SOME REMARKS ON BIEQUIDIMENSIONALITY
OF TOPOLOGICAL SPACES
AND NOETHERIAN SCHEMES

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ABSTRACT. There are many examples of the fact that dimension and codimension behave somewhat counterintuitively. In [**2**], it is stated that a topological space is equidimensional, equicodimensional and catenary if and only if every maximal chain of irreducible closed subsets has the same length. We construct examples that show that this is not even true for the spectrum of a Noetherian ring. This gives rise to two notions of biequidimensionality, and we show how these relate to the dimension formula and the existence of a codimension function.

1. Introduction. Unless otherwise stated, all topological spaces considered here are Noetherian T_0-spaces of finite dimension.

For a topological space X we define its *Krull dimension* and codimension in terms of maximal chains of irreducible closed subsets. Then the following definitions are standard, see for example, [**2,** Definitions (0.14.1.3), (0.14.2.1) and Proposition (0.14.3.2)].

Definition 1.1. Let X be a topological space.

(i) The space X is *equidimensional* if all irreducible components of X have the same dimension.

(ii) The space X is *equicodimensional* if all minimal irreducible closed subsets of X have the same codimension in X.

(iii) The space X is *catenary* if for all irreducible closed subsets $Y \subseteq Z$ all saturated chains of irreducible closed subsets that start with Y and end in Z have the same length.

In addition, we define the following.

Definition 1.2. Let X be a topological space.

(i) The space X is *weakly biequidimensional* if it is equidimensional, equicodimensional and catenary.

(ii) The space X is *biequidimensional* if all maximal chains of irreducible closed subsets of X have the same length.

We will see in Lemma 2.1 that every biequidimensional space is weakly biequidimensional. In [**2**, Proposition (0.14.3.3)], it is moreover claimed that a topological space is equidimensional, equicodimensional and catenary if and only if all maximal chains have the same length. Furthermore, they define a space to be biequidimensional if "those equivalent properties" hold. In Section 3, however, we construct examples that show that this is not the case even for Noetherian affine schemes.

The results on biequidimensional schemes stated in [**2**] are correct as long as biequidimensional is defined in the stronger sense as in Definition 1.2 (ii). In Section 4, we show, for example, that the *dimension formula* [**2**, Corollary (0.14.3.5)] need not hold for weakly biequidimensional schemes. Moreover, we prove in Section 5, that every biequidimensional space has a codimension function, whereas a weakly biequidimensional space need not.

The main reference for biequidimensional spaces and schemes is [**2**]. In accordance with the incorrect equivalence [**2**, Proposition (0.14.3.3)], many articles define biequidimensional as equidimensional, equicodimensional and catenary, and then they use properties like the dimension formula that only hold for spaces or schemes that are biequidimensional in the sense of Definition 1.2 (ii). In most cases, the spaces considered are even biequidimensional in that stronger sense, so the damage is relatively small. The purpose of this article is to raise awareness of the difference between the two concepts.

2. Biequidimensionality. Before constructing the examples, we discuss the two notions of biequidimensionality. As advertised earlier, we show first that the property weakly biequidimensional is indeed weaker.

Lemma 2.1. *Let X be a topological space. If X is biequidimensional, then X is weakly biequidimensional.*

Proof. Let Z be an irreducible component in X. Every maximal chain of length $\dim(Z)$ in Z is a maximal chain in X, and hence, it has length $\dim(X)$. Thus, X is equidimensional.

In the same way, it follows that every minimal irreducible closed subset has codimension $\dim(X)$, that is, the space X is equicodimensional.

Now, let $Y \subseteq Z$ be two irreducible closed subsets in X. All saturated chains between Y and Z can be completed by the same irreducible sets to maximal chains in X, and hence, they have the same length.   □

As we will see in Section 3, the converse implication does not hold. However, we have the following result.

Lemma 2.2. *Let X be a topological space. If X is equidimensional, catenary and every irreducible component of X is equicodimensional, then X is biequidimensional.*

Proof. Let $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k$ be a maximal chain of irreducible closed subsets in X. We have to show that it has length $\dim(X)$.

Since X is catenary, we have that $k = \operatorname{codim}(X_0, X_k)$. The minimal subset X_0 has codimension $\dim(X_k)$ in the equicodimensional component X_k. Finally, we have that $\dim(X_k) = \dim(X)$ as X is equidimensional.   □

Proposition 2.3. *Let X be a topological space. Then the following are equivalent.*

(i) *The space X is biequidimensional.*
(ii) *The space X is equidimensional, and for all irreducible closed subsets $Y \subseteq Z$ of X, we have that*

$$(2.1) \qquad \dim(Z) = \dim(Y) + \operatorname{codim}(Y, Z).$$

(iii) *The space X is equicodimensional, and for all irreducible closed subsets $Y \subseteq Z$ of X we have that*

$$(2.2) \qquad \operatorname{codim}(Y, X) = \operatorname{codim}(Y, Z) + \operatorname{codim}(Z, X).$$

Proof. These equivalences are shown in [**2**, Proposition (0.14.3.3)].
For the sake of completeness we give a proof here as well.

Suppose first that X is biequidimensional. We showed in Lemma 2.1
that X is equidimensional, equicodimensional and catenary. Let $Y \subseteq Z$
be two irreducible closed subsets. All maximal chains in Z can be
extended by the same irreducible sets containing Z to maximal chains
in X, and hence, they all have length $\dim(Z)$. In particular, this
holds for the chain obtained by composing a saturated chain of length
$\dim(Y)$ in Y and one of length $\mathrm{codim}(Y, Z)$ between Y and Z, and
equation (2.1) follows. Equation (2.2) can be shown in the same way.
Hence, assertion (i) implies both assertions (ii) and (iii).

For the converse implications, we show first that each of the equa-
tions (2.1) and (2.2) implies that X is catenary. Let $Y \subseteq Z \subseteq T$ be
irreducible closed subsets in X. Applying formulas (2.1) and (2.2),
respectively, to all three inclusions $Y \subseteq Z$, $Z \subseteq T$ and $Y \subseteq T$ gives

$$\mathrm{codim}(Y, T) = \mathrm{codim}(Y, Z) + \mathrm{codim}(Z, T).$$

By [**2**, Proposition (0.14.3.2)], this implies that X is catenary.

Now, let $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k$ be a maximal chain in X. Then X_0
is a closed point, and X_k is an irreducible component. Suppose first
that X is as in assertion (ii). Then X is catenary, and hence, we have
that $k = \mathrm{codim}(X_0, X_k)$. Since the closed point X_0 has dimension 0
and the irreducible component X_k has dimension $\dim(X)$ by equidi-
mensionality, equation (2.1) applied to the inclusion $X_0 \subsetneq X_k$ yields
that $k = \dim(X)$. This shows that assertion (ii) implies assertion (i).

Similarly, in the case of assertion (iii), we have that $\mathrm{codim}(X_k, X) = 0$
and, by equicodimensionality, that $\mathrm{codim}(X_0, X) = \dim(X)$. Hence,
equation (2.2) implies that $k = \dim(X)$, and assertion i holds.   □

Proposition 2.3 shows that assertions (a), (c) and (d) in [**2**, Propo-
sition (0.14.3.3)] are equivalent, and by Lemma 2.1, they imply asser-
tion (b).

In the case that X is a scheme, biequidimensionality can also be
described in the following way.

Lemma 2.4. *Let X be a scheme. Then the following are equivalent.*

(i) *The scheme X is biequidimensional.*
(ii) *For every closed point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is catenary and equidimensional of dimension $\dim(X)$.*

Proof. Let $x \in X$ be a point. Then there is an order-preserving bijection between the spectrum of the local ring $\mathcal{O}_{X,x}$ and the irreducible closed subsets Y of X containing $\overline{\{x\}}$. In particular, we see that X is catenary if and only if all local rings $\mathcal{O}_{X,x}$ are catenary. Moreover, a local ring $\mathcal{O}_{X,x}$ is equidimensional if and only if the subset $\overline{\{x\}}$ has the same codimension in every irreducible component of X that contains it.

Suppose first that X is biequidimensional. Then, X and hence all local rings are catenary by Lemma 2.1. We have, moreover, that $\dim(\mathcal{O}_{X,x}) = \operatorname{codim}(\overline{\{x\}}, X)$ and, as X is equicodimensional, for a closed point x, the latter number equals $\dim(X)$. Let Z be an irreducible component containing the closed point x. Every saturated chain between $\{x\}$ and Z is a maximal chain in X, and hence, it has length $\dim(X)$. This shows that every closed point has the same codimension in every irreducible component containing it.

Conversely, suppose that all local rings at closed points are catenary and equidimensional of dimension $\dim(X)$. Let $X_0 \subsetneq \cdots \subsetneq X_k$ be a maximal chain in X. Then $X_0 = \{x\}$ is a closed point, and there is an associated maximal chain in $\operatorname{Spec}(\mathcal{O}_{X,x})$. Since the local ring $\mathcal{O}_{X,x}$ is catenary and equidimensional, it follows that $k = \dim(\mathcal{O}_{X,x})$. Hence, all maximal chains in X have length $\dim(X)$. □

Remark 2.5. Note, however, that for a scheme to be biequidimensional, it is not sufficient for it to be equidimensional and that all local rings are catenary and equidimensional. Consider, for example, the spectrum X of the localization of $k[u, v, w, x, y, z]/(wx)$ away from the union of the prime ideals (u, v, w), (w, x) and (x, y, z). Then, X is equidimensional of dimension 2. Being essentially of finite type over a field, the scheme X and hence the local rings are catenary. The point corresponding to the maximal ideal (w, x) has codimension 1 in both irreducible components of X. As this is the only point that is contained in both components, we see that all local rings are equidimensional.

However, we have the following maximal chains of prime ideals in X:

$$
\begin{array}{ccc}
(w) & & (x) \\
| & \diagdown & \diagup \ | \\
(v,w) & & (x,y) \\
| & & | \\
(u,v,w) \quad (w,x) & & (x,y,z).
\end{array}
$$

Here the lines denote inclusion. We see that there are maximal chains of lengths 1 and 2. Hence, X is not biequidimensional.

Lemma 2.6. *Let X be an equidimensional scheme that is locally of finite type over a field k. Then X is biequidimensional.*

Proof. By Lemma 2.2, it suffices to show that every irreducible component Y of X is catenary and equicodimensional.

First, we observe that all local rings $\mathcal{O}_{Y,y}$ are localizations of finitely generated k-algebras, and hence, by [**2**, Corollary (0.16.5.12)], they are catenary. As there is a bijection between the prime ideals in $\mathcal{O}_{Y,y}$ and the irreducible closed subsets of Y containing $\overline{\{y\}}$, it follows that Y is catenary.
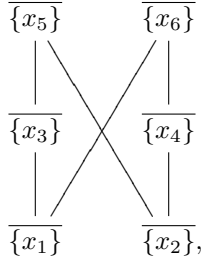
Let ξ be the generic point in Y. By [**3**, Proposition (5.2.1) and equation (5.2.1.1)], we have that $\dim(\mathcal{O}_{Y,y}) = \mathrm{tr}.\deg_k(\kappa(\xi)) = \dim(Y)$ for every closed point $y \in Y$. It follows that all closed points have the same codimension in Y, that is, Y is equicodimensional. $\square$

3. Construction of counterexamples. In this section, we construct examples of topological spaces, affine schemes and even Noetherian affine schemes that are weakly biequidimensional but not biequidimensional.

3.1. Topological spaces. Our first counterexample is the following finite topological space.

Example 3.1. Consider the finite topological space X with six points $x_1, \ldots, x_6$, each the generic point of an irreducible closed subset.

The relations between its irreducible closed subsets are given by the following diagram:

$$
\begin{array}{ccc}
\overline{\{x_5\}} & & \overline{\{x_6\}} \\
| & \times & | \\
\overline{\{x_3\}} & & \overline{\{x_4\}} \\
| & \times & | \\
\overline{\{x_1\}} & & \overline{\{x_2\}},
\end{array}
$$

where a line between two sets denotes inclusion, with the sets increasing from bottom to top. Then the topological space X is clearly equidimensional, equicodimensional and catenary. However, there are maximal chains of length 1 as well as chains of length 2.

3.2. Affine schemes. The properties biequidimensional and weakly biequidimensional are not equivalent, at least not in the given generality. This gives rise to the question whether they are equivalent at least in the particular case that X is the underlying topological space of an (affine) scheme.

However, even in this situation the answer is no. The next theorem classifies *spectral spaces*, that is, topological spaces that are the underlying topological space of an affine scheme.

Theorem 3.2 ([**5,** Theorem 6 and Proposition 10]). *Let X be a topological space. The following are equivalent.*

(i) *The space X is isomorphic to the underlying topological space of the spectrum of a ring.*
(ii) *The space X is the projective limit of finite T_0-spaces.*

In particular, this implies that every finite T_0-space is the spectrum of a ring, and we get the following counterexample.

Example 3.3. Consider the topological space X discussed in Example 3.1. By Theorem 3.2, it can be realized as the spectrum of a ring.

This gives an affine scheme that is weakly biequidimensional but not biequidimensional.

Therefore, the equivalence does not hold for general (affine) schemes, but it might still apply for Noetherian schemes.

3.3. Noetherian schemes. The space defined in Example 3.1 cannot be realized as the spectrum of a Noetherian ring by the next result.

Proposition 3.4. *Let A be a Noetherian ring, and let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ be two prime ideals such that $\mathrm{ht}_{A/\mathfrak{p}_1}(\mathfrak{p}_2/\mathfrak{p}_1) \geq 2$. Then there exist infinitely many prime ideals $\mathfrak{q}$ with $\mathfrak{p}_1 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}_2$.*

Proof. After replacing A by $(A/\mathfrak{p}_1)_{\mathfrak{p}_2}$, we can, without loss of generality, assume that A is a local domain of dimension ≥ 2, and it suffices to show that A has infinitely many prime ideals of height 1. Note that, by *Krull's principal ideal theorem*, every non-unit is contained in a prime ideal of height 1. As the maximal ideal $\mathfrak{p}_2$ consists of the set of non-units in A, it follows that $\mathfrak{p}_2$ is contained in the union of all prime ideals of height 1. If there are only finitely many prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_k$ of height 1, then *Prime avoidance* implies that $\mathfrak{p}_2$ is contained in one of the $\mathfrak{q}_i$, a contradiction. $\square$

The question of which topological space can be realized $\mathrm{Spec}(A)$ for a Noetherian ring A is unfortunately still open; an extensive survey of the state of the art can be found in [**7**]. For our particular case we can, however, make use of the next result.

Theorem 3.5 ([**1**, Theorem B]). *Let $\mathcal{A}$ be a finite partially ordered set. Then there exist a reduced Noetherian ring A and an embedding $i\colon \mathcal{A} \hookrightarrow \mathrm{Spec}(A)$ with the following properties.*

(i) *The map i establishes a bijection between the maximal (respectively, minimal) elements of $\mathcal{A}$ and the maximal (respectively, minimal) elements of $\mathrm{Spec}(A)$.*

(ii) *For all $a, a' \in \mathcal{A}$ such that $a < a'$ is saturated in $\mathcal{A}$, the chain $i(a) \subseteq i(a')$ is saturated in $\mathrm{Spec}(A)$.*

(iii) *For all $a, a' \in \mathcal{A}$, there exists a saturated chain of prime ideals of length r between $i(a)$ and $i(a')$ if and only if there exists a saturated chain of length r between a and a'.*

Example 3.6. Consider the finite partially ordered set $\mathcal{A}$ described by the diagram
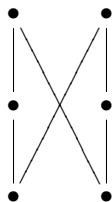
(where we again write elements in increasing order from bottom to top). There exist a Noetherian ring A and an embedding $i: \mathcal{A} \hookrightarrow \operatorname{Spec}(A)$ satisfying the properties of Theorem 3.5.

Let us take a closer look at the difference between $i(\mathcal{A})$ and $\operatorname{Spec}(A)$. Every additional point must lie between a minimal and a maximal element, but it cannot break a saturated chain. Moreover, we must have infinitely many prime ideals of height 1 in every component of dimension 2 by Proposition 3.4. This shows that the underlying topological space of $X = \operatorname{Spec}(A)$ is given as
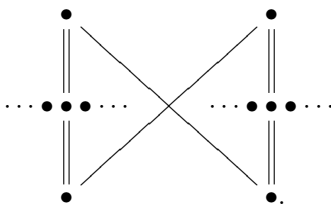
Instead of having only two prime ideals of height 1, there are infinitely many on each side. We see that X is equidimensional, equicodimensional and catenary. However, there are maximal chains of length 1 and of length 2.

This shows that even an affine Noetherian scheme that is weakly biequidimensional need not be biequidimensional.

Another counterexample, which is moreover essentially of finite type over a field, can be constructed in the following way.

Example 3.7. Let $B = k[v, w, x, y]/(vy, wy)$. Then the spectrum $Y = \mathrm{Spec}(B)$ is the scheme obtained by gluing $\mathbb{A}_k^3 = \mathrm{Spec}(k[v, w, x])$ and $\mathbb{A}_k^2 = \mathrm{Spec}(k[x, y])$ along the lines $v = w = 0$ and $y = 0$. Localizing B away from the union $(v, w, x, y - 1) \cup (v, w, y)$ of prime ideals gives a Noetherian ring A with two minimal and two maximal ideals. The spectrum X of A looks like

$$
\begin{array}{ccc}
(v, w) & & (y) \\
\| & \diagdown & \| \\
\cdots(v, w, x)\cdots & & \cdots(w, y)\cdots \\
\| & & \| \\
(v, w, x, y - 1) & & (v, w, y),
\end{array}
$$

with infinitely many prime ideals between (v, w) and $(v, w, x, y-1)$ and between (y) and (v, w, y).

We observe that X is equidimensional, equicodimensional and catenary. However, we have the maximal chains $(v, w) \subsetneq (v, w, y)$ of length 1 and $(v, w) \subsetneq (v, w, x) \subsetneq (v, w, x, y - 1)$ of length 2.

Note that, here, unlike in Example 3.6, the ring is given explicitly.

Remark 3.8. The scheme constructed in Example 3.7 is essentially of finite type over a field. Note that there are no counterexamples that are locally of finite type over a field. In fact, by Lemma 2.6, every equidimensional, and hence in particular, every weakly biequidimensional, scheme locally of finite type over a field is biequidimensional.

4. The dimension formula. Next, we show that the dimension formula holds in every biequidimensional space. However, a modification of Example 3.7 gives a weakly biequidimensional scheme where the dimension formula does not hold.

Proposition 4.1 ([2, Corollary (0.14.3.5)]). *Let X be a biequidimensional topological space. Then the* dimension formula *holds for every*

irreducible closed subset Y of X, that is, we have that

$$(4.1) \qquad \dim(X) = \dim(Y) + \operatorname{codim}(Y, X).$$

Proof. Let Y be an irreducible closed subset of X. We can choose maximal chains $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_l = Y$ and $Y = Y_0' \subsetneq Y_1' \subsetneq \cdots \subsetneq Y_k'$ of lengths $l = \dim(Y)$ and $k = \operatorname{codim}(Y, X)$, respectively. Then the composed chain $Y_0 \subsetneq \ldots \subsetneq Y_l \subsetneq Y_1' \subsetneq \cdots \subsetneq Y_k'$ is maximal. As X is biequidimensional, this chain has length $\dim(X)$. $\qquad\square$

Observe that the weakly biequidimensional Noetherian schemes that we constructed in Examples 3.6 and 3.7 satisfy the dimension formula (4.1). However, it does not hold in general for weakly biequidimensional spaces, as the following modification of Example 3.7 shows.

Example 4.2. Consider $B = k[u, v, w, x, y, z]/(uy, uz, vy, vz, wy, wz)$. This is the coordinate ring of the scheme obtained by gluing the affine spaces $\mathbb{A}_k^4 = \operatorname{Spec}(k[u, v, w, x])$ and $\mathbb{A}_k^3 = \operatorname{Spec}(k[x, y, z])$ along the lines $u = v = w = 0$ and $y = z = 0$. Localization away from the union of the prime ideals (u, v, w, y, z) and $(u, v, w, x, y - 1, z - 1)$ gives a Noetherian ring A of pure dimension 3. The spectrum $X = \operatorname{Spec}(A)$ is catenary, and it has two closed points corresponding to the prime ideals (u, v, w, y, z) and $(u, v, w, x, y-1, z-1)$, and both have codimension 3 in X. In particular, the scheme X is weakly biequidimensional. Moreover, we have the saturated chain of prime ideals in X:
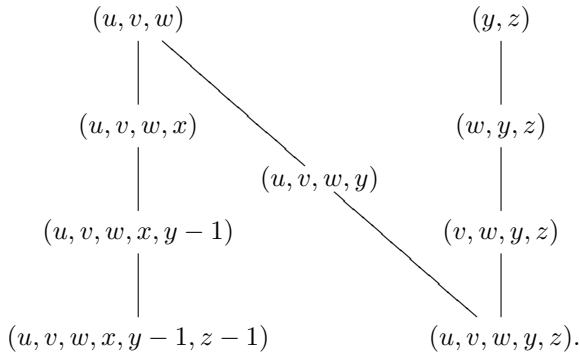
We see that the prime ideal $\mathfrak{p} = (u, v, w, y)$ has the property that

$$\mathrm{ht}(\mathfrak{p}) + \dim(A/\mathfrak{p}) = 1 + 1 = 2 < 3 = \dim(A),$$

that is, the dimension formula does not hold.

5. Existence of a codimension function. We conclude by show-ing that every biequidimensional topological space has a codimension function whereas a weakly biequidimensional space need not have it.

Definition 5.1 ([**4**, page 283, Definition]). Let X be a topological space. A *codimension function* is a function $d\colon X \to \mathbb{Z}$ such that

$$d(x') = d(x) + 1$$

holds for every specialization $x' \in \overline{\{x\}}$ such that $\mathrm{codim}(\overline{\{x'\}}, \overline{\{x\}}) = 1$.

Lemma 5.2. *The following properties hold:*

(i) *Let X be a scheme locally of finite type over a field. Then*

$$d(x) = -\dim(\overline{\{x\}})$$

defines a codimension function on X.

(ii) *Let X be a scheme essentially of finite type over a field k. Then*

$$d(x) = -\operatorname{tr.deg}_k(\kappa(x))$$

defines a codimension function on X.

Proof.

(i) We have to show that $\dim(\overline{\{x'\}}) = \dim(\overline{\{x\}}) - 1$ for all pairs of points $x, x' \in X$ such that $x' \in \overline{\{x\}}$ and $\mathrm{codim}(\overline{\{x'\}}, \overline{\{x\}}) = 1$. The irreducible subscheme $\overline{\{x\}}$ is locally of finite type and hence biequidimensional by Lemma 2.6. Then the statement is a direct consequence of the dimension formula, see Proposition 4.1.

(ii) Let $x, x' \in X$ be such that $x' \in \overline{\{x\}}$ and $\mathrm{codim}(\overline{\{x'\}}, \overline{\{x\}}) = 1$. After, if necessary, replacing X by an open affine neighborhood of x', we can assume without loss of generality that $X = \mathrm{Spec}(A)$, where $A = S^{-1}B$ for a finitely generated k-algebra B. Then x and x' correspond to points in $\mathrm{Spec}(B)$ having residue fields $\kappa(x)$ and $\kappa(x')$. Hence, it suffices to show that $d(x) = -\operatorname{tr.deg}_k(\kappa(x))$ is a codimension

function if X is a scheme of finite type over k. In this case, we have by [**3**, Proposition (5.2.1)] that $\operatorname{tr.deg}_k(\kappa(x)) = \dim(\overline{\{x\}})$, and $d(x)$ is the codimension function discussed in i. $\qquad\square$

Proposition 5.3. *Let X be a biequidimensional topological space. The map*

$$d(x) = \operatorname{codim}(\overline{\{x\}}, X)$$

defines a codimension function on X.

Proof. The statement is a direct application of equation (2.2) in Proposition 2.3. $\qquad\square$

Remark 5.4. Note that, by Proposition 4.1, the codimension function in Proposition 5.3 can be written as $d(x) = \dim(X) - \dim(\overline{\{x\}})$.

In particular, for a biequidimensional scheme locally of finite type over a field, the codimension functions defined in Lemma 5.2 (i) and in Proposition 5.3 differ only by the constant term $\dim(X)$.

Moreover, we have the following necessary and sufficient condition for the function $d(x) = \operatorname{codim}(\overline{\{x\}}, X)$ to be a codimension function.

Proposition 5.5. *Let X be a scheme. Then $d(x) = \operatorname{codim}(\overline{\{x\}}, X)$ is a codimension function if and only if all local rings are catenary and equidimensional.*

Proof. The existence of a codimension function directly implies that X and hence all the local rings are catenary.

Suppose first that $d(x) = \operatorname{codim}(\overline{\{x\}}, X)$ is a codimension function, and let $x \in X$. Let $\overline{\{x\}} = X_0 \subsetneq \cdots \subsetneq X_k$ and $\overline{\{x\}} = X_0' \subsetneq \cdots \subsetneq X_l'$ correspond to two maximal chains in $\operatorname{Spec}(\mathcal{O}_{X,x})$. Then the assumption on $d(x)$ implies that $\operatorname{codim}(\overline{\{x\}}, X) = \operatorname{codim}(X_k, X) + k$ as well as that $\operatorname{codim}(\overline{\{x\}}, X) = \operatorname{codim}(X_l', X) + l$. Both X_k and X_l' are irreducible components in X and hence $k = \operatorname{codim}(\overline{\{x\}}, X) = l$. This shows that the local ring $\mathcal{O}_{X,x}$ is equidimensional.

For the converse implication, we assume that all local rings in X are catenary and equidimensional. Let $x' \in \overline{\{x\}}$ be a direct specialization.

Let $\overline{\{x\}} = X_0 \subsetneq \cdots \subsetneq X_k$ be a saturated chain of length $\mathrm{codim}(\overline{\{x\}}, X)$. The extended chain $\overline{\{x'\}} \subsetneq X_0 \subsetneq \cdots \subsetneq X_k$ corresponds to a maximal chain in $\mathrm{Spec}(\mathcal{O}_{X,x'})$, and it has length $\dim(\mathcal{O}_{X,x'}) = \mathrm{codim}(\overline{\{x'\}}, X)$ since the local ring $\mathcal{O}_{X,x'}$ is catenary and equidimensional. It follows that $\mathrm{codim}(\overline{\{x'\}}, X) = \mathrm{codim}(\overline{\{x\}}, X) + 1$, that is, the function $d(x)$ is a codimension function. □

Remark 5.6. The results of Proposition 5.5 are stated in [**6,** page 266] in the context of catenary gradings on schemes.

As a consequence of Lemma 5.2, we see that the schemes constructed in Examples 3.7 and 4.2 do have codimension functions. The following discussion however shows that a codimension function need not exist for a weakly biequidimensional space.

Example 5.7. The scheme X constructed in Example 3.6 does not have any codimension function. Consider the irreducible closed subsets $Z_1, \ldots, Z_6$ in X satisfying the following inclusion relations

$$
\begin{array}{ccc}
Z_5 & & Z_6 \\
| & \times & | \\
Z_3 & \times & Z_4 \\
| & \times & | \\
Z_1 & & Z_2.
\end{array}
$$
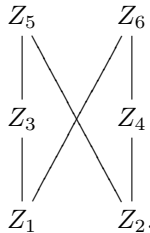
For $i = 1, \ldots, 6$, let z_i be the generic point of the irreducible set Z_i. Every codimension function $d \colon X \to \mathbb{Z}$ would then have to satisfy $d(z_1) = d(z_5) + 2 = d(z_6) + 1$ and $d(z_2) = d(z_5) + 1 = d(z_6) + 2$, which is impossible.

Note that X is not only weakly biequidimensional but it satisfies the dimension formula although it does not have any codimension function.

Furthermore, it follows from Lemma 5.2 that X is not essentially of finite type over a field.

As a dualizing complex on a locally Noetherian scheme gives rise to a codimension function, see [**4,** Proposition V.7.1], we see, moreover, that X does not have a dualizing complex.

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