we shall take a rather different point of view in dealing with the problem (1.1)-(1.4). Indeed, we use integral estimates in conjunction with Schauder estimate theory to derive an *a priori* estimate for the solution of (1.1)-(1.4). The method of continuity, which is similar to that applicable for a regular parabolic boundary value problem, is then applied to establish the global solvability of (1.1)-(1.4) in the classical sense.

The paper is organized as follows. In §2, we first deduce an *a priori* estimate for the solution and then prove the existence of the solution by means of the method of the continuity. We also include a useful regularity theorem. The continuous dependence of the solution upon the known data and uniqueness are established in §3.

The following basic hypotheses are assumed throughout the paper:

H(1). The functions $a(x,t,u,p), b(x,t,u,p)$ and $c(x,t,u,p)$ are differentiable with respect to all of their arguments. Furthermore,

(i) $a(x,t,u,p) \geq A_1 > 0$,

(ii) $|b(x,t,u,p)| \leq A_2[1+|u|+|p|]$,

(iii) $|c(x,t,u,p)| \leq A_3[1+|u|+|p|]$

for $(x,t,u,p) \in \overline{Q}_T \times R^2$, where A_1, A_2 and A_3 are three absolute constants.

H(2). $f_1(t)$ and $f_2(t) \in C^2[0,T], u_0(x) \in C^{2+\alpha}[0,1]$ and the consistency conditions

$$f_1(0) = u_0(0), \qquad f_2(0) = u_0(1),$$

$$f_1'(0) = a(0,0,u_0(0),u'(0))u_0''(0) + b(0,0,u_0(0),u_0'(0))$$

and

$$f_2'(0) = a(1,0,u_0(1),u_0'(1))u_0''(1) + b(1,0,u_0(1),u_0'(1))$$

are satisfied.

The notations of the norms in Banach spaces $C(\overline{Q}_T), C^{2,1}(\overline{Q}_T)$, etc. are those of Ladyzenskaya et al [**16**].

2. Existence and regularity. The following inequalities are well-known and are frequently used in this paper. We list them here for convenience.

1. Young's inequality: If $a \geq 0$ and $b \geq 0$, then, for any $\eta > 0$,

$$ab \leq \eta \frac{a^r}{r} + \eta^{-s/r} \frac{b^s}{s}, \tag{2.1}$$

where $r > 1, s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$.

2. Interpolation inequalities: If $u(x) \in H^1(0,1)$, then

$$||u||_{L^\infty(0,1)} \leq C||u||_{H^1(0,1)}^{2/3} ||u||_{L^1(0,1)}^{1/3}. \tag{2.2}$$

It is clear that the maximum principle for equation (1.1) is no longer valid in general. However, in the sequel we establish such a global *a priori* bound for $u(x,t)$ in the norm of the Banach space $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$. Our technique is based on integral calculations, imbedding inequalities and Schauder estimates under the hypotheses H(1)-H(2).

Let $T > 0$ be arbitrary and assume that $u(x,t)$ is an arbitrary solution of the problem (1.1)-(1.4). We first deduce the following result.

LEMMA 2.1. *Under the assumptions* H(1) *and* H(2), $u(x,t)$ *satisfies the following inequality*:

$$\int\int_{QT} u_{xx}^2 dxdt + \sup_{0 \leq t \leq T} \int_0^1 u_x^2(x,t)dx \leq C_1 \tag{2.3}$$

where C_1 *depends only on the* $A_i (i = 1,2,3)$, *the known data and the upper bound of* T.

PROOF. In what follows, various constants which appear during the process of the proof will be denoted by C; their dependency is the same as the final constants except for an additional explanation. Let $v(x,t)=(1-x)f_1(t)+xf_2(t)$ and $w(x,t)=u(x,t)-v(x,t), (x,t)\in\overline{Q}_T$. Then $w(x,t)$ is a solution of the following problem:

$$w_t = aw_{xx} + b - v_t + \int_0^t c(x,\tau,w+v,w_x+v_x)d\tau \tag{2.4}$$

$$w(0,t)=w(1,t)=0, \quad 0\le t\le T, \tag{2.5}$$

$$w(x,0)=u_0(x)-[(1-x)f_1(0)+xf_2(0)]\overset{\text{def}}{=}w_0(x), \quad 0\le x\le 1. \tag{2.6}$$

Multiplying equation (2.4) by w_{xx} and integrating it over Q_T, we obtain, employing the Cauchy-Schwarz inequality with a small parameter $\varepsilon>0$ and the assumption H(1), that

$$\begin{aligned} &A_1\int\int_{QT} w_{xx}^2dxdt - \int\int_{Q_T} w_tw_{xx}dxdt \\ &\le \varepsilon\int\int_{Q_T} w_{xx}^2dxdt + C(\varepsilon)\int\int_{Q_T}\Big\{1+w^2+w_x^2+ \\ &+\Big[\int_0^t A_3(1+|w|+|w_x|)d\tau\Big]^2\Big\}dxdt. \end{aligned} \tag{2.7}$$

Observe that

$$-\int\int_{QT} w_tw_{xx}dxdt = \frac{1}{2}\int_0^1 w_x(x,T)^2dx - \frac{1}{2}\int_0^1 w_0'(x)^2dx, \tag{2.8}$$

$$\int\int_{Q_T} w^2dxdt \le C\int\int_{Q_T} w_x^2dxdt, \tag{2.9}$$

and that

$$\begin{aligned} &\int\int_{Q_T}\Big[\int_0^t(1+|w|+|w_x|)d\tau\Big]^2dxdt \\ &\le \int_0^T\int_0^1\Big[2t\int_0^t(1+w^2+w_x^2)d\tau\Big]dxdt \\ &\le 2T\int_0^T\int_0^1\int_0^t[1+w^2+w_x^2]d\tau dxdt \\ &\equiv 2T\int_0^T\int_0^1[T-\tau][1+w^2+w_x^2]dxd\tau, \\ &\le 2T^2\int_0^T\int_0^1[1+w^2+w_x^2]dxdt \end{aligned} \tag{2.10}$$

Combining (2.8), (2.9) and (2.10) by choosing $\varepsilon = \frac{1}{4A_1}$, we have from (2.7) that

$$\frac{A_1}{2}\int\int_{Q_T} w_{xx}^2 dxdt + \int_0^1 w_x(x,T)^2 dxdt$$
$$\leq (1+T^2)C\int\int_{Q_T} w_x^2 dxdt + (1+T^2)C.$$

Since $T \geq 0$ is arbitrary, Gronwall's inequality implies that

$$\int_0^1 w_x(x,t)^2 dx \leq C(T)$$

Therefore,

$$\int\int_{Q_T} w_x(x,t)^2 dxdt \leq C \tag{2.11}$$

and

$$\int\int_{Q_T} w_{xx}^2 dxdt + \sup_{0\leq t\leq T}\int_0^1 w_x^2(x,t)dx \leq C \tag{2.12}$$

This concludes the estimate (2.3) since $u(x,t) = w(x,t) + v(x,t)$ on $\overline{Q}_T$. □

COROLLARY 2.1. *There exists a positive constant C_2 such that*

$$||u(x,t)||_{C(\overline{Q}_T)} \leq C_2 \tag{2.13}$$

where C_2 depends on the same quantities as C_1.

PROOF. This can be obtained directly from the estimate (2.3). □0

In order to estimate the norm of u_x, we need considerably more effort.

LEMMA 2.2. *There exists a constant C_3 such that*

$$\|u_x\|_{C(\overline{Q}_T)} \leq C_3 \tag{2.14}$$

where the dependency of C_3 is the same as C_1.

PROOF. Let $p > 2$ be an arbitrary even integer. Since

$$\begin{aligned}
&\int_0^T \frac{d}{dt}\Big[\int_0^1 w_x^p dx\Big] dt \\
&= \int_0^T \int_0^1 p w_x^{p-1} w_{xt} dx dt \\
&= -\int_0^T \int_0^1 p(p-1) w_x^{p-2} w_{xx} w_t dx dt + \int_0^T p w_x^{p-1} w_t |_{x=0}^{x=1} dt \\
&= -\int_0^T \int_0^1 p(p-1) w_x^{p-2} w_{xx} \Big[a w_{xx} + b - v_t + \int_0^t c d\tau\Big] dx dt,
\end{aligned} \tag{2.15}$$

it follows that
(2.16)

$$\begin{aligned}
&\int_0^1 w_x^p(x,T) dx + A_1 \int_0^T \int_0^1 p(p-1) w_x^{p-2} w_{xx}^2 dx dt \\
&\leq \int_0^1 w_0'(x)^p dx + \int_0^T \Big| p(p-1) w_x^{p-2} w_{xx} \Big(b - v_t + \int_0^t c d\tau\Big)\Big| dx dt \\
&\leq \int_0^1 w_0'(x)^p dx + \varepsilon \int_0^T \int_0^1 p(p-1) w_x^{p-2} w_{xx}^2 dx dt \\
&\quad + C(\varepsilon) \int_0^T \int_0^1 p(p-1) w_x^{p-2} \Big[b - v_t + \int_0^t c d\tau\Big]^2 dx dt.
\end{aligned}$$

Choosing $\varepsilon = A_1/2$ and using H(1), we find

$$\begin{aligned}
&\int_0^1 w_x^p(x,T) dx + A_1/2 \int_0^T \int_0^1 p(p-1) w_x^{p-2} w_{xx}^2 dx dt \\
&\leq \int_0^1 w_0'(x)^p dx + C \int_0^T \int_0^1 p(p-1) w_x^{p-2} \Big[1 + w^2 + w_x^2 + \\
&\Big(\int_0^t (1 + |w| + |w_x|) d\tau\Big)^2\Big] dx dt.
\end{aligned} \tag{2.17}$$

Let

$$I = \int_0^T \int_0^1 w_x^{p-2} \Big[\int_0^t (1 + |w| + |w_x|) d\tau \Big]^2 dx dt.$$

Then,

$$\begin{aligned} I &\le \int_0^T \int_0^1 w_x^{p-2} \Big[2T \Big(T + C_2^2 + \int_0^t w_x^2 d\tau \Big) \Big] dx dt \\ &\le CT(1+T) \int_0^T \int_0^1 w_x^{p-2} dx dt + 2T \int_0^T \int_0^1 w_x^{p-2} \Big(\int_0^t w_x^2 \tau \Big) dx dt \\ &\equiv CT(1+T) I_1 + 2T I_2 \end{aligned}$$

Using Young's inequality (2.1) with $r = \frac{p}{p-2}$, $s = \frac{p}{2}$ and $\eta = 1$, we have

$$\begin{aligned} I_2 &= \int_0^T \int_0^1 w_x^{p-2} \Big(\int_0^t w_x^2 d\tau \Big) dx dt \\ &\le \int_0^T \int_0^1 \Big[\frac{p-2}{p} w_x^p + \frac{2}{p} \Big(\int_0^t w_x^2 d\tau \Big)^{\frac{p}{2}} \Big] dx dt \\ &\le \int_0^T \int_0^1 w_x^p dx dt + \int_0^T \int_0^1 \Big[t^{\frac{p-2}{2}} \Big(\int_0^t w_x^p \Big) d\tau \Big] dx dt \\ &\le \int_0^T \int_0^1 w_x^p dx dt + T^{\frac{p-2}{2}} \int_0^T \int_0^1 \int_0^t w_x^p d\tau dx dt \\ &\le \int_0^T \int_0^1 w_x^p dx dt + T^{\frac{p-2}{2}} \int_0^T \int_0^1 (T - \tau) w_x^p dx d\tau \\ &\le (1 + T^{\frac{p}{2}}) \int_0^T \int_0^1 w_x^p dx dt, \end{aligned} \tag{2.18}$$

For the moment, we restrict T by $0 < T \le T_0 \overset{\text{def}}{=} 1$. Under this condition, it follows from (2.17)-(2.18) and $T \in [0, T_0]$ arbitrary that

$$\begin{aligned} &\sup_{0 \le t \le T} \int_0^1 w_x^p(x,t) dx + A_1/2 \int\int_{Q_T} p(p-1) w_x^{p-2} w_{xx} dx dt \\ &\le \int_0^1 w_0'(x)^p dx + C \int_0^T \int_0^1 p(p-1) w_x^{p-2} [1 + w_x^2] dx dt, \end{aligned} \tag{2.19}$$

where C depends only on C_2 and known data. Assume that $||w_x(x,t)||_{L\infty(Q_T)} \geq \max\{1, \frac{1}{T}||w_0'(x)||_0\}$ (Here $0 < T \leq T_0 = 1$ is a fixed number). Otherwise, we already have the estimate(2.14) on the interval $[0, T_0]$. Then

$$\begin{aligned}
&\sup_{0\leq t\leq T}\int_0^1 w_x^p dx + A_1/2\int_0^T\int_0^1 p(p-1)w_x^{p-2}w_{xx}^2 dxdt \\
&\leq C\int_0^T\int_0^1 p(p-1)w_x^p dxdt \qquad (2.20)\\
&\leq C\int_0^T p(p-1)||w_x(\cdot,t)||^p_{L^\infty(0,1)}dt
\end{aligned}$$

If the interpolation inequality (2.2) is employed we have

$$||w_x^{\frac{p}{2}}||_{L_\infty(0,1)} \leq C||w_x^{\frac{p}{2}}||^{2/3}_{H^1(0,1)}||w_x^{\frac{p}{2}}||^{1/3}_{L^1(0,1)}.$$

i.e.

$$\begin{aligned}
||w_x||^p_{L_\infty(0,1)} &\leq C||w_x^{\frac{p}{2}}||^{4/3}_{H^1(0,1)}||w_x^{\frac{p}{2}}||^{2/3}_{L^1(0,1)}\\
&\leq C\eta||w_x^{\frac{p}{2}}||^2_{H^1(0,1)} + C\eta^{-2}||w_x||^p_{L^{\frac{p}{2}}(0,1)},
\end{aligned}$$

where the last inequality is from Young's inequality (2.1) for $r = \frac{3}{2}$ and $s = 3$. Note that

$$||w_x^{\frac{p}{2}}||^2_{H^1(0,1)} = \int_0^1 [(p/2)w_x^{\frac{p}{2}-1}w_{xx}]^2dx + \int_0^1 w_x^p dx.$$

As a consequence,

$$\begin{aligned}
&\sup_{0\leq t\leq T_0}\int_0^1 w_x^p dx + (A_1/2)p(p-1)\int_0^T\int_0^1 w_x^{p-2}w_{xx}^2 dxdt\\
&\leq Cp(p-1)\Big[\frac{p^2}{4}\eta\int_0^T\int_0^1 w_x^{p-2}w_{xx}^2dxdt + \eta\int_0^T\int_0^1 w_x^p dxdt\Big]\\
&\quad + Cp(p-1)\eta^{-2}\int_0^T ||w_x||^p_{L^{\frac{p}{2}}(0,1)}dt.
\end{aligned}$$

If now η is chosen as $\eta = \min\{\frac{1}{2CT_0}, \frac{A_1}{p^2 C}\}$, then

$$\begin{aligned}\sup_{0\le t\le T}\int_0^1 w_x^p dx &+ p(p-1)\int_0^T\int_0^1 w_x^{p-2} w_{xx}^2 dxdt\\ &\le Cp(p-1)\eta^{-2}T\sup_{0\le t\le T}||w_x||^p_{L^{\frac{p}{2}}(0,1)}\\ &\le Cp^4\sup_{0\le t\le T}||w_x||^p_{L^{\frac{p}{2}}(0,1)},\end{aligned}$$

where C is constant which depends only on known data.

In order to complete our proof we will want to consider large value of p. To accomplish this, first let $p = p_k = 2^k$ and $\alpha_k = \sup_{\{0\le t\le T\}}\{\int_0^1 w_x^{p_k} dx\}^{\frac{1}{p_k}}$. If we take the p_k^{th} root of both sides of above inequality, we obtain

$$\alpha_k \le (Cp_k^4)^{\frac{1}{p_k}}\alpha_{k-1}.$$

Now

$$\prod_{k=1}^{+\infty} C^{\frac{1}{p_k}} = C^{\sum_{k=1}^{+\infty}\frac{1}{p_k}} = C^{\sum_{k=1}^{+\infty}\frac{1}{2^k}} \le C,$$

and

$$\prod_{k=1}^{+\infty} p_k^{\frac{4}{p_k}} = 2^{\sum_{k=1}^{+\infty}\frac{4k}{2^k}} \le C,$$

since

$$\sum_{k=1}^{+\infty}\frac{4k}{2^k} = 4\sum_{k=1}^{+\infty}\frac{k}{2^k}$$

is convergent. Thus it follows that, for $d_k = (Cp_k^4)^{\frac{1}{p_k}}$,

$$\alpha_k \le d_k\alpha_{k-1} \le [\prod_{l=1}^{k} d_l]\alpha_1 \le C\alpha_1.$$

As

$$\lim_{k\to+\infty}\alpha_k = ||w_x||_{L^\infty(Q_T)}$$

and $\alpha_1 \leq C_1$ by Lemma 2.1, it follows that

$$(2.21) \qquad ||w_x||_0^{\overline{Q}_T} \leq C\alpha_1 \leq C.$$

Note that for the interval $[T_0, 2T_0]$, we can repeat the above procedure and obtain previously the same inequality (2.21). After finitely many steps, one has the estimate (2.14). □

LEMMA 2.3. *There exist constants C_4 and $\alpha(0 < \alpha < 1)$, which depend on the same quantities as $C_i (i = 1, 2, 3)$, such that*

$$(2.22) \qquad ||u||_{C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{Q}_T)} \leq C_4,$$

and hence

$$(2.23) \qquad ||u_x||_{C^{\alpha,\alpha/2}(\overline{Q}_T)} \leq C_4.$$

PROOF. Let

$$\mu = \max_{\substack{(x,t)\in\overline{Q}_T, |u|\leq C_2,\\ |u_x|\leq C_3}} \left[|a(x,t,u,u_x)| + |b(x,t,u,u_x)| + \int_0^t |c(x,\tau,u,u_x)| d\tau\right]$$

Lemma 2.2 implies that μ is uniformly bounded and that the bound depends only on the known data. The desired result then follows from Theorem 5.1 (page 561) of Ladyzenskaya et al. [**16**] as a regular parabolic equation case. □

LEMMA 2.4. *There exists a constant C_5 such that*

$$(2.24) \qquad ||u||_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q}_T)} \leq C_5,$$

where C_5 depends only on the same quantities as $C_i, i = 1, \ldots, 4$.

PROOF. By Lemma 2.3, we know that $a(x,t,u(x,t),u_x(x,t))$ and $b(x,t,u(x,t),u_x(x,t))$ are uniformly Hölder continuous in $\overline{Q}_T$ with exponents α and $\frac{\alpha}{2}$ with respect to x and t, respectively. Considering equation (1.1) as a linear equation

$$u_t = au_{xx} + b + \int_0^t c d\tau$$

with initial-boundary conditions (1.2)-(1.4), we employ the Schauder estimate to obtain

$$||u||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)} \le C\Big[1 + \Big|\Big|\int_0^t c d\tau\Big|\Big|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)}\Big]. \tag{2.25}$$

Note that, for any function $g(x,t) \in C^{\alpha,\alpha/2}(\overline{Q}_T)$, we have the property (2.26)

$$\Big|\Big|\int_0^t g(x,\tau)d\tau\Big|\Big|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)} \le [||g(x,0)||_{C[0,1]} + (T+T^{1-\frac{\alpha}{2}})||g||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)}].$$

As a consequence

$$\begin{aligned}\Big|\Big|\int_0^t c(x,t,u,u_x)d\tau\Big|\Big|_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)} &\le C[1 + (T+T^{1-\frac{\alpha}{2}})||c(x,t,u,u_x)||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)}] \\ &\le C[1 + (T+T^{1-\frac{\alpha}{2}})||u||_{C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q}_T)}]\end{aligned}$$

is uniformly bounded by Lemma 2.3, and the bound depends only on the known data. Hence the estimate (2.24) follows (2.25) and the above inequality. □

With the above result in hand, we now can establish

THEOREM 2.1. *There exists a solution $u(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$ to the problem* (1.1)-(1.4) *under the conditions* H(1) *and* H(2).

PROOF. Let us define the operator L_λ by

$$L_\lambda u = u_t - \Big[au_{xx} + b + \lambda\int_0^t c d\tau\Big].$$

Let $\Sigma(\lambda) = \{\lambda \in [0,1]$: the problem $(1.1)_\lambda$-(1.4) is solvable$\}$, where $(1.1)_\lambda$ is the equation $L_\lambda u = 0$. By a standard continuation method ($\Sigma(\lambda)$ is not empty, $\Sigma(\lambda)$ is open and also closed), it follows that

$\Sigma(\lambda) \equiv [0,1]$. □

To conclude this section, we give a theorem on the regularity of the solution for the problem (1.1)-(1.4).

THEOREM 2.2. *Assume that $a(x,t,u,p), b(x,t,u,p)$ and $c(x,t,u,p)$ are infinitely differentiable in all of their arguments and that the boundary values $f_1(t)$ and $f_2(t)$ belong to $C^\infty(0,T]$. Then the solution $u(x,t)$ is infinitely differentiable with respect to x and t on the region $\overline{Q}_T \cap \{(x,t) : t > 0\}$.*

PROOF. Since $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$, we can differentiate equation (1.1) with respect to t and then $V = u_t$ satisfies

$$\begin{aligned} V_t = aV_{xx} + [a_p u_{xx} + b_p]V_x + [a_u u_{xx} + b_u]V \\ + [a_t u_{xx} + b_t + c(x,t,u,u_x)], \quad \text{in } Q_T, \end{aligned} \tag{2.27}$$

$$V(0,t) = f_1'(t), \quad 0 \le t \le T, \tag{2.28}$$

$$V(1,t) = f_2'(t), \quad 0 \le t \le T. \tag{2.29}$$

Since the coefficients of equation (2.27) are Hölder continuous with respect to x and t, the Schauder estimate for a parabolic equation implies that the solution

$$V \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T).$$

Hence,

$$u \in C^{4+\alpha,2+\frac{\alpha}{2}}(\overline{Q}_T).$$

We can redo the above procedure step-by-step to obtain

$$V \in C^{+\infty,+\infty}(\overline{Q}_T \cap \{t : t > 0\}).$$

It follows that

$$u(x,t) \in C^{+\infty,+\infty}(\overline{Q}_T \cap \{t : t > 0\}).$$

□

3. Continuous dependence and uniqueness.

THEOREM 3.1. *Assume that* $(f_1(t), f_2(t), u_0(x))$ *and* $(f_1^*(t), f_2^*(t), u_0^*(x))$ *are two known sets of initial-boundary values which satisfy* H(2). *Let* $u(x,t)$ *and* $u^*(x,t)$ *be two solutions of the problem (1.1)-(1.4) corresponding, respectively, to the above data. Then*

$$\begin{aligned} &\|u(x,t) - u^*(x,t)\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)} \\ (3.1) \qquad &\leq C[\|f_1(t) - f_1^*(t)\|_{C^{1+\frac{\alpha}{2}}[0,T]} + \|f_2(t) - f_2^*(t)\|_{C^{1+\frac{\alpha}{2}}[0,T]} \\ &\quad + \|u_0(x) - u_0^*(x)\|_{C^{2+\alpha}[0,1]}], \end{aligned}$$

where C *depends only on known data.*

PROOF. Let $w(x,t) = u(x,t) - u^*(x,t), (x,t) \in \overline{Q}_T$. Then $w(x,t)$ satisfies

$$w_t = aw_{xx} + b^*(x,t)w_x + c^*(x,t)w + d^*(x,t) \text{ in } Q_T, \tag{3.2}$$

$$w(0,t) = f_1(t) - f_1^*(t), \quad 0 \leq t \leq T, \tag{3.3}$$

$$w(1,t) = f_2(t) - f_2^*(t), \quad 0 \leq t \leq T, \tag{3.4}$$

$$w(x,o) = u_0(x) - u_0^*(x), \quad 0 \leq x \leq 1, \tag{3.5}$$

where

$$\begin{aligned} b^*(x,t) &= \int_0^1 b_p(x, tu^*, zu_x + (1-z)u_x^*)dz \\ &\quad + \Big[\int_0^1 a_p(x,t,u^*, zu_x + (1-z)u_x^*)dz)\Big] u_{xx}^*, \\ c^*(x,t) &= \int_0^1 b_u(x,t,zu + (1-z)u^*, u_x)dz \\ &\quad + \Big[\int_0^1 a_u(x,t,zu + (1-z)u^*, u_x)dz\Big] u_{xx}^*, \\ d^*(x,t) &= \int_0^t [d_1(x,\tau)w_x + d_2(x,\tau)w]d\tau, \\ d_1(x,t) &= \int_0^1 c_p(x,t,u^*, zu_x + (1-z)u_x^*)dz, \\ d_2(x,t) &= \int_0^1 c_z(x,t,zu + (1-z)u^*, u_x)dz. \end{aligned}$$

The estimate (2.22) implies that all the Hölder moduli of the coefficients in (3.2) are dominated by known data. From the Schauder estimate for the linear parabolic equation (3.2), we have

$$(3.6)\qquad \begin{aligned} ||u||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)} \leq C \sum_{i=1}^{i=1} ||fi(t) - f_i^*(t)||_{C^{1+\frac{\alpha}{2}}[0,T]} \\ + ||u_0(x) - u_0^*(x)||_{C^{2+\alpha}[0,1]} + ||d^*(x,t)||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)} \end{aligned}$$

The inequalities (2.24) and (2.26) yield

$$||d^*||_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)} \leq C[||w(x,0||_{C[0,1]} + (T + T^{1-\frac{\alpha}{2}})||u||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)}].$$

Therefore when T is small enough so that $(T + T^{1-\frac{\alpha}{2}})C \leq \frac{1}{2}$ we have the desired result. By taking a finite number of steps, therefore, we establish (3.1) for arbitrary T. □

COROLLARY 3.1. *The solution of the problem* (1.1)-(1.4) *is unique.* □

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