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A STRUCTURE THEOREM FOR MOST UNIONS OF COMPLETE INTERSECTIONS

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ABSTRACT. Using the connections among almost complete intersection schemes, arithmetically Gorenstein schemes and schemes that are a union of complete intersections, we give a structure theorem for the arithmetically Cohen-Macaulay union of two complete intersections of codimension 2, of type (d_1, e_1) and (d_2, e_2) such that $\min\{d_1, e_1\} \neq \min\{d_2, e_2\}$. We apply the results for computing Hilbert functions and graded Betti numbers for such schemes.

Introduction. The simplest projective schemes which one can study are those whose defining ideals can be generated by the minimal number of equations with respect to their codimension c. These schemes are complete intersection schemes whose defining ideals are minimally generated by exactly c elements. In particular, this implies that they can be generated by a regular sequence. Now, there are different ways to generalize such a notion. For instance, one may study arithmetically Cohen Macaulay schemes which can be generated by a number of elements equal to one more than the codimension. These kinds of schemes are usually denominated almost complete intersections and are recently studied, for instance, in **[6, 9, 10, 14, 15, 16]**. Another possibility is to study schemes which, as the complete intersections, have Cohen-Macaulay type 1 or equivalently with principal last syzygies module. In this case, we have arithmetically Gorenstein schemes and we have a large literature on this theme (see, for instance, **[1, 3, 7, 12, 13]**). Finally, from a more geometric point of view, one can study schemes which are a finite union of complete intersections with some kind of generic property for realizing such unions.

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All of these generalizations are strictly related, as we show in this paper in the codimension 2 case. Indeed, using information on an almost complete intersection and on a Gorenstein scheme directly linked to it, we obtain nice information on a union of two complete intersection schemes. The idea is very simple. Start from a union of two complete intersections X_1 and X_2 of codimension 2 in $\mathbb{P}^r$ which realizes an arithmetically Cohen-Macaulay scheme X (this, in particular, implies that $X_1 \cap X_2$ is aCM of codimension 3. The sum of their defining ideals I_{X_1} and I_{X_2} in the polynomial ring $R = k[x_0, \ldots, x_r]$ defines an almost complete intersection of codimension 3. Now, if we link $I_{X_1} + I_{X_2}$ in a complete intersection generated by three of its generators, we get a Gorenstein scheme G of codimension 3.

Now, we use the pfaffian resolution of this Gorenstein scheme to obtain a free resolution of the given union I_X obtaining in this way a structure theorem for such schemes (Theorem 1.4). In order to do so we need the assumption that, if X_1 has type (d_1, e_1) and X_2 has type (d_2, e_2), then $\min\{d_1, e_1\} \neq \min\{d_2, e_2\}$. As applications of this result we obtain a description of the Hilbert functions of these schemes, in particular, the starting degree and the Castelnuovo-Mumford regularity for I_X, and much information regarding their graded Betti numbers (indeed, all up to very few cancellations). In many cases, our resolutions are minimal; thus, in these cases we obtain the Hilbert-Burch matrix of the defining ideal of these schemes.

1. Union of complete intersections of codimension two. Let k be an algebraically closed field, and let $R := k[x_0, \ldots, x_r]$. We consider the standard grading on R and we consider only homogeneous ideals in R.

An ideal $I_Q \subset R$ is said to be an *almost complete intersection* ideal of codimension c if I_Q is perfect and it is minimally generated by less or equal to $c+1$ forms (note that we include complete intersections in this definition).

Every almost complete intersection ideal I_Q of codimension c is directly linked in a complete intersection to a Gorenstein ideal $I_G \subset R$. Indeed, if $I_Z \subseteq I_Q$ is generated by c minimal generators of I_Q, which form a regular sequence, then $I_G := I_Z : I_Q$ is a Gorenstein ideal. By liaison theory (see **[11]** for a complete discussion on this argument) we also have $I_Q = I_Z : I_G$.

Gorenstein and almost complete intersection ideals in codimension 3 were extensively studied. In particular, it is well known that the Gorenstein ideals of codimension 3 are generated by the $(n-1)$-pfaffians of an alternating matrix of odd size n, see **[1]**. If A is an alternating matrix, we will denote by $\operatorname{pf} A$ its pfaffian and by $\operatorname{Pf}_r(A)$ the ideal generated by the r-pfaffians of A.

Let $X_1, X_2 \subset \mathbb{P}^r$, $r \geq 2$, be two complete intersection schemes of codimension 2 without common components. Assume that $X_1 \cup X_2$ is aCM. For instance, note that this always occurs for a disjoint union of two zero-dimensional complete intersection schemes of $\mathbb{P}^2$.

Remark 1.1. By the standard exact sequence,

$$0 \longrightarrow I_{X_1} \cap I_{X_2} \longrightarrow I_{X_1} \oplus I_{X_2} \longrightarrow I_{X_1} + I_{X_2} \longrightarrow 0, \tag{1.1}$$

we see that the homological dimension of $R/(I_{X_1} + I_{X_2})$ is less than or equal to 3 (by mapping cone); hence, since by assumption, X_1 and X_2 have no common components, it is exactly 3. Consequently, the ideal $I_{X_1} + I_{X_2}$ is Artinian for $r = 2$, and it is a saturated ideal for $r \geq 3$; precisely, when $r \geq 3$, $I_{X_1} + I_{X_2} = I_{X_1 \cap X_2}$. Therefore, $X_1 \cap X_2$ is aCM of codimension 3.

We give a structure theorem for schemes of the type $X_1 \cup X_2$, with a light restriction about the degrees.

We begin by collecting some well-known facts about pfaffians and determinants of skew-symmetric matrices which will be useful for proving our results.

We introduce the following notation. If M is a matrix, we denote by $M_{[i_1,\ldots,i_r;j_1,\ldots,j_s]}$ the submatrix of M obtained by deleting the rows labeled by the integers $i_1, \ldots, i_r$ and the columns labeled by the integers $j_1, \ldots, j_s$. By $M_{[i_1,\ldots,i_r;-]}$, we will denote the submatrix of M obtained by deleting only the rows, and analogously for $M_{[-;j_1,\ldots,j_s]}$. Moreover, $M_{[i_1,\ldots,i_r;i_1,\ldots,i_r]}$ will be denoted by $M_{(i_1,\ldots,i_r)}$.

In the sequel, if $a_1 < \cdots < a_n$ are integers and $(b_1, \ldots, b_n)$ is a permutation of them, we use $\operatorname{sgn}(b_1, \ldots, b_n)$ as the sign of the permutation $(b_1, \ldots, b_n)$ with respect to $(a_1, \ldots, a_n)$.

Lemma 1.2. *Let A be an alternating matrix of odd size n. Then*:

(i) $\det A_{[i;j]} = \operatorname{pf} A_{(i)} \operatorname{pf} A_{(j)}$.

(ii) $\det A_{[i,j;h,k]} = \operatorname{sgn}(i,j)\operatorname{sgn}(h,k)(\operatorname{sgn}(i,j,h)\operatorname{pf} A_{(i,j,h)} \operatorname{pf} A_{(k)} - \operatorname{sgn}(i,j,k)\operatorname{pf} A_{(i,j,k)} \operatorname{pf} A_{(h)}) = \operatorname{sgn}(h,k)\operatorname{sgn}(i,j)(\operatorname{sgn}(h,k,j)\operatorname{pf} A_{(h,k,j)} \times \operatorname{pf} A_{(i)} - \operatorname{sgn}(h,k,i)\operatorname{pf} A_{(h,k,i)} \operatorname{pf} A_{(j)})$.

Proof. The first result is due to Cayley, see [**2**]. The second is a revision of a generalization due to Heymans, see [**5,** formula (3.31)]. □

Lemma 1.3. *Let A be an alternating matrix of odd size n, and let $p_i := (-1)^i \operatorname{pf} A_{(i)}$. Let*

$$B := \begin{pmatrix} 0 & a_1 & \cdots & a_n \\ b_1 & & & \\ \vdots & & A & \\ b_n & & & \end{pmatrix}.$$

Then

$$\det B = \sum_{i=1}^{n} a_i p_i \sum_{j=1}^{n} b_j p_j.$$

Proof. Using Lemma 1.2 (i), we know that

$$\det A_{[i;j]} = \operatorname{pf} A_{(i)} \operatorname{pf} A_{(j)}.$$

Thus, to obtain the result, it is sufficient to compute $\det B$ by using the Laplace rule with respect to the first row and the first column. □

Theorem 1.4.

(i) *Let $X_1, X_2 \subset \mathbb{P}^r$, $r \geq 2$, be two complete intersection schemes of codimension 2 without common components of type, respectively, (d_1, e_1) and (d_2, e_2), with $d_1 \geq e_1$, $d_2 \geq e_2$, and assume that $e_1 > e_2$. Suppose that $X := X_1 \cup X_2$ is* aCM. *Then, there exists an alternating matrix A of odd size n with entries in R and $3n$ forms α_i, β_i, γ_i, $1 \leq i \leq n$, such that I_X is the ideal generated by the maximal minors* *of the matrix*

$$M = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ \gamma_1 & & & & & \\ \gamma_2 & & & & & \\ \vdots & & & A & & \\ \gamma_{n-1} & & & & & \\ \gamma_n & & & & & \end{pmatrix}.$$

Precisely, it is possible to choose four forms f_1, g_1, f_2 *and* g_2, *with* g_2 *of minimal degree among the four forms, such that* $I_{X_1} = (f_1, g_1)$, $I_{X_2} = (f_2, g_2)$ *and* (f_1, g_1, f_2) *is a regular sequence, and we can choose* M *in such a way that*

$$f_1 = \sum_{i=1}^{n} \alpha_i p_i, \qquad g_1 = \sum_{i=1}^{n} \beta_i p_i,$$

and

$$f_2 = \sum_{i=1}^{n} \gamma_i p_i, \qquad g_2 = \operatorname{pf} \overline{A},$$

where $p_i = (-1)^i \operatorname{pf} A_{(i)}$ *and*

$$\overline{A} = \begin{pmatrix} 0 & 0 & 0 & \gamma_1 & \cdots & \gamma_n \\ 0 & 0 & 0 & \beta_1 & \cdots & \beta_n \\ 0 & 0 & 0 & \alpha_1 & \cdots & \alpha_n \\ -\gamma_1 & -\beta_1 & -\alpha_1 & & & \\ \vdots & \vdots & \vdots & & A & \\ -\gamma_n & -\beta_n & -\alpha_n & & & \end{pmatrix}.$$

(ii) *If M is a matrix as above where* $\mathrm{Pf}_{n-1} A$ *is an ideal of height* 3*, we set*

$$f_1 := \sum_{i=1}^{n} \alpha_i p_i,$$
$$g_1 := \sum_{i=1}^{n} \beta_i p_i,$$
$$f_2 := \sum_{i=1}^{n} \gamma_i p_i,$$

where the p_i's are as in (i)*. If (f_1, g_1, f_2) is a regular sequence and f_2 and* $\mathrm{pf}\,\overline{A}$ *are coprime, then the ideal generated by the maximal minors of M defines a scheme which is a union of two complete intersections of codimension* 2*.*

Proof.

(i) Let $I_{X_1} = (f_1, g_1)$ and $I_{X_2} = (f_2, g_2)$, with $d_i = \deg f_i$ and $e_i = \deg g_i$ for $i = 1, 2$. Consequently, we may choose f_1, g_1, f_2 such that they form a regular sequence (this can be done since the codimension of $X_1 \cap X_2$ is 3). Denote by $I_Q = I_{X_1} + I_{X_2}$ and $I_Z = (f_1, g_1, f_2)$. Let $I_G := I_Z : I_Q$, and note that, since I_Q is the ideal of an almost complete intersection and I_Z is the ideal of a complete intersection contained within it, I_G is the ideal of a Gorenstein scheme of codimension 3. By the structure theorem of Buchsbaum and Eisenbud, see [**1**], there exists an alternating matrix A of odd size n such that $I_G = \mathrm{Pf}_{n-1}(A)$. Thus, $I_G = (p_1, \ldots, p_n)$, where $p_i = (-1)^i \,\mathrm{pf}\, A_{(i)}$. Since $f_1, g_1, f_2 \in I_G$, we can write

$$f_1 = \sum_{i=1}^{n} \alpha_i p_i,$$
$$g_1 = \sum_{i=1}^{n} \beta_i p_i,$$
$$f_2 = \sum_{i=1}^{n} \gamma_i p_i.$$

Furthermore, in this setting, since $e_1 > e_2$, using [**14,** Theorem 3.3], we have that $(f_2, g_2) = (f_2, \mathrm{pf}\,\overline{A})$ (in particular, if $e_2 < d_2$, then $g_2 = \mathrm{pf}\,\overline{A}$, up to a unit). Then, we set

$$M := \begin{pmatrix} 0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ \gamma_1 & & & & & \\ \gamma_2 & & & & & \\ \vdots & & & \mathrm{A} & & \\ \gamma_{n-1} & & & & & \\ \gamma_n & & & & & \end{pmatrix}.$$

We want to show that $I_X = I_{n+1}(M)$. At first, we show that

$$I_{n+1}(M) \subseteq I_X = I_{X_1} \cap I_{X_2}.$$

By Lemma 1.3, we immediately obtain that

$$\det M_{[1;-]} = f_1 f_2, \qquad \det M_{[2;-]} = g_1 f_2.$$

Now let t be an integer, $3 \le t \le n+2$. We want to compute $\det M_{[t;-]}$. In order to do so, we apply the Laplace rule with respect to the first row, the second row and the first column. Thus, if we set

$$\sigma_{ij} := \begin{cases} (-1)^{i+j} & \text{if } i < j \\ (-1)^{i+j+1} & \text{if } i > j \end{cases}$$

$$\tau_{ij} := \begin{cases} (-1)^{i+1} & \text{if } i < j \\ (-1)^{i} & \text{if } i > j \end{cases}$$

and $S_t := \{(i,j,h) \mid 1 \le i,j,h \le n,\ i \ne j,\ h \ne t-2\}$, $3 \le t \le n+2$, we obtain the following expansion

$$\det M_{[t,-]} = \sum_{S_t} \sigma_{ij} \tau_{h,t-2} \alpha_i \beta_j \gamma_h \det A_{[t-2,h;i,j]}.$$

Applying Lemma 1.2 (ii), we have

$$\begin{aligned} &\det M_{[t,-]} \\ &\quad = \sum_{S_t} \sigma_{ij} \tau_{h,t-2} \alpha_i \beta_j \gamma_h \operatorname{sgn}(t-2,h) \\ &\quad \times \operatorname{sgn}(i,j)(\operatorname{sgn}(t-2,h,i) \operatorname{pf} A_{(t-2,h,i)} \operatorname{pf} A_{(j)} \\ &\qquad - \operatorname{sgn}(t-2,h,j) \operatorname{pf} A_{(t-2,h,j)} \operatorname{pf} A_{(i)}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{S_t} (-1)^{i+h} \operatorname{sgn}(t-2,h,i)\alpha_i\beta_j\gamma_h \operatorname{pf} A_{(t-2,h,i)} p_j \\
&\quad - \sum_{S_t} (-1)^{j+h} \operatorname{sgn}(t-2,h,j)\alpha_i\beta_j\gamma_h \operatorname{pf} A_{(t-2,h,j)} p_i \\
&= \sum_j \Bigg(\sum_{i\neq j; h\neq t-2} (-1)^{i+h} \operatorname{sgn}(t-2,h,i)\alpha_i\gamma_h \operatorname{pf} A_{(t-2,h,i)} \Bigg) \beta_j p_j \\
&\quad - \sum_i \Bigg(\sum_{j\neq i; h\neq t-2} (-1)^{j+h} \operatorname{sgn}(t-2,h,j)\beta_j\gamma_h \operatorname{pf} A_{(t-2,h,j)} \Bigg) \alpha_i p_i.
\end{aligned}
$$

Now we sum, subtract the quantity

$$\sum_{k; h\neq t-2} (-1)^{k+h} \operatorname{sgn}(t-2,h,k) \operatorname{pf} A_{(t-2,h,k)} \alpha_k\gamma_h\beta_k p_k$$

and obtain

$$
\begin{aligned}
&\sum_j \Bigg(\sum_{i; h\neq t-2} (-1)^{i+h} \operatorname{sgn}(t-2,h,i)\alpha_i\gamma_h \operatorname{pf} A_{(t-2,h,i)} \Bigg) \beta_j p_j \\
&\quad - \sum_i \Bigg(\sum_{j; h\neq t-2} (-1)^{j+h} \operatorname{sgn}(t-2,h,j)\beta_j\gamma_h \operatorname{pf} A_{(t-2,h,j)} \Bigg) \alpha_i p_i;
\end{aligned}
$$

if we set

$$\lambda := \sum_{i; h\neq t-2} (-1)^{i+h} \operatorname{sgn}(t-2,h,i)\alpha_i\gamma_h \operatorname{pf} A_{(t-2,h,i)}$$

and

$$\mu := \sum_{j; h\neq t-2} (-1)^{j+h} \operatorname{sgn}(t-2,h,j)\beta_j\gamma_h \operatorname{pf} A_{(t-2,h,j)},$$

we have

$$\lambda \sum_j \beta_j p_j - \mu \sum_i \alpha_i p_i = \lambda g_1 - \mu f_1 \in (f_1, g_1) = I_{X_1}.$$

On the other hand,

$$\det M_{[t,-]} = \sum_{S_t} \sigma_{ij}\tau_{h,t-2}\alpha_i\beta_j\gamma_h \det A_{[t-2,h;i,j]}.$$

Applying the second equality in Lemma 1.2 (ii), we have

$$\begin{aligned}
&\det M_{[t,-]}\\
&\quad=\sum_{S_t}\sigma_{ij}\tau_{h,t-2}\alpha_i\beta_j\gamma_h\operatorname{sgn}(h,t-2)\operatorname{sgn}(i,j)\\
&\quad\times(\operatorname{sgn}(i,j,t-2)\operatorname{pf}A_{(i,j,t-2)}\operatorname{pf}A_{(h)}\\
&\qquad-\operatorname{sgn}(i,j,h)\operatorname{pf}A_{(i,j,h)}\operatorname{pf}A_{(t-2)})\\
&\quad=\sum_{S_t}(-1)^{i+j+1}\operatorname{sgn}(i,j,t-2)\alpha_i\beta_j\gamma_h\operatorname{pf}A_{(i,j,t-2)}p_h\\
&\qquad-\sum_{S_t}(-1)^{i+j+h+1}\operatorname{sgn}(i,j,h)\alpha_i\beta_j\gamma_h\operatorname{pf}A_{(i,j,h)}\operatorname{pf}A_{(t-2)}).
\end{aligned}$$

Now sum and subtract the quantity

$$\sum_{i,j;i\neq j}(-1)^{i+j+1}\operatorname{sgn}(i,j,t-2)\alpha_i\gamma_j\beta_{t-2}\operatorname{pf}A_{(i,j,t-2)}p_{t-2}.$$

Then, we obtain

$$\begin{aligned}
&\sum_h\bigg(\sum_{i,j;i\neq j}(-1)^{i+j+1}\operatorname{sgn}(i,j,t-2)\alpha_i\beta_j\operatorname{pf}A_{(i,j,t-2)}\bigg)\gamma_h p_h\\
&\quad-\operatorname{pf}A_{(t-2)}\sum_{i,j,h;i\neq j}(-1)^{i+j+h+1}\operatorname{sgn}(i,j,h)\alpha_i\beta_j\gamma_h\operatorname{pf}A_{(i,j,h)}\\
&=\bigg(\sum_{i,j;i\neq j}(-1)^{i+j+1}\operatorname{sgn}(i,j,t-2)\alpha_i\beta_j\operatorname{pf}A_{(i,j,t-2)}\bigg)\\
&\times\sum_h\gamma_h p_h-\operatorname{pf}A_{(t-2)}\operatorname{pf}\overline{A}\\
&=\bigg(\sum_{i,j;i\neq j}(-1)^{i+j+1}\operatorname{sgn}(i,j,t-2)\alpha_i\beta_j\operatorname{pf}A_{(i,j,t-2)}\bigg)f_2\\
&\quad-\operatorname{pf}A_{(t-2)}\operatorname{pf}\overline{A}\in(f_2,\operatorname{pf}\overline{A})=I_{X_2}.
\end{aligned}$$

Note that, in the last step, we computed $\operatorname{pf}\overline{A}$ by the Laplace rule with respect to the 3×3 minors of the first three rows. Thus, $I_{n+1}(M)$ is contained in $I_{X_1}\cap I_{X_2}$.

Now let Y be the scheme defined by the ideal $I_{n+1}(M)$. In order to complete the proof, it will be enough to show that $\deg Y=\deg X=\deg X_1+\deg X_2$, i.e., we must show that $\deg Y=d_1e_1+d_2e_2$.

From M, we can now deduce the degrees of a set of generators and of the corresponding syzygies of I_Y. The degrees of generators are

$$d_1 + d_2, e_1 + d_2, e_2 + \pi_1, \dots, e_2 + \pi_n,$$

where $\pi_i = \deg p_i$. The degrees of syzygies are

$$e_2 + d_2, d_1 + e_1 + d_2 - \pi_1, \dots, d_1 + e_1 + d_2 - \pi_n.$$

Now, by the degrees of generators and syzygies, the Hilbert function of Y may be computed, from which one can easily deduce the degree of Y. We obtain

$$\begin{aligned} 2\deg Y = (e_2 + d_2)^2 & \\ &+ \sum_{i=1}^{n} (d_1 + e_1 + d_2 - \pi_i)^2 - (d_1 + d_2)^2 - (e_1 + d_2)^2 \\ &- \sum_{i=1}^{n} (e_2 + \pi_i)^2. \end{aligned}$$

Since the p_is are minimal generators for $I_G = I_Z : I_Q$, we have that

$$2 \sum_{i=1}^{n} \pi_i = (n-1)(d_1 + e_1 + d_2 - e_2),$$

see, for instance, **[12]**. Now a straightforward computation shows that

$$2 \deg Y = 2d_1 e_1 + 2d_2 e_2,$$

and we are done.

(ii) Let $I_G := \mathrm{Pf}_{n-1} A$. I_G defines an aG scheme of codimension 3. Of course, I_G contains the complete intersection ideal $I_Z := (f_1, g_1, f_2)$. Now, we set $I_Q := I_Z : I_G$. Using **[14,** Theorem 3.3], $I_Q = (f_1, g_1, f_2, g_2)$, where $g_2 := \mathrm{pf}\, \overline{A}$. Let $I_{X_1} := (f_1, g_1)$ and $I_{X_2} := (f_2, g_2)$. By the hypotheses, X_1 and X_2 are complete intersection schemes. By (i), $I_{n+1}(M) = I_{X_1 \cup X_2}$. $\square$

Remark 1.5. The hypothesis $e_1 > e_2$ is essential for our construction. Indeed, if we consider $I_{X_1} := (x^2, y^2)$ and $I_{X_2} := (t^2, (x+y+t)^2)$ as ideals in $k[x, y, t]$, our construction produces the ideal $I_{X_1} \cap I_{X_2'}$ where $I_{X_2'} := (t^2, xy + xt + yt)$ which is different from $I_{X_1} \cap I_{X_2}$ although $I_{X_1} + I_{X_2} = I_{X_1} + I_{X_2'}$.

Corollary 1.6. *Using the same hypotheses of Theorem* 1.4, *the ideal* $I_{X_1} \cap I_{X_2}$ *admits the following free graded resolution* (*not necessarily minimal*):

$$(1.2)\qquad 0 \longrightarrow R(-(d_2+e_2)) \oplus \bigoplus_{i=1}^{n} R(-(d_1+e_1+d_2-\pi_i))$$
$$\xrightarrow{M} R(-(d_1+d_2)) \oplus R(-(e_1+d_2)) \oplus \bigoplus_{i=1}^{n} R(-(e_2+\pi_i))$$
$$\longrightarrow I_{X_1} \cap I_{X_2} \longrightarrow 0.$$

Proof. The result follows by the degree matrix of M. □

2. Applications to Hilbert functions and graded Betti numbers. From Theorem 1.4, we are able to describe all possible Hilbert functions for aCM schemes which are a union of two complete intersection schemes of codimension 2 without common components. For this, we need no assumption on the degrees.

In the sequel, we will set $(a)_+ := \max\{0, a\}$.

Proposition 2.1. *Let* $X_1, X_2 \subset \mathbb{P}^r$, $r \geq 2$, *be two complete intersection schemes of codimension 2, without common components, of type, respectively,* (d_1, e_1) *and* (d_2, e_2), *with* $d_1 \geq e_1$ *and* $d_2 \geq e_2$ *and say* $e_2 = \min\{e_1, e_2\}$. *Then, the Hilbert function of the Artinian reduction* A *of* $I_{X_1} \cap I_{X_2}$ *is*:

(2.1)
$$\begin{aligned} H_A(t) = t+1 &- \sum_{i=1}^{n} (t+1-e_2-\pi_i)_+ \\ &- (t+1-d_1-d_2)_+ - (t+1-e_1-d_2)_+ \\ &+ \sum_{i=1}^{n} (t+1-d_1-e_1-d_2+\pi_i)_+ + (t+1-d_2-e_2)_+, \end{aligned}$$

where the π_i*s are the minimal generators degrees of an* aG *scheme linked to* $X_1 \cap X_2$ *in a complete intersection of type* (e_1, d_1, d_2).

Proof. If $e_1 > e_2$, then this result follows immediately by Corollary 1.2. Otherwise, the construction of Theorem 1.4 produces the scheme $X_1 \cup X_2'$ (where X_2' is a complete intersection of the same type of X_2) such that $I_{X_1} + I_{X_2'} = I_{X_1} + I_{X_2}$. Thus, using sequence (1.1), we obtain that the resolution of $I_{X_1} \cap I_{X_2}$ is up to cancelations the same as the resolution of $I_{X_1} \cap I_{X_2'}$, from which we get the assertion. $\square$

Remark 2.2. By the formula (2.1), we deduce some facts about the ideal $I_{X_1} \cap I_{X_2}$. For this, we use the following setting $d_1 \geq e_1$ and $\pi_i \leq \pi_{i+1}$ for every i.

(i) The degree of the first generator is $e_2+\pi_1$; indeed, since $I_Z \subseteq I_G$, we have in particular that $\pi_1 \leq e_1 \leq d_1$; thus, $e_2+\pi_1 \leq e_1+d_2 \leq d_1+d_2$.

(ii) The degree of the second generator is $e_2 + \pi_2$; indeed, by the previous observation, $\pi_2 \leq \max\{e_1, d_2\}$; thus, $e_2 + \pi_2 \leq e_1 + d_2 \leq d_1 + d_2$.

(iii) Let

$$\sigma := \max\{t \mid H_A(t) > 0\}$$

(the socle degree of the Artinian algebra A). Note that, since $I_Z \subseteq I_G$, if we arrange (e_1, d_1, d_2) in a manner that is not decreasing, we have that they are, respectively, greater than or equal to $\pi_1 \leq \pi_2 \leq \pi_3$. In any case, $\sigma \leq e_1 + d_1 + d_2 - \pi_1 - 2$ since the biggest degree of a minimal syzygy of A is less than or equal to $e_1 + d_1 + d_2 - \pi_1$ because $d_2+e_2 \leq e_1+d_1+d_2-\pi_1$ (as $e_1 \geq \pi_1$). This bound is sharp if and only if $e_1 > \pi_1$. Whenever $e_1 = \pi_1$, $\sigma \leq e_1 + d_1 + d_2 - \pi_2 - 2$ since, in this case, the biggest degree of a minimal syzygy of A is less than or equal to $e_1+d_1+d_2-\pi_2$ because $d_2+e_2 \leq e_1+d_1+d_2-\pi_2$ (as $d_1 \geq \pi_2$). This bound is sharp if and only if $e_1 = \pi_1$ and $d_1 > \pi_2$. Whenever $e_1 = \pi_1$ and $d_1 = \pi_2$, $\sigma \leq \max\{e_1 + d_1 + d_2 - \pi_3 - 2, d_2 + e_2 - 2\}$; moreover, if $\max\{e_1 + d_1 + d_2 - \pi_3 - 2, d_2 + e_2 - 2\} = e_1 + d_1 + d_2 - \pi_3 - 2$, then $\sigma = e_1 + d_1 + d_2 - \pi_3 - 2$.

(iv) Although the complete intersections have Hilbert function of decreasing type, this is no longer true, in general, for the unions of two of them, as we will see in the Proposition 2.3.

In Proposition 2.3 and Corollary 2.4 we consider some special cases.

Proposition 2.3. *Let $X_1, X_2 \subset \mathbb{P}^r$, $r \geq 2$, be two complete intersection schemes of codimension 2 of type, respectively, (d_1, e_1) and (d_2, e_2).*

Assume that $X_1 \cup X_2$ *is* aCM. *Let* $I_{X_1} = (f_1, g_1)$ *and* $I_{X_2} = (f_2, g_2)$, $\deg f_i = d_i$ *and* $\deg g_i = e_i$. *Suppose that* $f_2 \in (f_1, g_1, g_2)$, *say* $f_2 = a_1 f_1 + b_1 g_1 + b_2 g_2$. *Then the Hilbert-Burch matrix of* $I_{X_1} \cap I_{X_2}$ *is*

$$M = \begin{pmatrix} -g_2 & 0 \\ b_1 & f_1 \\ a_1 & -g_1 \end{pmatrix}.$$

Consequently, $I_{X_1} \cap I_{X_2} = (f_2 - b_2 g_2, g_1 g_2, f_1 g_2)$.

The graded minimal free resolution of $I_{X_1} \cap I_{X_2}$ *is*

$$\begin{aligned} 0 \longrightarrow R(-(d_2+e_2) \oplus R(-(e_1+e_2+d_1) \\ \longrightarrow R(-d_2) \oplus R(-(e_1+e_2)) \oplus R(-(d_1+e_2)). \end{aligned}$$

The Hilbert function of the Artinian reduction A *of* $I_{X_1} \cap I_{X_2}$ *is*

$$H_A(t) = H_{A_1}(t - e_2) + H_{A_2}(t),$$

where A_i *is the Artinian reduction of* I_{X_i}.

Proof. Let $I_Y = I_2(M)$. Note that $\operatorname{ht} I_Y = \operatorname{ht}(f_2 - b_2 g_2, g_1 g_2, f_1 g_2) = 2$; namely, if $\operatorname{ht} I_Y = 1$, the three generators should have a common factor. Since (f_1, g_1) is a regular sequence, f_2 should have a common factor with g_2, a contradiction.

Trivially, $I_Y \subseteq I_{X_1} \cap I_{X_2}$; on the other hand, an easy computation shows that $\deg I_Y = d_1 e_1 + d_2 e_2 = \deg I_{X_1} \cap I_{X_2}$. Thus, $I_Y = I_{X_1} \cap I_{X_2}$.

A minimal set of generators of $I_{X_1} \cap I_{X_2}$, and its resolution can be deduced immediately from the matrix M.

With regard to the Hilbert function, we have, for every $t \in \mathbb{Z}$, that

$$\begin{aligned} H_A(t) = (t+1)_+ - (t+1-d_2)_+ - (t+1-e_1-e_2)_+ - (t+1-d_1-e_2)_+ \\ + (t+1-d_2-e_2)_+ + (t+1-e_1-e_2-d_1)_+; \end{aligned}$$

$$\begin{aligned} H_{A_1}(t-e_2) = (t+1-e_2)_+ - (t+1-e_2-d_1)_+ \\ - (t+1-e_2-e_1)_+ + (t+1-e_2-d_1-e_1)_+; \end{aligned}$$

$$H_{A_2}(t) = (t+1)_+ - (t+1-d_2)_+ - (t+1-e_2)_+ + (t+1-d_2-e_2)_+,$$

from which we obtain our formula. $\square$

Corollary 2.4. *Let* $X_1, X_2 \subset \mathbb{P}^r$, $r \geq 2$, *be two complete intersection schemes of codimension* 2 *of type, respectively,* (d_1, e_1) *and* (d_2, e_2). *Assume that* $X_1 \cup X_2$ *is* aCM. *Let* $I_{X_1} = (f_1, g_1)$ *and* $I_{X_2} = (f_2, g_2)$, $\deg f_i = d_i$ *and* $\deg g_i = e_i$. *We suppose that* $d_2 \geq d_1 + e_1 + e_2 - 2$, *and* (f_1, g_1, g_2) *is a regular sequence. Then, the same conclusions of Proposition* 2.3 *hold.*

In particular, when $d_2 > d_1 + e_1 + e_2$, *then* H_A *is no longer of decreasing type, where* A *is the Artinian reduction of* $I_{X_1} \cap I_{X_2}$.

Proof. Our assumptions imply that $I_{X_1} + I_{X_2}$ is an aCM ideal of height 3. Let B be the Artinian reduction of the complete intersection ideal (f_1, g_1, g_2). Then, $B_t = 0$ for $t \geq d_1 + e_1 + e_2 - 2$; thus, since $d_2 \geq d_1 + e_1 + e_2 - 2$, $f_2 \in (f_1, g_1, g_2)$. Now, applying Proposition 2.3, we obtain our assertion.

If $d_2 > d_1 + e_1 + e_2$, then

$$H_A(t) = H_{A_1}(t - e_2) + H_{A_2}(t) = e_2 < e_1 + e_2$$

for every t such that $d_1 + e_1 + e_2 - 1 \leq t \leq e_2 - 1$, i.e., H_A takes the same value (less than $e_1 + e_2$) in at least two adjacent degrees. $\square$

In the next proposition, we collect results on the graded Betti numbers of our schemes which are consequences of Theorem 1.4.

Proposition 2.5. *Using the same assumptions of Theorem* 1.4, *we have*

(i) *The graded Betti numbers can be obtained by the resolution* (1.2), *only deleting at most three terms in degrees* $d_1 + d_2$, $e_1 + d_2$ *and* $e_2 + d_2$.

(ii) *In any case, two among the products* $f_1 f_2$, $f_1 g_2$, $g_1 f_2$ *or* $g_1 g_2$ *are minimal generators for* $I_{X_1} \cap I_{X_2}$.

Proof. (i) It is enough to observe that the only units in matrix M can appear in the first two rows or in the first column.

(ii) If the resolution (1.2) is minimal, then $f_1 f_2$ and $g_1 f_2$ are the first two maximal minors of the matrix M. Otherwise, suppose that, say $f_1 f_2$ is not a minimal generator for $I_{X_1} \cap I_{X_2}$. This implies that $e_1 = \pi_i$ for some i and g_1 is a minimal generators for I_G; thus, we can replace p_i with g_1. Note that $\deg\det M_{[i+2;-]} = \pi_i + e_2 = e_1 + e_2$, so $\det M_{[i+2;-]}$ can be chosen as a minimal generator for $I_{X_1} \cap I_{X_2}$. Now $\det M_{[i+2;-]} = q + g_1 \operatorname{pf} \overline{A}$ (see the computation in the proof of Theorem 1.4). However, $q + g_1 \operatorname{pf} \overline{A} = q + g_1 g_2 + \lambda f_2 g_1$ for some q and λ. Then $g_1 g_2$ can replace it as a minimal generator for $I_{X_1} \cap I_{X_2}$. Analogously, when $g_1 f_2$ is not a minimal generator for our ideal, the same argument shows that we can take $f_1 g_2$ as a minimal generator. $\square$

Example 2.6. An example is given here in which the resolution (1.2) is minimal free for $I_{X_1} \cap I_{X_2}$. In $R = k[x_0, \ldots, x_8]$, we consider the following two complete intersections $I_{X_1} = (f_1, g_1)$ and $I_{X_2} = (f_2, g_2)$, where

$$\begin{aligned} f_1 &= x_0^3 x_1^3 x_7, \\ g_1 &= x_2^3 x_5^3 x_6 \\ f_2 &= (x_0^3 + x_2^3 + x_4^3) x_3^3 x_8 + (x_0^3 x_7 - x_5^3 x_8) x_1^3, \\ g_2 &= (x_0^3 + x_2^3 + x_4^3) x_6 x_7 x_8. \end{aligned}$$

Let $I_G := (f_1, g_1, f_2) : (f_1, g_1, f_2, g_2)$; I_G is a Gorenstein ideal with five generators in degree 6. Consequently, the resolution (1.2) is

$$0 \to R(-13) \oplus R(-15)^5 \to R(-12)^5 \oplus R(-14)^2 \to I_{X_1} \cap I_{X_2} \to 0,$$

which is clearly minimal.

Remark 2.7. Note that, concerning the cancellations, we describe all the possibilities which could occur in Proposition 2.5. Indeed, it will be sufficient to choose for a suitable Gorenstein G a complete intersection containing it and whose generators are or are not minimal generators for G.

Remark 2.8. Note that, in resolution (1.2), a syzygy of degree $d_2 + e_2$ appears. It induces the trivial syzygy on I_{X_2} via the map in the exact sequence (1.1). This implies that we will have a cancelation in degree $d_2 + e_2$ in the mapping cone in the exact sequence (1.1). In fact, we have

$$I_{X_1} \cap I_{X_2} = (f_1 f_2, g_1 f_2, h_1, \ldots, h_n),$$

where $h_i := \det M_{[i+2;-]}$ (see the proof of Theorem 1.4). The syzygy of degree $d_2 + e_2$, in the same notation of Theorem 1.4, is $(0, 0, \gamma_1, \ldots, \gamma_n)$.

From the proof of Theorem 1.4, we have that $h_i = \lambda_i f_2 + p_i \operatorname{pf} \overline{A}$; thus, we obtain

$$\sum_{i=1}^{n} \gamma_i(\lambda_i f_2 + p_i \operatorname{pf} \overline{A}) = 0 \Longrightarrow \sum_{i=1}^{n} \gamma_i \lambda_i f_2 + \sum_{i=1}^{n} \gamma_i p_i \operatorname{pf} \overline{A} = 0.$$

Since

$$\sum_{i=1}^{n} \gamma_i p_i = f_2,$$

we are done.

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