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ARITHMETICAL RANK OF STRINGS AND CYCLES

KYOUKO KIMURA AND PAOLO MANTERO

ABSTRACT. Let R be a polynomial ring over a field K. To a given squarefree monomial ideal $I \subset R$, one can associate a hypergraph $\mathcal{H}(I)$. In this article, we prove that the arithmetical rank of I is equal to the projective dimension of R/I when $\mathcal{H}(I)$ is a string or a cycle hypergraph.

Introduction. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and I a squarefree monomial ideal of R. The *arithmetical rank* of I, denoted by $\operatorname{ara} I$, is defined as the minimum number u of elements $q_1, \ldots, q_u \in R$ such that the equality

$$\sqrt{(q_1, \ldots, q_u)} = \sqrt{I} \ (= I)$$

holds. When this is the case one says that $q_1, \ldots, q_u$ generate I up to radical. Let $G(I)$ denote the minimal set of monomial generators of I, and set $\mu(I) = \#G(I)$. Then $\operatorname{ara} I \leq \mu(I)$ holds. On the other hand, Lyubeznik **[15]** proved that $\operatorname{ara} I \geq \operatorname{pd} R/I$, where $\operatorname{pd} R/I$ denotes the projective dimension of R/I. Therefore, we have

$$\operatorname{height} I \leq \operatorname{pd} R/I \leq \operatorname{ara} I \leq \mu(I).$$

From the above inequalities, it is natural to ask when $\operatorname{ara} I = \operatorname{pd} R/I$ holds. Many authors including **[1]**–**[11, 13, 16, 17, 18, 19]** investigated this problem. In particular, in **[10, 11]** (see also **[7]**), Terai, Yoshida and the first author attacked the problem for ideals I with $\mu(I) - \operatorname{height} I \leq 2$. Their idea is to classify these squarefree monomial ideals using hypergraphs (this classification is also used in **[12]**). The association of a hypergraph to a squarefree monomial ideal I of R with

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$G(I) = \{m_1, \dots, m_\mu\}$ is defined by setting

$$\mathcal{H}(I) := \big\{\{j \in [\mu] : x_i \mid m_j\} : i = 1, \dots, n\big\}.$$

$\mathcal{H}(I)$ is indeed a (separated) hypergraph on the vertex set $[\mu] := \{1, 2, \dots, \mu\}$. On the other hand, given a separated hypergraph $\mathcal{H}$, one can construct a squarefree monomial ideal I with $\mathcal{H}(I) = \mathcal{H}$; see Section 1 for more details.

We focus on the squarefree monomial ideals I such that $\mathcal{H}(I)$ is a string or a cycle. For these ideals, Lin and the second author **[14]** found an explicit formula expressing the projective dimension of R/I in terms of purely combinatorial invariants of the hypergraph $\mathcal{H}(I)$, namely,

$$\operatorname{pd}(R/I) = \mu(I) - b(\mathcal{H}(I)) + M(\mathcal{H}(I)), \tag{0.1}$$

see the discussion before Theorem 2.3 for the definition of $b(\mathcal{H}(I))$ and $M(\mathcal{H}(I))$.

In the present work, we study the arithmetical rank of these ideals I. We prove that $\operatorname{pd} R/I$ elements can be chosen so that they generate I up to radical and have "small" monomial support. To be more precise, let us recall that the *binomial arithmetical rank* of I, denoted by $\operatorname{biara} I$, is the minimum number u of monomials or binomials $q_1, \dots, q_u \in R$ which generate I up to radical. Here, we also define the *trinomial arithmetical rank* of I as the minimum number u of monomials, binomials or trinomials $q_1, \dots, q_u \in R$ which generate I up to radical. We denote this by $\operatorname{triara} I$. Clearly, we have $\operatorname{ara} I \le \operatorname{triara} I \le \operatorname{biara} I$. Our main result is the next theorem.

Theorem 0.1. *Let I be a squarefree monomial ideal of R.*

(1) *Assume that $\mathcal{H}(I)$ is a string hypergraph. Then* $\operatorname{ara} I = \operatorname{biara} I = \operatorname{pd} R/I$.
(2) *Assume that $\mathcal{H}(I)$ is a cycle hypergraph. Then* $\operatorname{ara} I = \operatorname{triara} I = \operatorname{pd} R/I$.

In particular, the arithmetical rank of these ideals is independent of the characteristic of the field K. Crucial ingredients of our proof of Theorem 0.1 are a lemma by Schmitt and Vogel ([**18**], Lemma 3.2) and formula (0.1) for the projective dimension (Theorem 2.3).

Now we explain the organization of this article. In Section 1, we recall the definition of the (separated) hypergraph associated to a squarefree monomial ideal, first introduced in [**10**]. In Section 2, we recall a few results by Lin and the second author [**14**] that will be employed in the subsequent sections. Then, in Sections 3 and 4, we prove Theorem 0.1 (1) and (2), respectively.

1. Hypergraphs. In this section, we recall the construction of a separated hypergraph associated to any squarefree monomial ideal. The construction was introduced in [**10**], see also [**7, 11, 12, 14**].

Set $V = [\mu]$. A collection $\mathcal{H} \subset 2^V$ is called a *hypergraph* on vertex set V if $V = \bigcup_{F\in\mathcal{H}} F$. An element $F \in \mathcal{H}$ is called a *face* of $\mathcal{H}$. A vertex $j \in V$ is called *closed*, respectively, *open*, if $\{j\} \in \mathcal{H}$, respectively, $\{j\} \notin \mathcal{H}$. A hypergraph is called *saturated* if $\{j\} \in \mathcal{H}$ for all $j \in V$. Let $i, j \in V$ be two vertices of $\mathcal{H}$. We say that i is a neighbor of j if there exists a face $F \in \mathcal{H}$ containing both i and j.

A hypergraph $\mathcal{H}$ on V is said to be *separated* if, for all vertices $i, j \in V$, $i \neq j$, there exist faces $F, G \in \mathcal{H}$ such that $i \in F \setminus G$ and $j \in G \setminus F$. Let I be a squarefree monomial ideal of $R = K[x_1, \ldots, x_n]$ with $G(I) = \{m_1, \ldots, m_\mu\}$. The hypergraph associated to I is defined as

$$\mathcal{H}(I) := \big\{\{j \in [\mu] \ : \ x_i \mid m_j\} : i = 1, \ldots, n\big\},$$

which is a separated hypergraph on $[\mu]$.

Conversely, let $\mathcal{H}$ be a separated hypergraph on $[\mu]$. Then, we can construct a squarefree monomial ideal I with $\mathcal{H}(I) = \mathcal{H}$ in a polynomial ring with enough variables as follows: for each $F \in \mathcal{H}$, take a squarefree monomial m_F such that m_F and m_G are coprime if $F \neq G$. For each $j \in [\mu]$, set $m_j = \prod_{F\in\mathcal{H},\, j\in F} m_F$. Then $I = (m_1, \ldots, m_\mu)$ is a squarefree monomial ideal with $\mathcal{H}(I) = \mathcal{H}$. This construction implies that there are many ideals I (in various polynomial rings) with $\mathcal{H}(I) = \mathcal{H}$. We set $I(\mathcal{H})$ to be the ideal obtained from the above construction by setting each m_F to be a variable x_F in a polynomial ring $R(\mathcal{H}) := K[x_F : F \in \mathcal{H}]$.

The above correspondence between squarefree monomial ideals and separated hypergraphs yields the classification of squarefree monomial ideals mentioned in the introduction. The next proposition shows the usefulness of this association for our purpose.

Proposition 1.1 ([**7,** Proposition 3.2], [**14,** Corollary 2.4])**.** *Let I_1 and I_2 be squarefree monomial ideals with $\mathcal{H}(I_1) = \mathcal{H}(I_2)$. Then* $\operatorname{pd} I_1 = \operatorname{pd} I_2$ *and* $\operatorname{ara} I_1 = \operatorname{ara} I_2$ *hold.*

Let I be a squarefree monomial ideal of R. Set $\mathcal{H} = \mathcal{H}(I)$. By Proposition 1.1, the following notation is well defined:

$$\operatorname{pd}(\mathcal{H}) := \operatorname{pd} R/I, \qquad \operatorname{ara}(\mathcal{H}) := \operatorname{ara}(I).$$

We call $\operatorname{pd}(\mathcal{H})$, respectively, $\operatorname{ara}(\mathcal{H})$, the projective dimension, respectively, arithmetical rank, of $\mathcal{H}$. We will compute $\operatorname{pd}(\mathcal{H})$ and $\operatorname{ara}(\mathcal{H})$ by computing $\operatorname{pd} R(\mathcal{H})/I(\mathcal{H})$ and $\operatorname{ara} I(\mathcal{H})$, respectively.

Remark 1.2. The statement of Proposition 1.1 remains true if we replace the arithmetical rank by the binomial or the trinomial arithmetical rank. Hence, we use the similar notation $\operatorname{biara}(\mathcal{H})$ and $\operatorname{triara}(\mathcal{H})$.

2. Projective dimensions of a string hypergraph and a cycle hypergraph. In this section, we collect results about the projective dimensions of a string hypergraph and a cycle hypergraph. These results are proved by Lin and the second author in [**14**].

We first recall the definitions of a string hypergraph and a cycle hypergraph.

Definition 2.1 ([**14,** Definition 2.13])**.** Fix $\mu \geq 2$. A hypergraph $\mathcal{H}$ on $V = [\mu]$ is a string if $\{j, j+1\} \in \mathcal{H}$ for all $j = 1, \dots, \mu - 1$, and the only other possible faces of $\mathcal{H}$ are of the form $\{j\}$, for some $j \in V$.

For a string hypergraph $\mathcal{H}$ on $[\mu]$, we call the vertices 1 and μ the endpoints of $\mathcal{H}$. Note that, if $\mathcal{H}$ is separated, then both endpoints are closed vertices.

Definition 2.2 ([**14,** Definition 4.1])**.** Fix $\mu \geq 3$. A hypergraph $\mathcal{H}$ on $V = [\mu]$ is a μ-cycle if $\mathcal{H}$ can be written as $\mathcal{H} = \widetilde{\mathcal{H}} \cup \{\{\mu, 1\}\}$ where $\widetilde{\mathcal{H}}$ is a string hypergraph on $[\mu]$.

To introduce the explicit formula for the projective dimension of a string hypergraph and a cycle hypergraph in terms of invariants of the hypergraph we need some more definitions.

A hypergraph $\mathcal{H}$ on $[\mu]$ is called a *string of opens* if $\mathcal{H}$ is a string hypergraph with $\mu \geq 3$ whose only closed vertices are its endpoint.

First, we assume that $\mathcal{H}$ is a string hypergraph. We set $s = s(\mathcal{H})$ to be the number of strings of opens inside $\mathcal{H}$. We number the strings of opens in $\mathcal{H}$ from one endpoint to another and set $n_i(\mathcal{H})$ to be the number of open vertices in the ith string of opens. We say that $\mathcal{H}$ is a 2-*special configuration* if $s \geq 2$, $\mathcal{H}$ does not contain two adjacent closed vertices, $n_1 \equiv n_s \equiv 1 \mod 3$, and $n_i \equiv 2 \mod 3$ for $i = 2, \ldots, s-1$. Two 2-special configurations contained in $\mathcal{H}$ are said to be *disjoint* if they do not have a common open vertex. The *modularity* of $\mathcal{H}$, denoted by $M(\mathcal{H})$, is the maximum number of pairwise disjoint 2-special configurations contained in $\mathcal{H}$.

Next, we assume that $\mathcal{H}$ is a cycle hypergraph. If $\mathcal{H}$ contains at least two closed vertices, we define $s = s(\mathcal{H})$ and $n_1(\mathcal{H}), \ldots, n_s(\mathcal{H})$ analogously to the case of a string hypergraph. If $\mathcal{H}$ contains at most one closed vertex, we set $s = s(\mathcal{H}) = 1$ and $n_1(\mathcal{H}) = \mu(\mathcal{H}) - 1$. In either case, the definition of a 2-special configuration $\mathcal{S}$ in $\mathcal{H}$ is the same as in the case of a string hypergraph, except for allowing that the two extremal vertices of $\mathcal{S}$ coincide. The modularity $M(\mathcal{H})$ is defined in the same way as in the case of a string hypergraph.

Let $\mathcal{H}$ be a string hypergraph or a cycle hypergraph. Set

$$b(\mathcal{H}) = s(\mathcal{H}) + \sum_{i=1}^{s(\mathcal{H})} \left\lfloor \frac{n_i(\mathcal{H}) - 1}{3} \right\rfloor .$$

Theorem 2.3 ([**14,** Theorems 3.4, 4.3])**.** *Let $\mathcal{H}$ be a string hypergraph or a cycle hypergraph. Then*

$$\mathrm{pd}(\mathcal{H}) = \mu(\mathcal{H}) - b(\mathcal{H}) + M(\mathcal{H}).$$

We also state some inductive results about the projective dimension.

Let I be a squarefree monomial ideal with $G(I) = \{m_1, \ldots, m_\mu\}$. Then, we set $I_i := (m_{i+1}, \ldots, m_\mu)$ and $\mathcal{H}_i := \mathcal{H}(I_i)$. Also, we set $J_1 := I_1 : m_1$ and $\mathcal{Q}_1 := \mathcal{H}(J_1)$.

Lemma 2.4 ([**14,** Lemmas 2.6, 2.11]). *Let $\mathcal{H}$ be a hypergraph on $[\mu]$ with $\mu \geq 2$. Assume that $\{1\} \in \mathcal{H}$. Then,* $\operatorname{pd}(\mathcal{H}) = \max\{\operatorname{pd}(\mathcal{H}_1), \operatorname{pd}(\mathcal{Q}_1) + 1\}$. *Moreover, if all the neighbors of* 1 *are closed vertices, then* $\operatorname{pd}(\mathcal{H}) = \operatorname{pd}(\mathcal{H}_1) + 1$.

Finally, for a string hypergraph $\mathcal{H}$, we will use the following results that allow us to compare $\operatorname{pd}(\mathcal{H})$ with the projective dimension of a smaller string hypergraph.

Lemma 2.5 ([**14,** Lemma 2.14 (ii)]). *Let $\mathcal{H}$ be a string hypergraph on $[\mu]$ with $\mu \geq 3$. Then* $\operatorname{pd}(\mathcal{H}) \leq \operatorname{pd}(\mathcal{H}_2) + 2$.

Lemma 2.6 ([**14,** Proposition 2.15]). *Let $\mathcal{H}$ be a string hypergraph on $[\mu]$ with $\mu \geq 4$. Assume $\{2\} \notin \mathcal{H}$. Then* $\operatorname{pd}(\mathcal{H}) = \operatorname{pd}(\mathcal{H}_3) + 2$.

3. Strings. In this section, we consider string hypergraphs. The goal of this section is to prove the next result.

Theorem 3.1. *Let $\mathcal{H}$ be a string hypergraph. Then* $\operatorname{ara}(\mathcal{H}) = \operatorname{biara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$.

Before proving the theorem, we introduce a useful lemma by Schmitt and Vogel [**18**].

Lemma 3.2 ([**18,** page 249, Lemma]). *Let R be a commutative ring and P a finite subset of R. Let $P_0, P_1, \ldots, P_u$ be subsets of P satisfying the following three conditions*:

- (SV1) $\bigcup_{\ell=0}^{u} P_\ell = P$.
- (SV2) $\# P_0 = 1$.
- (SV3) *For any integer $\ell > 0$ and elements $p, p'' \in P_\ell$ with $p \neq p''$, there exist an integer $\ell' < \ell$ and an element $p' \in P_{\ell'}$ such that $pp'' \in (p')$.*

Let I be an ideal of R generated by P, and set

$$q_\ell = \sum_{p \in P_\ell} p, \qquad \ell = 0, 1, \ldots, u.$$

Then $q_0, q_1, \ldots, q_u$ generate I up to radical.

We first see the case where the number of vertices is ≤ 3.

Lemma 3.3. *Let $\mathcal{H}$ be a string hypergraph on $[\mu]$. If $\mu \leq 3$, then* $\operatorname{ara}(\mathcal{H}) = \operatorname{biara}(\mathcal{H}) = \operatorname{pd}(H)$.

Proof. If $\mathcal{H}$ is saturated, then $\operatorname{pd}(\mathcal{H}) = \mu$, and there is nothing to prove. The remaining case is that $\mu = 3$ and the vertex 2 of $\mathcal{H}$ is open. Then, $I(\mathcal{H}) = (y_1x_1, x_1x_2, y_3x_2)$. In this case, $\operatorname{pd}(\mathcal{H}) = 2$. By Lemma 3.2, we have $x_1x_2, y_1x_1 + y_3x_2$ generate $I(\mathcal{H})$ up to radical. $\square$

Next we assume $\mu \geq 4$. We divide the proof into two cases, depending on whether vertex 2 is closed or open.

Lemma 3.4. *Let $\mathcal{H}$ be a string hypergraph on $[\mu]$. Assume the neighbor* 2 *of endpoint* 1 *of $\mathcal{H}$ is closed. If* $\operatorname{biara}(\mathcal{H}_1) = \operatorname{pd}(\mathcal{H}_1)$, *then* $\operatorname{biara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$.

Proof. We first note that $\operatorname{biara}(\mathcal{H}) \leq \operatorname{biara}(\mathcal{H}_1) + 1$ since $I(\mathcal{H})$ has one more generator than $I(\mathcal{H}_1)$. We then have the chain of inequalities

$$\operatorname{biara}(\mathcal{H}) \leq \operatorname{biara}(\mathcal{H}_1) + 1 = \operatorname{pd}(\mathcal{H}_1) + 1 = \operatorname{pd}(\mathcal{H}) \leq \operatorname{biara}(\mathcal{H}),$$

where the last equality follows by Lemma 2.4. Therefore, $\operatorname{biara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$. $\square$

Lemma 3.5. *Let $\mathcal{H}$ be a string hypergraph on $[\mu]$ with $\mu \geq 4$. Assume the neighbor* 2 *of endpoint* 1 *of $\mathcal{H}$ is open. If* $\operatorname{biara}(\mathcal{H}_3) = \operatorname{pd}(\mathcal{H}_3)$, *then* $\operatorname{biara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$.

Proof. Write $I(\mathcal{H}) = I_3 + I'$, where $I_3 = I(\mathcal{H}_3) = (m_4, \ldots, m_\mu)$ and $I' = (m_1, m_2, m_3)$. Note that $\mathcal{H}(I')$ is a string hypergraph on vertex set $[3]$. Since the vertex 2 of $\mathcal{H}(I')$ is open, we have $\operatorname{biara} I' = 2$ by Lemma 3.3. We then have

$$\begin{aligned}\operatorname{biara}(\mathcal{H}) &= \operatorname{biara}(I_3 + I') \leq \operatorname{biara}(I_3) + \operatorname{biara}(I') \\ &= \operatorname{biara}(I_3) + 2 = \operatorname{pd}(\mathcal{H}_3) + 2.\end{aligned}$$

Since $\operatorname{pd}(\mathcal{H}_3) + 2 = \operatorname{pd}(\mathcal{H})$ by Lemma 2.6, and $\operatorname{pd}(\mathcal{H}) \leq \operatorname{biara}(\mathcal{H})$ always holds, we have $\operatorname{biara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$. $\square$

We can now prove Theorem 3.1.

Proof of Theorem 3.1. We prove it by induction on the number μ of vertices of $\mathcal{H}$.

If $\mu \leq 3$, then the statement follows by Lemma 3.3. We may then assume $\mu \geq 4$, and the statement is proved for string hypergraphs with $< \mu$ vertices. Then both $\operatorname{biara}(\mathcal{H}_1) = \operatorname{pd}(\mathcal{H}_1)$ and $\operatorname{biara}(\mathcal{H}_3) = \operatorname{pd}(\mathcal{H}_3)$ hold, and the assertion follows from Lemmas 3.4 and 3.5. $\square$

4. Cycles. In this section, we consider cycle hypergraphs. The goal of this section is to prove the next result.

Theorem 4.1. *Let $\mathcal{H}$ be a cycle hypergraph. Then* $\operatorname{ara}(\mathcal{H}) = \operatorname{triara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$.

We first consider the case where $\mathcal{H}$ contains at most one closed vertex.

Lemma 4.2. *Let $\mathcal{H}$ be a cycle hypergraph. If $\mathcal{H}$ contains at most* 1 *closed vertex, then* $\operatorname{triara}(\mathcal{H}) = \operatorname{biara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$.

If $\mathcal{H}$ does not contain any closed vertex, then $I(\mathcal{H})$ is also the edge ideal of a cycle. In [**2,** Propositions 2.2, 2.3, 2.4], Barile, et al., constructed monomials and binomials which generate this ideal up to radical. Below, we show that the same construction with minor modifications also works for $\mathcal{H}$, which contains precisely one closed vertex.

Proof of Lemma 4.2. Let $\mathcal{H}$ be a μ-cycle. By assumption, we may assume that the monomial generators of $I(\mathcal{H})$ are of the forms:

$$yx_1x_\mu,\ x_1x_2,\ x_2x_3, \dots, x_{\mu-1}x_\mu,$$

where $x_1, x_2, \dots, x_\mu$ are pairwise distinct variables and y is either a variable which is different from $x_1, x_2, \dots, x_\mu$ or $y = 1$. By Theorem 2.3, we have

$$\operatorname{pd}(\mathcal{H}) = \mu - \left(1 + \left\lfloor \frac{\mu - 2}{3} \right\rfloor\right).$$

We distinguish three cases.

Case 1. $\mu = 3m$ $(m \geq 1)$. In this case, $\mathrm{pd}(\mathcal{H}) = 2m$. Consider the $2m$ elements:

$$\begin{cases} q_0 = x_1x_2, \\ q_1 = yx_1x_\mu + x_2x_3, \end{cases}$$

$$\begin{cases} q_{2i} = x_{3i+1}x_{3i+2}, \\ q_{2i+1} = x_{3i}x_{3i+1} + x_{3i+2}x_{3i+3}, \end{cases} \qquad i = 1, 2, \ldots, m-1.$$

Lemma 3.2, see also [**2,** Proposition 2.2], yields that $q_0, q_1, \ldots, q_{2m-1}$ generate $I(\mathcal{H})$ up to radical.

Case 2. $\mu = 3m+1$ $(m \geq 1)$. In this case, $\mathrm{pd}(\mathcal{H}) = 2m+1$. Consider the following $2m$ elements:

$$\begin{cases} q_{2i} = x_{3i+2}x_{3i+3}, \\ q_{2i+1} = x_{3i+1}x_{3i+2} + x_{3i+3}x_{3i+4}, \end{cases} \qquad i = 0, 1, 2, \ldots, m-1.$$

Set $q_{2m} = yx_1x_{3m+1}$.

Lemma 3.2, see also [**2,** Proposition 2.3], now yields that $q_0, q_1, \ldots, q_{2m}$ generate $I(\mathcal{H})$ up to radical.

Case 3. $\mu = 3m+2$ $(m \geq 1)$. In this case, $\mathrm{pd}(\mathcal{H}) = 2m+1$. Consider the following $2m$ elements:

$$\begin{cases} q_0 = x_1x_2, \\ q_1 = x_2x_3 + x_4x_5, \end{cases}$$

$$\begin{cases} q_{2i} = x_{3i}x_{3i+1} + x_{3i+2}x_{3i+3}, \\ q_{2i+1} = x_{3i+2}x_{3i+3} + x_{3i+4}x_{3i+5}, \end{cases} \qquad i = 1, 2, \ldots, m-1.$$

Set $q_{2m} = yx_1x_{3m+2} + x_{3m}x_{3m+1}$, see also [**2,** Proposition 2.4].

Set $J = (q_0, q_1, \ldots, q_{2m})$. We claim $\sqrt{J} = I(\mathcal{H})$. It is clear that $J \subset I(\mathcal{H})$. Thus, we prove $\sqrt{J} \supset I(\mathcal{H})$.

We first prove $x_1I(\mathcal{H}) \subset \sqrt{J}$. Since one has $q_0, q_1 \in J$, then $x_1 \cdot x_1x_2, x_1x_2x_3, x_1x_4x_5 \in \sqrt{J}$. We claim that

$$x_1x_{3i}x_{3i+1}, x_1x_{3i+2}x_{3i+3}, x_1x_{3i+4}x_{3i+5} \in \sqrt{J}, \qquad i = 1, 2, \ldots, m-1. \tag{4.1}$$

We prove this by induction on i.

For the case $i = 1$, we need to prove that $x_1x_3x_4, x_1x_5x_6, x_1x_7x_8 \in \sqrt{J}$. Since $x_1q_2 = x_1x_3x_4 + x_1x_5x_6 \in J$ and $x_1x_4x_5 \in \sqrt{J}$, Lemma 3.2 yields $x_1x_3x_4, x_1x_5x_6 \in \sqrt{J}$. Then, since $x_1q_3 = x_1x_5x_6 + x_1x_7x_8 \in J$ and $x_1x_5x_6 \in \sqrt{J}$, we also have $x_1x_7x_8 \in \sqrt{J}$.

Assume that equation (4.1) is true for $i - 1$. Then, since $x_1q_{2i} = x_1x_{3i}x_{3i+1}+x_1x_{3i+2}x_{3i+3} \in J$ and $x_1x_{3i+1}x_{3i+2} = x_1x_{3(i-1)+4}x_{3(i-1)+5} \in \sqrt{J}$, Lemma 3.2 yields $x_1x_{3i}x_{3i+1}, x_1x_{3i+2}x_{3i+3} \in \sqrt{J}$. Then $x_1q_{2i+1} = x_1x_{3i+2}x_{3i+3} + x_1x_{3i+4}x_{3i+5} \in J$ and $x_1x_{3i+2}x_{3i+3} \in \sqrt{J}$; hence, we have $x_1x_{3i+4}x_{3i+5} \in \sqrt{J}$, as required. Therefore, equation (4.1) holds true for all i. Moreover,

$$q_{2m} = yx_1x_{3m+2} + x_3x_{3m+1} \in J$$

and

$$x_1x_{3m+1}x_{3m+2} = x_1x_{3(m-1)+4}x_{3(m-1)+5} \in \sqrt{J}.$$

These two facts imply

$$x_1 \cdot yx_1x_{3m+2}, x_1x_{3m}x_{3m+1} \in \sqrt{J}.$$

Hence, we have $x_1I(\mathcal{H}) \subset \sqrt{J}$.

Next, we prove $I(\mathcal{H}) \subset \sqrt{J}$. Since $x_1I(\mathcal{H}) \subset \sqrt{J}$, we have $yx_1^2x_{3m+2} \in \sqrt{J}$, whence $yx_1x_{3m+2} \in \sqrt{J}$. Since $q_{2m} \in J$, we also have $x_{3m}x_{3m+1} \in \sqrt{J}$.

We now prove

$$x_{3i}x_{3i+1},\ x_{3i+2}x_{3i+3},\ x_{3i+4}x_{3i+5} \in \sqrt{J}, \quad i = 1, 2, \ldots, m-1 \tag{4.2}$$

by descending induction on i.

When $i = m - 1$, since $x_{3m}x_{3m+1} \in \sqrt{J}$ and $q_{2(m-1)+1} = x_{3m-1}x_{3m}+x_{3m+1}x_{3m+2} \in J$, Lemma 3.2 gives $x_{3m-1}x_{3m}, x_{3m+1}x_{3m+2} \in \sqrt{J}$. Also, since $q_{2(m-1)} = x_{3m-3}x_{3m-2} + x_{3m-1}x_{3m} \in J$, we have $x_{3m-3}x_{3m-2} \in \sqrt{J}$.

Next, assume that equation (4.2) holds true for $i + 1$. Since $q_{2i+1} = x_{3i+2}x_{3i+3} + x_{3i+4}x_{3i+5} \in J$ and $x_{3i+3}x_{3i+4} = x_{3(i+1)}x_{3(i+1)+1} \in \sqrt{J}$, then Lemma 3.2 yields $x_{3i+2}x_{3i+3}, x_{3i+4}x_{3i+5} \in \sqrt{J}$. Then $q_{2i} = x_{3i}x_{3i+1} + x_{3i+2}x_{3i+3} \in J$; so we have $x_{3i}x_{3i+1} \in \sqrt{J}$, as required.

Note that $x_1x_2 = q_0 \in J$. Also, since $q_1 = x_2x_3 + x_4x_5$ and $x_3x_4 \in \sqrt{J}$, then $x_4x_5 \in \sqrt{J}$. This completes the proof. $\square$

Next, we consider the case where the number of vertices is at most four. In this case, we know that $\operatorname{ara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$ by **[10]**. We prove the following, slightly more precise lemma.

Lemma 4.3. *Let $\mathcal{H}$ be a cycle hypergraph on $[\mu]$ with $\mu \leq 4$. Then* $\operatorname{triara}(\mathcal{H}) = \operatorname{biara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$.

Proof. We first assume that $\operatorname{pd}(\mathcal{H}) = \mu$. In this case, we can choose μ monomial generators. Next, we assume that $\operatorname{pd}(\mathcal{H}) < \mu$. In this case, we can easily check that $\operatorname{pd}(\mathcal{H}) = \mu - 1$.

When $\mu = 3$, then the three generators of $I(\mathcal{H})$ can be written as x_1x_2, $y_1x_1x_3$ and $y_2x_2x_3$, where each y_i can possibly be 1. By Lemma 3.2, $x_1x_2, y_1x_1x_3 + y_2x_2x_3$ generate $I(\mathcal{H})$ up to radical.

When $\mu = 4$, then the four generators of $I(\mathcal{H})$ can be written as x_1x_2, $y_1x_1x_4$, $y_2x_2x_3$ and $y_3x_3x_4$, where each y_i is possibly 1. Lemma 3.2 yields that the elements x_1x_2, $y_1x_1x_4 + y_2x_2x_3$ and $y_3x_3x_4$ generate $I(\mathcal{H})$ up to radical. $\square$

Thus, we can assume that the number of vertices of a cycle hypergraph is at least five.

Lemma 4.4. *Let $\mathcal{H}$ be a cycle hypergraph on $[\mu]$ with $\mu \geq 5$. If $\mathcal{H}$ contains two adjacent closed vertices, then* $\operatorname{triara}(\mathcal{H}) = \operatorname{biara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$.

Proof. Without loss of generality, we may assume 1 and μ are two adjacent closed vertices.

We first assume that the vertex 2 is also closed. Then we have $\operatorname{pd}(\mathcal{H}) = \operatorname{pd}(\mathcal{H}_1) + 1$, by Lemma 2.4. Since $\mathcal{H}_1$ is a string hypergraph, we have $\operatorname{biara}(\mathcal{H}_1) = \operatorname{pd}(\mathcal{H}_1)$, by Theorem 3.1. Now, the equality $\operatorname{biara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$ follows because the monomial m_1 corresponding to the vertex 1, together with elements which generate $I(\mathcal{H}_1)$ up to radical, generate $I(\mathcal{H})$ up to radical, i.e. if $\sqrt{I(\mathcal{H}_1)} = \sqrt{(a_1, \ldots, a_r)}$, then $\sqrt{I(\mathcal{H})} = \sqrt{(m_1, a_1, \ldots, a_r)}$.

We may then assume that the vertex 2 is open. Then the monomials corresponding to vertices 1, 2 and 3 can be written as $y_1x_1x_\mu$, x_1x_2 and $y_3x_2x_3$, respectively, where y_3 is possibly 1. Note that $\mathcal{Q}_1$ is the disjoint union of $\mathcal{H}_3$ and a closed vertex. Thus, $\operatorname{pd}(\mathcal{Q}_1) = \operatorname{pd}(\mathcal{H}_3) + 1$. By Lemma 2.4, we have

$$\operatorname{pd}(\mathcal{H}) = \max\{\operatorname{pd}(\mathcal{H}_1), \operatorname{pd}(\mathcal{Q}_1) + 1\} = \max\{\operatorname{pd}(\mathcal{H}_1), \operatorname{pd}(\mathcal{H}_3) + 2\}.$$

Since $\mathcal{H}_1$ is a string hypergraph, we have $\operatorname{pd}(\mathcal{H}_1) \leq \operatorname{pd}(\mathcal{H}_3) + 2$ by Lemma 2.5, and thus, $\operatorname{pd}(\mathcal{H}) = \operatorname{pd}(\mathcal{H}_3) + 2$. Also, since $\mathcal{H}_3$ is a string hypergraph, Theorem 3.1 shows that $\operatorname{biara}(\mathcal{H}_3) = \operatorname{pd}(\mathcal{H}_3)$.

Since the elements x_1x_2 and $y_1x_1x_\mu + y_3x_2x_3$, together with elements which generate I_3 up to radical, generate $I(\mathcal{H})$ up to radical, we obtain $\operatorname{biara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$. □

In order to prove the next lemma, we use Theorem 2.3.

Lemma 4.5. *Let $\mathcal{H}$ be a cycle hypergraph. Suppose that there is a string of opens with n_0 open vertices, with $n_0 \equiv 0 \mod 3$ in $\mathcal{H}$. Then* $\operatorname{triara}(\mathcal{H}) = \operatorname{biara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$.

Proof. By Lemma 4.2, we may assume that $\mathcal{H}$ contains at least 2 closed vertices. Let $\mathcal{S}_0$ be the string of opens with n_0 open vertices, and let u_1, u_2, u_3 be three adjacent open vertices in $\mathcal{S}_0$ such that u_1 is adjacent to a closed vertex v. Let v' be the other neighbor of u_3. We consider the ideal I'' with $G(I'') = G(I) \setminus \{u_1, u_2, u_3\}$. Then, $\mathcal{H}'' := \mathcal{H}(I'')$ is a string hypergraph whose endpoints are v and v', i.e., $\mathcal{H}''$ is obtained by *deletion* of the vertices u_1, u_2 and u_3 from $\mathcal{H}$ and changing v' to be closed if v' is open in $\mathcal{H}$. We claim that $\operatorname{pd}(\mathcal{H}) = \operatorname{pd}(\mathcal{H}'') + 2$. Then, since we know that $\operatorname{biara}(\mathcal{H}'') = \operatorname{ara}(\mathcal{H}'') = \operatorname{pd}(\mathcal{H}'')$, we can conclude that $\operatorname{biara}(\mathcal{H}) = \operatorname{ara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$, because $\operatorname{ara}(\mathcal{H}'')$ elements which generate I'' up to radical, together with u_2 and $u_1 + u_3$, generate I up to radical.

Hence, we only need to prove the equality $\operatorname{pd}(\mathcal{H}) = \operatorname{pd}(\mathcal{H}'') + 2$. We first note that $\mu(\mathcal{H}'') = \mu(\mathcal{H}) - 3$ and that v' is a closed vertex in $\mathcal{H}''$ (independently of whether it is closed or not in $\mathcal{H}$).

If v' is closed in $\mathcal{H}$, then $s(\mathcal{H}'') = s(\mathcal{H}) - 1$. Since $\lfloor (n_0 - 1)/3 \rfloor = 0$, we have $b(\mathcal{H}'') = b(\mathcal{H}) - 1$. Moreover, $M(\mathcal{H}'') = M(\mathcal{H})$, because $\mathcal{S}_0$

does not belong to any 2-special configuration in $\mathcal{H}$. Therefore, we have $\mathrm{pd}(\mathcal{H}) = \mathrm{pd}(\mathcal{H}'') + 2$, by Theorem 2.3.

If v' is open in $\mathcal{H}$, then $s(\mathcal{H}'') = s(\mathcal{H})$. Let n_0'' be the number of open vertices in the string of opens $\mathcal{H}''$, one of whose endpoints is v'. Then, $n_0'' = n_0 - 4 \equiv 2 \mod 3$. Note that $\lfloor (n_0 - 1)/3 \rfloor = n_0/3 - 1$ and $\lfloor (n_0'' - 1)/3 \rfloor = n_0/3 - 2$. Thus, $b(\mathcal{H}'') = b(\mathcal{H}) - 1$. Moreover, we have $M(\mathcal{H}'') = M(\mathcal{H})$, because both strings of opens do not belong to any 2-special configuration. Therefore, by Theorem 2.3, we have $\mathrm{pd}(\mathcal{H}) = \mathrm{pd}(\mathcal{H}'') + 2$. □

By Lemma 4.5, we may then assume that each string of opens in $\mathcal{H}$ contains a number of open vertices that is either congruent to $2 \mod 3$ or $1 \mod 3$.

Lemma 4.6. *If we prove that* $\mathrm{ara}(\mathcal{H}) = \mathrm{triara}(\mathcal{H}) = \mathrm{pd}(\mathcal{H})$ *for a cycle hypergraph* $\mathcal{H}$ *whose strings of opens all have at most* 2 *open vertices, then Theorem* 4.1 *follows.*

Proof. Let $\mathcal{H}$ be a μ-cycle. By Lemma 4.2, we may assume $\mathcal{H}$ has at least two closed vertices. By Lemma 4.3, we may assume $\mu \geq 5$. Moreover, by Lemma 4.4, we may assume that there are no two adjacent closed vertices in $\mathcal{H}$.

Suppose that $\mathcal{H}$ contains a string of opens $\mathcal{S}$ with $n_0 \geq 3$ open vertices. By Lemma 4.5, we may assume that $n_0 \equiv 1, 2 \mod 3$.

We first assume that $n_0 \equiv 1 \mod 3$. Let v be an endpoint of $\mathcal{S}$, and let u_1, u_2, u_3, u_4 be adjacent open vertices following v. Let $\mathcal{H}'$ be the cycle hypergraph obtained by turning u_2 into a closed vertex. We claim that $\mathrm{pd}(\mathcal{H}) = \mathrm{pd}(\mathcal{H}')$.

Indeed, by the change we made, the string of opens $\mathcal{S}$ in $\mathcal{H}$ is now divided into two strings of opens $\mathcal{S}_1$ and $\mathcal{S}_2$ (in $\mathcal{H}'$), with 1 and $n_0 - 2$ open vertices, respectively. It is easy to see that $\mu(\mathcal{H}') = \mu(\mathcal{H})$ and $s(\mathcal{H}') = s(\mathcal{H}) + 1$. Also, since $\lfloor (n_0 - 1)/3 \rfloor = (n_0 - 1)/3$, $\lfloor (1-1)/3 \rfloor + \lfloor ((n_0 - 2) - 1)/3 \rfloor = (n_0 - 1)/3 - 1$, we have $b(\mathcal{H}') = b(\mathcal{H})$. Moreover, the modularity is also unchanged because the change does not affect the number of 2-special configurations. Now, equality $\mathrm{pd}(\mathcal{H}) = \mathrm{pd}(\mathcal{H}')$ follows from the formula of Theorem 2.3.

Next, assume that $n_0 \equiv 2 \mod 3$. Let v be an endpoint of $\mathcal{S}$, and let u_1, u_2, u_3, u_4 and u_5 be adjacent open vertices following v. Let $\mathcal{H}'$ be the cycle hypergraph obtained by turning u_3 into a closed vertex. We claim that $\mathrm{pd}(\mathcal{H}) = \mathrm{pd}(\mathcal{H}')$.

By the change, the string of opens $\mathcal{S}$ in $\mathcal{H}$ is now divided into two strings of opens $\mathcal{S}_1$ and $\mathcal{S}_2$ (in $\mathcal{H}'$), with 2 and $n_0 - 3$ open vertices, respectively. It is easy to see that $\mu(\mathcal{H}') = \mu(\mathcal{H})$ and $s(\mathcal{H}') = s(\mathcal{H}) + 1$. Since $\lfloor (n_0 - 1)/3 \rfloor = (n_0 - 2)/3$, $\lfloor (2-1)/3 \rfloor + \lfloor ((n_0 - 3) - 1)/3 \rfloor = (n_0 - 2)/3 - 1$, we have $b(\mathcal{H}') = b(\mathcal{H})$. Furthermore, the modularity is also unchanged because the change does not affect the number of 2-special configurations. All of the above together with the formula of Theorem 2.3 implies the equality $\mathrm{pd}(\mathcal{H}) = \mathrm{pd}(\mathcal{H}')$.

Moreover, in either case, if $\mathrm{triara}(\mathcal{H}') = \mathrm{pd}(\mathcal{H}')$, then also $\mathrm{triara}(\mathcal{H}) = \mathrm{pd}(\mathcal{H})$ holds, as can be seen by substituting 1 for the variables corresponding to the vertices which we caused to become closed.

Then, this procedure produces a new hypergraph $\widetilde{\mathcal{H}}$ (obtained by causing selected open vertices of $\mathcal{H}$ to become closed) and all strings of opens in $\widetilde{\mathcal{H}}$ have at most two open vertices. Moreover, the above shows that, if $\mathrm{triara}(\widetilde{\mathcal{H}}) = \mathrm{pd}(\widetilde{\mathcal{H}})$, then $\mathrm{triara}(\mathcal{H}) = \mathrm{pd}(\mathcal{H})$ also holds. The statement now follows. $\square$

By the above results, we may then assume that $\mathcal{H}$ is a cycle not containing two consecutive closed vertices and whose strings of opens have at most two open vertices. Note that, for such a graph, $b(\mathcal{H}) = s(\mathcal{H})$ holds.

Next, we prove the case where there are strings of opens with precisely two open vertices.

Lemma 4.7. *Assume that $\mathcal{H}$ contains a closed-open-open-closed string $\mathcal{S}$, where the two closed vertices of $\mathcal{S}$ are distinct. Let $\mathcal{H}'$ be the cycle hypergraph obtained by removing the two open vertices of $\mathcal{S}$ from $\mathcal{H}$ and identifying the two closed vertices of $\mathcal{S}$.*

If $\mathrm{triara}(\mathcal{H}') = \mathrm{pd}(\mathcal{H}')$, *then* $\mathrm{triara}(\mathcal{H}) = \mathrm{pd}(\mathcal{H})$.

Proof. Let $\mathcal{H}$ be a cycle hypergraph on $[\mu]$. By Lemma 4.4, we may assume that there are no two adjacent closed vertices in $\mathcal{H}$, and by

Lemma 4.6, all strings of opens have at most two open vertices. We first claim that $\mathrm{pd}(\mathcal{H}) = \mathrm{pd}(\mathcal{H}') + 2$.

It is easy to see that $\mu(\mathcal{H}) = \mu(\mathcal{H}') + 3$ and $s(\mathcal{H}) = s(\mathcal{H}') + 1$. Since the removed string of opens has two open vertices, the modularity is unchanged. Hence, the claim follows by the formula of Theorem 2.3. To prove the statement, we show that $\mathrm{triara}(\mathcal{H}) \le \mathrm{triara}(\mathcal{H}') + 2$.

Let $1, \mu, \mu-1, \mu-2$ be the vertices of string $\mathcal{S}$. We set the monomials corresponding to these vertices to be

$$y_1x_1x_\mu, x_{\mu-1}x_\mu, x_{\mu-2}x_{\mu-1}, y_{\mu-2}x_{\mu-3}x_{\mu-2}. \tag{4.3}$$

We set

$$\begin{cases} g_0 = y_1y_{\mu-2}x_1x_{\mu-3}x_{\mu-2}x_\mu, \\ g_1 = y_1x_1x_\mu + x_{\mu-2}x_{\mu-1}, \\ g_2 = y_{\mu-2}x_{\mu-3}x_{\mu-2} + x_{\mu-1}x_\mu. \end{cases}$$

We claim that

$$y_1x_1x_\mu,\ x_{\mu-1}x_\mu,\ x_{\mu-2}x_{\mu-1},\ y_{\mu-2}x_{\mu-3}x_{\mu-2} \in \sqrt{(g_0, g_1, g_2)}.$$

Indeed, since

$$x_{\mu-2}x_{\mu-1} \cdot x_{\mu-1}x_\mu = (g_1 - y_1x_1x_\mu)(g_2 - y_{\mu-2}x_{\mu-3}x_{\mu-2}) \in (g_0, g_1, g_2),$$

we have $x_{\mu-2}x_{\mu-1}x_\mu \in \sqrt{(g_0, g_1, g_2)}$. Then the claim follows by Lemma 3.2.

Let I_0 be the squarefree monomial ideal which is generated by all monomials in $G(I(\mathcal{H}))$ except for the four monomials in equation (4.3). Then $I(\mathcal{H}) = I_0 + (y_1x_1x_\mu, x_{\mu-1}x_\mu, x_{\mu-2}x_{\mu-1}, y_{\mu-2}x_{\mu-3}x_{\mu-2})$. Let I' be the squarefree monomial ideal defined as

$$I' = I_0 + (y_1y_{\mu-2}x_1x_{\mu-3}x_{\mu-2}x_\mu),$$

and note that $\mathcal{H}(I') = \mathcal{H}'$. Since $g_0 = y_1y_{\mu-2}x_1x_{\mu-3}x_{\mu-2}x_\mu \in I'$ it follows that $\mathrm{ara}(\mathcal{H}')$ elements which generate I' up to radical, together with g_1, g_2 generate $I(\mathcal{H})$ up to radical. $\square$

Therefore, we reduce to the case of cycle hypergraphs in which closed vertices and open vertices appear alternately.

Lemma 4.8. *Let $\mathcal{H}$ be a cycle hypergraph in which closed vertices and open vertices appear alternately. Then we have* $\operatorname{ara}(\mathcal{H}) = \operatorname{triara}(\mathcal{H}) = \operatorname{pd}(\mathcal{H})$.

Proof. We first note that the number μ of vertices of $\mathcal{H}$ is even.

Case 1. $\mu = 4m$. By Theorem 2.3, we have $\operatorname{pd}(\mathcal{H}) = 3m$, because $s(\mathcal{H}) = 2m$ and $M(\mathcal{H}) = m$. We now divide the vertices into disjoint groups of four adjacent vertices. In other words, there exist m strings of the shape *closed-open-closed-open* in $\mathcal{H}$. It suffices to show that the ideal associated to any such string is generated up to radical by three polynomials, each of which has at most three terms. So, letting m_1, m_2, m_3 and m_4 be monomials corresponding to the four vertices of the string, we can write m_1, m_2, m_3 and m_4 as $y_1x_1x_\mu, x_1x_2, y_3x_2x_3, x_3x_4$. By Lemma 3.2, the following three polynomials

$$x_1x_2,\ y_1x_1x_\mu + y_3x_2x_3,\ x_3x_4$$

generate (m_1, m_2, m_3, m_4) up to radical, whence the statement follows.

Case 2. $\mu = 4m + 2$ $(m \geq 1)$. In this case, we prove the statement by induction on m. First assume $m = 1$. Then $I(\mathcal{H})$ is generated by the following six monomials:

$$y_1x_1x_6,\ x_1x_2,\ y_3x_2x_3,\ x_3x_4,\ y_5x_4x_5,\ x_5x_6.$$

By Theorem 2.3, we have $\operatorname{pd}(\mathcal{H}) = 4$, and by Lemma 3.2, the following 4 polynomials generate $I(\mathcal{H})$ up to radical:

$$\begin{cases} x_1x_2, \\ x_3x_4, \\ x_5x_6, \\ y_1x_1x_6 + y_3x_2x_3 + y_5x_4x_5. \end{cases}$$

Now we assume that $m \geq 2$. In this case, $\mathcal{H}$ contains a 2-special configuration $\mathcal{S}$: *closed-open-closed-open-closed*. Let $1, \mu, \mu-1, \mu-2, \mu-3$ be the vertices of the string $\mathcal{S}$. We set the monomials corresponding to these vertices to be

$$y_1x_1x_\mu,\ x_{\mu-1}x_\mu,\ y_{\mu-1}x_{\mu-2}x_{\mu-1},\ x_{\mu-3}x_{\mu-2},\ y_{\mu-3}x_{\mu-4}x_{\mu-3}. \tag{4.4}$$

Let $\mathcal{H}'$ be the cycle hypergraph obtained by removing the three inner vertices $\mu, \mu-1, \mu-2$ of $\mathcal{S}$ from $\mathcal{H}$ and identifying the two endpoints 1 and $\mu-3$ of $\mathcal{S}$. Then $\mathcal{H}'$ is the cycle hypergraph with $\mu(\mathcal{H}') = 4(m-1)+2$ in which closed vertices and open vertices appear alternately. Note that Theorem 2.3 yields $\operatorname{pd}(\mathcal{H}) = \operatorname{pd}(\mathcal{H}')+3$, because $\mu(\mathcal{H}) = \mu(\mathcal{H}')+4$, $s(\mathcal{H}) = s(\mathcal{H}') + 2$, and $M(\mathcal{H}) = M(\mathcal{H}') + 1$.

Let I_0 be the squarefree monomial ideal which is generated by all monomials in $G(I(\mathcal{H}))$ except for the five monomials in equation (4.4). Then

$$I(\mathcal{H}) = I_0+(y_1x_1x_\mu,\ x_{\mu-1}x_\mu,\ y_{\mu-1}x_{\mu-2}x_{\mu-1},\ x_{\mu-3}x_{\mu-2},\ y_{\mu-3}x_{\mu-4}x_{\mu-3}).$$

We set $I' = I_0 + (y_1y_{\mu-3}x_1x_{\mu-4}x_{\mu-3}x_\mu)$. Note that $\mathcal{H}(I') = \mathcal{H}'$ and $y_1y_{\mu-3}x_1x_{\mu-4}x_{\mu-3}x_\mu$ is the monomial corresponding to the vertex 1 of $\mathcal{H}(I')$. Since $y_1y_{\mu-3}x_1x_{\mu-4}x_{\mu-3}x_\mu \in I'$, the following three polynomials, together with $\operatorname{ara}(\mathcal{H}')$ elements which generate I' up to radical, generate $I(\mathcal{H})$ up to radical:

$$\begin{cases} x_{\mu-1}x_\mu, \\ x_{\mu-3}x_{\mu-2}, \\ y_1x_1x_\mu + y_{\mu-1}x_{\mu-2}x_{\mu-1} + y_{\mu-3}x_{\mu-4}x_{\mu-3}. \end{cases} \qquad \square$$

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Shizuoka University, Department of Mathematics, Faculty of Science, 836 Ohya, Suruga-ku, Shizuoka, 422-8529 Japan
Email address: **kimura.kyoko.a@shizuoka.ac.jp**

University of Arkansas, Department of Mathematical Sciences, Fayetteville, AR 72701
Email address: **pmantero@uark.edu**