

EFFECTIVE BOUNDS FOR THE SINGULAR VALUES OF INTEGRAL OPERATORS

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I. Introduction. In [1, 11] results on the asymptotic behavior of condition numbers of matrices arising in the numerical treatment of integral equations of the first kind are presented. These are partially based on a theorem of Chang [2], which provides information on the asymptotic properties of singular values of certain integral operators. Chang's paper is a natural outgrowth of a long sequence of investigations of eigenvalues of such operators (see, e.g., [4, 8, 9]).

From the viewpoint of the numerical analyst, the estimates in [1, 11], being asymptotic, are rather unsatisfactory. Much more valuable would be actual bounds. A careful examination of Chang's work [3] reveals that such bounds are available there, but at the cost of a great deal of labor. We present here a simplified version of the proof of Chang. The desired bounds are an immediate by-product.

In Section II we prove a result on singular values of products of operators. (This maybe extracted from theorems of Weyl and Horn [6, 10]. We provide an elementary proof.) In §III this result is applied to the integral operators of primary interest, but under a somewhat restrictive hypothesis. This assumption is completely removed in §IV, and the desired bounds are obtained. In the final section, the possible extension of results to more general integral operators is discussed. Some classical eigenvalue bounds are obtained in the Appendix.

II. A basic lemma. We consider integral operators K, L , and M where

$$(2.1) \quad K. = \int_0^1 K(x, y) \cdot dy,$$

with similar representations for L and M . All kernels are assumed to be in L_2 . Denote by κ_j the singular values of K , with λ_j and μ_j the

This work was performed under the auspices of the U.S. Department of Energy. Received by the editors on March 27, 1986.

corresponding values for L and M . We assume the singular values ordered by magnitude.

LEMMA 2.1. *Suppose $K = LM$. Then*

$$(2.2) \quad \kappa_1 \kappa_2 \dots \kappa_n \leq (\lambda_1 \lambda_2 \dots \lambda_n)(\mu_1 \mu_2 \dots \mu_n), \quad n = 1, 2, \dots$$

PROOF. Let φ_j and ψ_j be the normalized singular functions belonging to κ :

$$(2.3) \quad K\varphi_j = \kappa_j \psi_j, \quad j = 1, 2, \dots$$

Consider the space S generated by $\{M\varphi_1, M\varphi_2, \dots, M\varphi_n\}$. Assume for the moment that S is of dimension n , and let $\theta_1, \theta_2, \dots, \theta_n$ be an orthonormal set spanning S . Form the n by n Galerkin matrices

$$(2.4) \quad \begin{aligned} \hat{K} &= (\psi_j, K\varphi_j), \\ \hat{L} &= (\psi_j, L\varphi_j), \\ \hat{M} &= (\theta_j, M\varphi_j) \end{aligned}$$

with singular values $\hat{\kappa}_m, \hat{\lambda}_m, \hat{\mu}_m$. Obviously, $\hat{\kappa}_m = \kappa_m$. Also

$$(2.5) \quad \kappa_1 \kappa_2 \dots \kappa_n = \det \hat{K}.$$

By a theorem of linear algebra [5],

$$(2.6) \quad \hat{K} = \hat{L}\hat{M}.$$

Therefore,

$$(2.7) \quad \begin{aligned} \kappa_1 \kappa_2 \dots \kappa_n &= \det \hat{L} \det \hat{M} \\ &= (\hat{\lambda}_1 \hat{\lambda}_2 \dots \hat{\lambda}_n)(\hat{\mu}_1 \hat{\mu}_2 \dots \hat{\mu}_n). \end{aligned}$$

It is proved in [1] by an application of the mini-max principle [3, 7] that

$$(2.8) \quad \hat{\lambda}_m \leq \lambda_m, \quad \hat{\mu}_m \leq \mu_m, \quad m = 1, 2, \dots, n.$$

The result is now immediate. \square

The condition that S be of dimension n must be removed. If it is of lower dimension, then M annihilates a linear combination of $\varphi_1, \varphi_2, \dots, \varphi_n$. Therefore, $\mu_j = 0$ for some $j \leq n$. Hence all succeeding singular values are zero, and the same holds for the κ 's. Thus (2.2) continues to hold.

III. The preliminary theorem. We begin with another lemma.

LEMMA 3.1.

$$\kappa_n \leq \frac{\varepsilon_n \|K\|}{\sqrt{n}},$$

where $\|\dots\|$ indicates the usual L_2 norm and $\varepsilon_n \rightarrow 0$, $\varepsilon_n \leq 1$, $n = 1, 2, \dots$

PROOF. It is a classical result that (see [3])

$$(3.1) \quad \sum_{j=1}^{\infty} \kappa_j^2 \leq \|K\|^2.$$

Hence

$$(3.2) \quad n\kappa_n^2 \leq \sum_{j=1}^n \kappa_j^2 \leq \|K\|^2$$

and

$$(3.3) \quad \kappa_n \leq \frac{\|K\|}{\sqrt{n}}.$$

For $\varepsilon > 0$, there exists N such that, for $n, m \geq N$,

$$(3.4) \quad \sum_m^n \kappa_j^2 < \varepsilon.$$

Thus

$$(3.5) \quad (n - m)\kappa_n^2 < \varepsilon.$$

Fix m and choose n large enough so that $m\kappa_n^2 < \varepsilon$. From (3.5),

$$(3.6) \quad n\kappa_n^2 \leq 2\varepsilon,$$

and the proof is complete. \square

LEMMA 3.2. *The operator*

$$(3.7) \quad L \cdot = \int_0^x 1 \cdot dy$$

has singular values $\lambda_n = ((n + 1/2)\pi)^{-1}$.

PROOF. A simple computation provides the result. \square

LEMMA 3.3. *Let $K(0, y) = \frac{\partial K}{\partial x}(0, y) = \dots = \frac{\partial^s K}{\partial x^s}(0, y) = 0$ for almost all y . Further suppose that $\frac{\partial^{s+1}}{\partial x^{s+1}}K(x, y) \in L_2$ and that*

$$(3.8) \quad \frac{\partial^s K}{\partial x^s}(x, y) = \int_0^x \frac{\partial^{s+1} K}{\partial z^{s+1}}(z, y) dz$$

for almost all y . Then

$$(3.9) \quad \kappa_1, \kappa_2 \dots \kappa_n \leq \frac{\varepsilon_n \left\| \frac{\partial^{s+1} K}{\partial x^{s+1}} \right\|^n}{\pi^{n(s+1)} (n!)^{1/2}} \prod_{p=1}^n (p + 1/2)^{-(s+1)},$$

where $\varepsilon_n \rightarrow 0$, $\varepsilon \leq 1$, $n = 1, 2, \dots$

PROOF. We first establish the result for $s = 0$. In Lemma 2.1 select L as in Lemma 3.2. Let M be the integral operator with kernel $\partial K / \partial x$. From Lemma 3.1 we obtain

$$(3.10) \quad \mu_n = \varepsilon'_n \left\| \frac{\partial K}{\partial x} \right\| n^{-1/2}, \quad \varepsilon'_n \rightarrow 0, \quad \varepsilon'_n \leq 1, \quad n = 1, 2, \dots$$

Thus

$$\begin{aligned}
 \kappa_1 \kappa_2 \dots \kappa_n &\leq \prod_{p=1}^n (\pi(p+1/2))^{-1} \prod_{p=1}^n \left(\varepsilon'_p \left\| \frac{\partial K}{\partial x} \right\| p^{-1/2} \right) \\
 (3.11) \quad &= \frac{\varepsilon_n \left\| \frac{\partial K}{\partial x} \right\|^n}{\pi^n (n!)^{1/2}} \prod_{p=1}^n (p+1/2)^{-1}, \\
 \varepsilon_n &= \prod_{p=1}^n \varepsilon'_p.
 \end{aligned}$$

Clearly (3.11) agrees with (3.9) for $s = 0$. Proceeding inductively, we suppose the result holds for $s = s'$. Choose L as before and select M as the operator with kernel $\frac{\partial^{s'} K}{\partial x^{s'}}$. Then

$$\begin{aligned}
 \kappa_1 \kappa_2 \dots \kappa_n &\leq \prod_{p=1}^n ((p+1/2)\pi)^{-1} \cdot \frac{\varepsilon_n \left\| \frac{\partial^{s'+1} K}{\partial x^{s'+1}} \right\|^n}{\pi^{ns'} (n!)^{1/2} \prod_{p=1}^n (p+1/2)^{s'}} \\
 &\leq \frac{\varepsilon_n \left\| \frac{\partial^{s'+1} K}{\partial x^{s'+1}} \right\|^n}{\pi^{n(s'+1)} (n!)^{1/2}} \prod_{p=1}^n (p+1/2)^{-(s'+1)}.
 \end{aligned}$$

This completes the induction. \square

We have a preliminary theorem.

THEOREM 3.1. *If $K(x, y)$ satisfies the hypotheses of Lemma 3.3, then*

$$\begin{aligned}
 (3.13) \quad \kappa_n &\leq \frac{\varepsilon_n \left\| \frac{\partial^{s+1} K}{\partial x^{s+1}} \right\|^n}{\pi^{s+1} (n!)^{1/2n}} \prod_{p=1}^n (p+1/2)^{-(1+s')/n} \\
 &= B_n^s, \quad \varepsilon_n \leq 1, \varepsilon_n \rightarrow 0.
 \end{aligned}$$

PROOF. We observe that $\kappa_1 \kappa_2 \dots \kappa_n \geq k_n^n$. The result follows from (3.12) except that we get $\varepsilon_n^{1/n}$. Note, however, that (see (3.11))

$$\varepsilon_n^{1/n} = \left\{ \prod_{p=1}^n \varepsilon'_p \right\}^{1/n} \leq 1/n \sum_{p=1}^n \varepsilon'_p \rightarrow 0.$$

Clearly $\varepsilon_n^{1/n} \leq 1$. \square

IV. Removal of the restrictions on K at $x = 0$. We now suppose the condition $\frac{\partial^j K}{\partial x^j}(0, y) = 0$, $j = 0, 1, 2, \dots, s$ fails to hold. Define

$$(4.1) \quad \tilde{K}(x, y) = K(x, y) - N(x, y),$$

where

$$(4.2) \quad N(x, y) = \sum_{j=0}^s \frac{1}{j!} \frac{\partial^j K}{\partial x^j}(0, y) x^j.$$

Observe that

$$(4.3) \quad \frac{\partial^p \tilde{K}}{\partial x^p}(x, y) = \frac{\partial^p K}{\partial x^p}(x, y) - \sum_{j=0}^{s-p} \frac{1}{j!} \frac{\partial^{j+p} K(0, y)}{\partial x^{j+p}} x^j,$$

$$p = 0, 1, 2, \dots, s,$$

so that

$$(4.4) \quad \frac{\partial^p \tilde{K}(0, y)}{\partial x^p} = 0, \quad p = 0, 1, 2, \dots, s.$$

Assume that $\frac{\partial^{s+1} K(x, y)}{\partial x^{s+1}} \in L_2$ and that

$$(4.5) \quad \frac{\partial^s K}{\partial x^s}(x, y), \quad -\frac{\partial^s K(0, y)}{\partial x^s} = \int_0^x \frac{\partial^{s+1} K(z, y) dz}{\partial z^{s+1}}$$

for almost all y . Then $\tilde{K}(x, y)$ satisfies the hypotheses of Lemma 3.3, and the conclusions of the lemma apply to the singular values $\tilde{\kappa}_j$ of \tilde{K} .

LEMMA 4.1. *Let T_1 and T_2 be two compact linear operators in L_2 . Denote their singular values by $\sigma_j(T_k)$, $k = 1, 2, j = 1, 2, \dots$. Then if $T = T_1 + T_2$,*

$$(4.6) \quad \sigma_{p+q-1}(T) \leq \sigma_p(T_1) + \sigma_q(T_2), \quad p + q > 1.$$

PROOF. The analogue of this theorem for eigenvalues can be established by the mini-max principle (see [7]). The proof for singular values is virtually the same. \square

THEOREM 4.1. Let $K(0, y), \frac{\partial K}{\partial x}(0, y), \dots, \frac{\partial^s K(0, y)}{\partial x^s}$ be in L_2 . Let $m(s)$ be the number of these functions that are not zero for almost all y . Further, suppose that $\frac{\partial^{s+1} k}{\partial x^{s+1}}(x, y) \in L_2$ and that

$$(4.7) \quad \frac{\partial^s K}{\partial x^s}(x, y) - \frac{\partial^s K}{\partial x^s}(0, y) = \int_0^x \frac{\partial^{s+1} K}{\partial z^{s+1}}(z, y) dz.$$

Then

$$(4.8) \quad \kappa_{n+m(s)} \leq B_n^s, \quad n = 1, 2, \dots,$$

where B_n^s is given by (3.13).

PROOF. Write, in the notation of Lemma 4.1,

$$(4.9) \quad \sigma_{p+q-1}(K) \leq \sigma_p(\tilde{K}) + \sigma_q(N).$$

Observe that N has separable kernel of order at most $m(s)$, and is therefore equivalent to a matrix operator of the same order. The operator N can thus have no more than $m(s)$ non-zero singular values. Thus

$$(4.10) \quad \sigma_j(N) = 0, \quad j \geq m(s) + 1.$$

In (4.9), choose $p = n, q = m(s) + 1$. This yields

$$(4.11) \quad \begin{aligned} \kappa_{n+m(s)} &= \sigma_{n+m(s)}(K) \\ &\leq \sigma_n(\tilde{K}) + \sigma_{m(s)+1}(N) \\ &= \tilde{\kappa}_n \leq B_n^s. \end{aligned}$$

The proof is complete. \square

It should be noted that we have chosen to concentrate interest on κ_n rather than on $\kappa_1 \kappa_2 \dots \kappa_n$. Thus some information has been lost in our

development. Further simplification can be achieved by estimating B_n^s .

THEOREM 4.3. *The quantity B_n^s satisfies*

$$(4.12) \quad \begin{aligned} B_1^s &\leq (2/3)^{1+s} \left\| \frac{\partial^{s+1} K}{\partial x^{s+1}} \right\| \pi^{-s-1}, \\ B_n^s &\leq \varepsilon_n \sqrt{\pi} (.910)^{s+3/2} \left\| \frac{\partial^{s+1} K}{\partial x^{s+1}} \right\| n^{-(s+3/2)}, \quad n = 2, 3, \dots \end{aligned}$$

PROOF. From (3.13),

$$(4.13) \quad \begin{aligned} B_n^s &= \frac{\varepsilon_n \left\| \frac{\partial^{s+1} K}{\partial x^{s+1}} \right\|}{\pi^{s+1} (n!)^{1/2n}} \prod_{p=1}^n (p+1/2)^{-(1+s)/n} \\ &< \frac{\varepsilon_n \left\| \frac{\partial^{s+1} K}{\partial x^{s+1}} \right\|}{\pi^{s+1} (n!)^{(s+3/2)/n}}. \end{aligned}$$

By Stirling's formula

$$(4.14) \quad n! \geq \sqrt{2\pi} e^{-n-1} (n+1)^{n+1/2},$$

so that

$$(4.15) \quad (n!)^{1/n} \geq \left(\frac{\sqrt{2\pi}}{e} \right)^{1/n} e^{-1} n.$$

Now, for $n \geq 2$,

$$(4.16) \quad \frac{1}{e} \left(\frac{\sqrt{2\pi}}{e} \right)^{\frac{1}{n}} > .350.$$

Thus, for $n \geq 2$,

$$(4.17) \quad B_n^s < \varepsilon_n \sqrt{\pi} \left(\frac{.910}{n} \right)^{s+3/2} \left\| \frac{\partial^{s+1} K}{\partial x^{s+1}} \right\|.$$

The result for B_1^s is obtained directly from (4.13). This completes the proof. It should be observed that the result of Chang [2] follows at

once from Theorem 4.3. \square

V. Summary and remarks. We have presented a new and relatively transparent proof of a theorem of Chang. The approach has easily produced effective bounds on the singular values of integral operators. Chang's efforts actually represented a partial extension to singular values of results of Hille and Tamarkin on eigenvalues (see the Appendix). The methods of this article can be extended in such a way as to obtain a much more complete extension. That matter will be covered in a forthcoming paper.

APPENDIX

A CLASSICAL RESULT OF HILLE AND TAMARKIN

In the early studies, the behavior of the eigenvalues of integral operators, rather than of the singular values, was investigated. Methods often used the Fredholm determinant and rather deep results from analytic function theory. In particular, Hille and Tamarkin [4] showed that if $K(x, y)$ satisfies the hypotheses of Theorem 4.1 and Λ_j are the eigenvalues of K_j arranged so that $|\Lambda_{i+1}| \leq |\Lambda_i|$, then

$$(A.1) \quad |\Lambda_n| = o(n^{-s-3/2}).$$

(see also Chang [2].) An easy proof is now available.

According to a result of Weyl [10],

$$(A.2) \quad |\Lambda_1 \Lambda_2 \dots \Lambda_n| \leq \kappa_1 \kappa_2 \dots \kappa_n, \quad n = 1, 2, \dots$$

Now, for $n > m(s)$,

$$(A.3) \quad \begin{aligned} \prod_{j=1}^n \kappa_j &= \prod_{j=1}^{m(s)} \kappa_j \prod_{j=1}^{n-m(s)} \kappa_{j+m(s)} \\ &\leq M_s \prod_{j=1}^{n-m(s)} \tilde{\kappa}_j \\ &\leq \frac{M_s \left(\prod_{p=1}^{n-m(s)} \varepsilon'_p \right) \left\| \frac{\partial^{s+1} K}{\partial x^{s+1}} \right\|^{n-m(s)}}{\pi^{(n-m(s))(s+1)} ((n-m(s))!)^{1/2}} \cdot \prod_{p=1}^{n-m(s)} (p+1/2)^{-(s+1)}. \end{aligned}$$

Here, M_s is a constant and we have used Lemma 3.3 applied to $\tilde{\kappa}$. Upon noting that

$$(A.4) \quad |\Lambda_n| \leq \prod_{j=1}^n \kappa_j$$

and making use of the kinds of estimates used in Theorem 4.3, we readily obtain (A.1).

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