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DIRECT SUMMANDS OF INFINITE-DIMENSIONAL POLYNOMIAL RINGS

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ABSTRACT. Let k be a field and R a pure subring of the infinite-dimensional polynomial ring $k[X_1,\ldots]$. If R is generated by monomials, then we show that the equality of height and grade holds for all ideals of R. Also, we show R satisfies the weak Bourbaki unmixed property. As an application, we give the Cohen-Macaulay property of the invariant ring of the action of a linearly reductive group acting by k-automorphism on $k[X_1,\ldots]$. This provides several examples of non Noetherian Cohen-Macaulay rings (e.g., Veronese, determinantal and Grassmanian rings).

1. Introduction. In this paper, we are interested in the following property of finite-dimensional polynomial rings which is a version of Hochster-Roberts theorem (see [**16**]):

Theorem 1.1. *Let $S := k[X_1,\ldots,X_n]$ be a polynomial ring over a field k, and let R be an $\mathbb{N}$-graded subring of S which is pure in S. Then R is Cohen-Macaulay.*

The historical reason for this interest comes from the Cohen-Macaulayness of the invariant ring of the action of linearly reductive groups on polynomial rings. For more details, see [**6,** Theorem 6.5.1]. Suppose a ring R is pure in a Noetherian regular ring which contains a field. As a result of the existence of balanced big Cohen-Macaulay algebras, R is Cohen-Macaulay, see [**15,** Theorem 2.3].

Recently, the notion of Cohen-Macaulayness was generalized to the non-Noetherian situation, see [**5, 11**]. One difficulty is the failure of

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several classical ideal theory results such as the principal ideal theorem. In the absence of these ideal theory results, the relationship between dimension theory and homological algebra is given by the following two samples. Denote the *Koszul* grade by K. grade. The first easy example is the following inequality

$$\mathrm{K.\,grade}_R(\mathfrak{a}, R) \leq \mathrm{ht}_R(\mathfrak{a}),$$

which was proved in [**5,** Lemma 3.2]. If equality is achieved for all ideals, we say R is Cohen-Macaulay in the sense of ideals. The second example is the Čech cohomology that was used by Hamilton and Marley to define the notion of strong parameter sequence. A ring R is called Cohen-Macaulay in the sense of *Hamilton-Marley* if each strong parameter sequence on R is a regular sequence on R. For more details, see Definition 3.2.

Theorem 1.1 can be extended in two different directions. First, we focus on non Noetherian finite-dimensional subrings of $k[X_1, \ldots, X_n]$. This kind of investigation was initiated in [**3, 4**]. Second, we focus on the infinite-dimensional version of Theorem 1.1.

Notation 1.2. By $R[X_1, \ldots]$, we mean $\bigcup_{i=1}^{\infty} R[X_1, \ldots, X_i]$.

We refer the reader to [**2, 12**] and the references therein to see some properties of infinite-dimensional polynomial rings via algebraic statistics and chemistry motivations.

In this paper, we attempt to obtain, mostly by a direct limit argument, results on the widely unknown realm of the infinite-dimensional ring $k[X_1, \ldots]$. More explicitly, we are interested in the next question.

Question 1.3. *Suppose that k is a field and R is a pure subring of $S := k[X_1, \ldots]$. Is R Cohen-Macaulay?*

More generally, let $\mathcal{P}$ be a property of commutative Noetherian rings. There is a *cut-paste* idea to extend this property to the realm of non-Noetherian rings. To explain the idea, let R be a non-Noetherian ring. We refer to $\mathcal{P}$ as the *cut* property when R is written as a direct limit of Noetherian rings satisfying $\mathcal{P}$. If the property $\mathcal{P}$ behaves nicely with respect to the direct limit, we refer to it as the *paste* property. We apply a cut-paste idea to give a positive answer to Question 1.3 when

$$R \cap k[X_1, \dots, X_n] \hookrightarrow k[X_1, \dots, X_n]$$

is pure for sufficiently large n, see Theorem 3.4. It is worth noting that this condition holds when R is generated by monomials (see Corollary 3.6). For an application, recall that a linear algebraic group over k is called *linearly reductive* if every G-module V is a direct sum of irreducible G-submodules. For more details, see Remark 3.7. Then Theorem 3.4 can be restated as follows.

Corollary 1.4 (see Corollary 3.8)**.** *Let k be an algebraically closed field and $A = k[X_1, \dots]$. Suppose that G is a linearly reductive group over k acting on A (in the sense of Remark* 3.7 (ii)*) by a degree preserving action. Then A^G is Cohen-Macaulay in the sense of each part of Definition* 3.2.

Veronese, determinantal and Grassmanian rings are important sources of Cohen-Macaulay rings, see **[7]**. They are subrings of a finite-dimensional polynomial ring over a field. Their definitions may be extended to the case of an infinite-dimensional polynomial ring. We study their Cohen-Macaulayness in Section 4.

2. Preliminary lemmas. This section contains five lemmata. They do not involve any Cohen-Macaulay concept. We will use them in the next section.

Lemma 2.1. *Let k be a field and I a finitely generated ideal of $S = k[X_1, \dots]$. Then each minimal prime ideal of I is finitely generated.*

Proof. Let $\{f_1, \dots, f_n\}$ be a generating set for I and $\mathfrak{p} \in \min_S(I)$. Take m to be such that $f_i \in R := k[X_1, \dots, X_m]$ for all $1 \leq i \leq n$. Observe that $(\mathfrak{p} \cap R)S$ is prime, because

$$R[X_{m+1}, \dots]/\mathfrak{q}R[X_{m+1}, \dots] \cong R/\mathfrak{q}[X_{m+1}, \dots],$$

for all $\mathfrak{q} \in \operatorname{Spec}(R)$. Also, $I = (I \cap R)S$. In view of

$$I = (I \cap R)S \subseteq (\mathfrak{p} \cap R)S \subseteq \mathfrak{p},$$

we see that $\mathfrak{p} = (\mathfrak{p} \cap R)S$. Clearly, $\mathfrak{p} \cap R$ is finitely generated as an ideal of R. So, $\mathfrak{p}$ is a finitely generated ideal of S. $\square$

Let I be an ideal of a ring R. By $\mathrm{Var}_R(I)$, we mean the set of all prime ideals of R containing I. Also, $\min_R(I)$ denotes the set of all minimal prime ideals of I.

Lemma 2.2. *Let $R \to S$ be a pure ring homomorphism and I an ideal of R. Let $\mathfrak{p} \in \min_R(I)$. Then there exists an $\mathfrak{q} \in \min_S(IS)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. In particular, if $\min_S(IS)$ is finite, then $\min_R(I)$ is finite.*

Proof. Since $IS \cap R = I$, we have a natural injective homomorphism $R/I \hookrightarrow S/IS$. Note that $\mathfrak{p} \in \min_R(I)$. By [**17,** page 41, Example 1], there exists a $\mathfrak{q}\prime \in \mathrm{Var}_S(IS)$ such that $\mathfrak{q}\prime \cap R = \mathfrak{p}$. Let $\mathfrak{q} \in \min_S(IS)$ be such that $\mathfrak{q} \subseteq \mathfrak{q}\prime$. Then

$$I \subseteq \mathfrak{q} \cap R \subseteq \mathfrak{q}\prime \cap R = \mathfrak{p}.$$

Thus, $\mathfrak{q} \cap R = \mathfrak{p}$. This completes the proof. $\square$

Lemma 2.3. *Let $R \hookrightarrow S := k[X_1, \ldots]$ be a pure ring homomorphism and I a finitely generated ideal of R. Then $\min_R(I)$ is finite.*

Proof. In view of Lemma 2.1, elements of $\min_S(IS)$ are finitely generated. By [**1,** Theorem], $\min_S(IS)$ is finite. The claim now follows by Lemma 2.2. $\square$

Lemma 2.4. *Let k be a field, $R \hookrightarrow k[X_1, \ldots]$ a pure ring homomorphism and I an ideal of R. If $0 \leq n \leq \mathrm{ht}_R(I)$, then there are $x_1, \ldots, x_n \in I$ such that*

$$i \leq \mathrm{ht}_R((x_1, \ldots, x_i)R)$$

for all $0 \leq i \leq n$.

Proof. We prove the claim by induction on n. The first step of induction is obvious. Now, suppose $n > 0$ and the claim has been proved for all $j < n$. Suppose $j < n$. By inductive hypothesis, we can find $x_1, \ldots, x_j \in I$ such that

$$i \leq \mathrm{ht}_R((x_1, \ldots, x_i)R)$$

for all $1 \leq i \leq j$. Set

$$X := \{Q \in \min((x_1, \dots, x_j)R) : \mathrm{ht}_R(Q) = j\}.$$

Suppose $X = \emptyset$. Then, $j + 1 \leq \mathrm{ht}_R((x_1, \dots, x_j)R)$. Hence, we have

$$j + 1 \leq \mathrm{ht}_R((x_1, \dots, x_j, x_{j+1})R)$$

for all $x_{j+1} \in I$. Thus, without loss of the generality, we may and do assume that $X \neq \emptyset$. It follows by Lemma 2.3 that X is finite. Note that

$$I \nsubseteq \bigcup_{Q \in X} Q,$$

unless $\mathrm{ht}_R(I) \leq j < n$, which is impossible by our assumptions. Set

$$x_{j+1} \in I \setminus \bigcup_{Q \in X} Q.$$

Thus,

$$j \leq \mathrm{ht}_R((x_1, \dots, x_j)R) \leq \mathrm{ht}_R((x_1, \dots, x_{j+1})R).$$

But, $\mathrm{ht}_R((x_1, \dots, x_{j+1})R) \neq j$. So, $j + 1 \leq \mathrm{ht}_R((x_1, \dots, x_{j+1})R)$. $\square$

The module case of the next result (when the base ring is fixed) is well known.

Lemma 2.5. *Let R, S and T be commutative rings. Let $\varphi : R \to S$ and $\theta : S \to T$ be ring homomorphisms. The following hold.*

(i) *If φ and θ are pure, then $\theta\varphi$ is pure.*
(ii) *If $\theta\varphi$ is pure, then φ is pure.*

Proof. Let M be an R-module. Set $\psi := \theta \otimes \mathrm{id}_{S \otimes_R M}$. Then the next diagram is commutative:

$$\begin{tikzcd}
R \otimes_R M \arrow[r, "\varphi \otimes Id_M"] & S \otimes_R M \arrow[r, "\theta \otimes Id_M"] \arrow[d] & T \otimes_R M \arrow[d] \\
& (S \otimes_S S) \otimes_R M \arrow[r] \arrow[d] & (T \otimes_S S) \otimes_R M \arrow[d] \\
& S \otimes_S (S \otimes_R M) \arrow[r, "\psi"] & T \otimes_S (S \otimes_R M),
\end{tikzcd}$$

where the columns are isomorphic.

Now, we prove the lemma.

(i) If φ and θ are pure, then $\varphi \otimes \mathrm{id}_M$ and ψ are one-to-one. So $\varphi \otimes \mathrm{id}_M$ and $\theta \otimes \mathrm{id}_M$ are one-to-one. It is now clear that $\theta\varphi$ is pure, because $(\theta \otimes \mathrm{id}_M)(\varphi \otimes \mathrm{id}_M) = \theta\varphi \otimes \mathrm{id}_M$.

(ii) If $\theta\varphi$ is pure, then $(\theta \otimes \mathrm{id}_M)(\varphi \otimes \mathrm{id}_M) = \theta\varphi \otimes \mathrm{id}_M$ is one-to-one. Hence, $\varphi \otimes \mathrm{id}_M$ is one-to-one. Therefore, φ is pure. □

3. Infinite-dimensional Cohen-Macaulay rings. Our main result in this section is Theorem 3.4 and its corollaries. Let $\mathfrak{a}$ be an ideal of a ring R and M an R-module. Suppose first that $\mathfrak{a}$ is finitely generated with the generating set $\underline{x} := x_1, \ldots, x_r$. Denote the Koszul complex of R with respect to $\underline{x}$ by $\mathbb{K}_\bullet(\underline{x})$. Koszul grade of $\mathfrak{a}$ on M is defined by

$$\mathrm{K.grade}_R(\mathfrak{a}, M) := \inf\{i \in \mathbb{N} \cup \{0\} \mid H^i(\mathrm{Hom}_R(\mathbb{K}_\bullet(\underline{x}), M)) \neq 0\}.$$

Note that by [**6,** Corollary 1.6.22] and [**6,** Proposition 1.6.10 (d)], this does not depend on the choice of generating sets of $\mathfrak{a}$. Suppose now that $\mathfrak{a}$ is a general ideal (not necessarily finitely generated). Take Σ to be the family of all finitely generated subideals $\mathfrak{b}$ of $\mathfrak{a}$. The Koszul grade of $\mathfrak{a}$ on M can be defined by

$$\mathrm{K.grade}_R(\mathfrak{a}, M) := \sup\{\mathrm{K.grade}_R(\mathfrak{b}, M) : \mathfrak{b} \in \Sigma\}.$$

By using [**6,** Proposition 9.1.2 (f)], this definition coincides with the original definition for finitely generated ideals.

Remark 3.1.

(i) A system $\underline{x} = x_1, \ldots, x_\ell$ of elements of R is called a weak regular sequence on M if x_i is a nonzero-divisor on $M/(x_1, \ldots, x_{i-1})M$ for all $i = 1, \ldots, \ell$. The classical grade of an ideal $\mathfrak{a}$ on M is defined to be the supremum of the lengths of all weak regular sequences on M contained in $\mathfrak{a}$.

(ii) Recall that the classical grade coincides with the Koszul grade if the ring and the module both are Noetherian.

(iii) Let R be a ring, M an R-module and $\underline{x} = x_1, \ldots, x_\ell$ a sequence of elements of R. For each $m \geq n$, there is a natural chain map $\varphi_n^m(\underline{x}) : \mathbb{K}_\bullet(\underline{x}^m) \to \mathbb{K}_\bullet(\underline{x}^n)$, see [**11,** page 346]. Recall from [**18**] that $\underline{x}$ is weak proregular if, for each $n > 0$, there exists an $m \geq n$ such that the maps

$$H_i(\varphi_n^m(\underline{x})) : H_i(\mathbb{K}_\bullet(\underline{x}^m)) \longrightarrow H_i(\mathbb{K}_\bullet(\underline{x}^n))$$

are 0 for all $i \geq 1$.

Now, we recall the following key definitions.

Definition 3.2 (see [**5,** Definition 3.1] and references therein)**.** Let R be a ring.

(i) R is called Cohen-Macaulay in the sense of *Glaz* if, for each prime ideal $\mathfrak{p}$ of R,

$$\operatorname{ht}_R(\mathfrak{p}) = \operatorname{K.grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, R_\mathfrak{p}).$$

(ii) Recall that a prime ideal $\mathfrak{p}$ is *weakly associated* to a module M if $\mathfrak{p}$ is minimal over $(0 :_R m)$ for some $m \in M$. We denote the set of weakly associated primes of M by $\operatorname{wAss}_R M$. Let $\mathfrak{a}$ be a finitely generated ideal of R. Set $\mu(\mathfrak{a})$ for the minimal number of elements of R that needs to generate $\mathfrak{a}$. Assume that, for each ideal $\mathfrak{a}$ with the property $\operatorname{ht}(\mathfrak{a}) \geq \mu(\mathfrak{a})$, we have $\min(\mathfrak{a}) = \operatorname{wAss}_R(R/\mathfrak{a})$. A ring with such a property is called *weak Bourbaki unmixed* (WB). For more details, see [**10**].

(iii) By $H^i_{\underline{x}}(M)$, we mean the ith cohomology of the Čech complex of M with respect to $\underline{x} := x_1, \dots, x_\ell$. Adopt the above notation. Then $\underline{x}$ is called a *parameter sequence* on R, if:
(1) $\underline{x}$ is a weak proregular sequence;
(2) $(\underline{x})R \neq R$; and
(3) $H^\ell_{\underline{x}}(R)_\mathfrak{p} \neq 0$ for all $\mathfrak{p} \in \operatorname{V}(\underline{x}R)$.
Also, $\underline{x}$ is called a *strong parameter sequence* on R if $x_1, \dots, x_i$ is a parameter sequence on R for all $1 \leq i \leq \ell$. R is called Cohen-Macaulay in the sense of *Hamilton-Marley* (HM) if each strong parameter sequence on R is a regular sequence on R. For more details, see [**11**].

(iv) Let $\mathcal{A}$ be a non-empty class of ideals of a ring R. R is called Cohen-Macaulay in the sense of $\mathcal{A}$, if $\operatorname{ht}_R(\mathfrak{a}) = \operatorname{K.grade}_R(\mathfrak{a}, R)$ for all $\mathfrak{a} \in \mathcal{A}$. We denote this property by $\mathcal{A}$. The classes that we are interested in are $\operatorname{Spec}(R)$, $\max(R)$, the class of all ideals and the class of all finitely generated ideals.

Remark 3.3. The diagram below was proven in [**5,** 3.2. Relations]:

$$\text{Max} \Longleftarrow \text{Spec} \Longleftrightarrow \text{ideals} \Longrightarrow \text{Glaz} \Longrightarrow \text{f.g. ideals} \Longrightarrow \text{HM} \Longleftarrow \text{WB}\,.$$

Also, when the base ring is coherent, Spec $\Rightarrow$ WB.

The following will play an essential role in the proof of Corollary 3.6.

Theorem 3.4. *Let k be a field and R a subring of $k[X_1,\ldots]$ containing k. Assume that there is a strictly increasing infinite sequence $\{b_n\}_{n\in\mathbb{N}}$ of positive integers such that $R\cap k[X_1,\ldots,X_{b_n}] \hookrightarrow k[X_1,\ldots,X_{b_n}]$ is pure for all $n\in\mathbb{N}$. Then R is Cohen-Macaulay in the sense of each part of Definition* 3.2.

Proof. We denote $R\cap k[X_1,\ldots,X_{b_n}]$ by R_n for all $n\in\mathbb{N}$. In view of Remark 3.3, we need to show that R is Cohen-Macaulay in the sense of ideals and R is weak Bourbaki unmixed. Let $n\in\mathbb{N}$. One has $R_n\cap(Jk[X_1,\ldots,X_{b_n}])=J$ for every ideal J of R_n. So R_n is a Noetherian ring. Therefore, we deduce from [**6,** Theorem 10.4.1, Remark 10.4.2] that R_n is a Noetherian Cohen-Macaulay ring. Keep in mind that the ring homomorphism

$$k[X_i:1\leq i\leq b_n]\to k[X_i:1\leq i<\infty]$$

is pure. By looking at the next commutative diagram and Lemma 2.5,

$$\begin{array}{ccc} R_n & \longrightarrow & k[X_i:1\leq i\leq b_n] \\ \downarrow & & \downarrow \\ R & \longrightarrow & k[X_i:1\leq i<\infty], \end{array}$$

the ring homomorphism $R_n\to R$ is pure. Also, by [**4,** Lemma 3.9],

$$R=\bigcup_{n\in\mathbb{N}} R_n\to k[X_i:1\leq i<\infty]=\bigcup_{n\in\mathbb{N}} k[X_1,\ldots,X_{b_n}]$$

is a pure ring homomorphism.

(i) First, we show that R is Cohen-Macaulay in the sense of ideals. Let I be an ideal of R such that $n\leq \operatorname{ht}_R(I)$. We use Lemma 2.4 to find elements $a_1,\ldots,a_n\in I$ such that

$$i\leq \operatorname{ht}_R((a_1,\ldots,a_i)R),\quad \text{for all } 1\leq i\leq n.$$

Now we claim that, for each $1 \leq i \leq n$, there exists $l_i \in \mathbb{N}$ such that $a_1/1, \ldots, a_i/1$ is a regular sequence in $(R_k)_{\mathfrak{q}}$ for every $k \geq l_i$ and $\mathfrak{q} \in \operatorname{Var}_{R_k}((a_1, \ldots, a_i)R_k)$. To this end, let $1 \leq i \leq n$. In view of Lemma 2.3, $\min_R(a_1, \ldots, a_i)R$ is finite. Denote it by $\{Q_1, \ldots, Q_m\}$. We have the following chain of prime ideals

$$P_{j_0} \subsetneqq \ldots \subsetneqq P_{j_i} = Q_j$$

for all $1 \leq j \leq m$. Pick $b_{j_t} \in P_{j_t} \setminus P_{j_{t-1}}$ for all $1 \leq j \leq m$ and $1 \leq t \leq i$. Set

$$Y := \{b_{jt} | 1 \leq j \leq m, 1 \leq t \leq i\}.$$

Since Y is finite, there exists $\ell_i \in \mathbb{N}$ such that $Y \subseteq R_{\ell_i}$ and $\{a_1, \ldots, a_n\} \subseteq R_{\ell_i}$. We use this to deduce that

$$i \leq \operatorname{ht}_{R_k}(Q_j \cap R_k)$$

for all $1 \leq j \leq m$ and $\ell_i \leq k$. Let $\ell_i \leq k$. By Lemma 2.2, for each $\mathfrak{p} \in \min_{R_k}((a_1, \ldots, a_i)R_k)$, there is a $1 \leq j \leq m$ such that $Q_j \cap R_k = \mathfrak{p}$. Hence, $\operatorname{ht}_{R_k}((a_1, \ldots, a_i)R_k) \geq i$. The reverse inequality holds because R_k is Noetherian. So

$$\operatorname{ht}_{(R_k)_{\mathfrak{q}}}((a_1, \ldots, a_i)(R_k)_{\mathfrak{q}})) = i$$

for all $\mathfrak{q} \in \operatorname{Var}_{R_k}((a_1, \ldots, a_i)R_k)$. Since $(R_k)_{\mathfrak{q}}$ is a Noetherian Cohen-Macaulay local ring, $a_1/1, \ldots, a_i/1$ is a regular sequence in $(R_k)_{\mathfrak{q}}$. This proves the claim.

Set $l := \max\{\ell_1, \ldots, \ell_n\}$ and fix $k \geq l$. Then $a_1/1, \ldots, a_i/1$ is a regular sequence in $(R_k)_{\mathfrak{q}}$ for all $1 \leq i \leq n$ and $\mathfrak{q} \in \operatorname{Var}_{R_k}((a_1, \ldots, a_i)R_k)$. Then $a_1, \ldots, a_n$ is a regular sequence in R_k for all $k \geq l$. Hence, $a_1, \ldots, a_n$ is a weak regular sequence in R. Consequently, $n \leq \operatorname{K.grade}_R(I, R)$. So $\operatorname{ht}_R(I) \leq \operatorname{K.grade}_R(I, R)$. The reverse inequality is always true by [**5,** Lemma 3.2].

(ii) Here we show that R is weak Bourbaki unmixed.

Let $\mathfrak{a}$ be a proper finitely generated ideal of R with the property that $\operatorname{ht}(\mathfrak{a}) \geq \mu(\mathfrak{a})$. Set $\ell := \mu(\mathfrak{a})$, and let $\underline{y} := y_1, \ldots, y_\ell$ be a generating set for $\mathfrak{a}$. In view of Lemma 2.4, there exists $\underline{x} := x_1, \ldots, x_l$ in $\mathfrak{a}$ such that

$$i \leq \operatorname{ht}_R((x_1, \ldots, x_i)R),$$

for each $1 \leq i \leq \ell$. Let $1 \leq i \leq \ell$, and set $\mathfrak{a}_i := (x_1, \dots, x_i)R$. In view of part (i),

$$i \leq \operatorname{ht} \mathfrak{a}_i = \mathrm{K.\,grade}_R(\mathfrak{a}_i, R) \leq \mu(\mathfrak{a}_i) \leq i.$$

So, by [**11,** Proposition 3.3(e)], $\underline{x}$ is a strong parameter sequence on R. In view of Remark 3.3, R is Cohen-Macaulay in the sense of Hamilton-Marley. Therefore, $\underline{x}$ is a regular sequence on R.

There are $r_{ij} \in R$ such that $x_i = \sum_{1 \leq j \leq l} r_{ij} y_j$ for all $1 \leq i \leq l$. Recall that $R_m = R \cap k[X_1, \dots, X_{b_m}]$ for all m. Take $n \in \mathbb{N}$ be such that all of r_{ij}, $\underline{x}$ and $\underline{y}$ belong to R_m for all $m \geq n$.

Suppose $\mathfrak{p} \in \mathrm{wAss}(R/\mathfrak{a})$. Clearly, $\underline{x}$ is a regular sequence on $R_{\mathfrak{p}}$. Set $\mathfrak{p}_m := \mathfrak{p} \cap R_m$ for all $m \geq n$. The purity of $R_m \to R$ implies that $\underline{x}$ is a regular sequence on R_m, see [**6,** Proposition 6.4.4]. Then $\underline{x}$ is a regular sequence on $R_{m(P_m)}$. Note that $\underline{x}R_m \subseteq \underline{y}R_m$. Thus,

$$\ell \leq \operatorname{ht}_{R_{m(P_m)}}(\underline{x}R_{m(P_m)}) \leq \operatorname{ht}_{R_{m(P_m)}}(\underline{y}R_{m(P_m)}) \leq \ell.$$

Since $R_{m(P_m)}$ is a Noetherian Cohen-Macaulay local ring, we see $\underline{y}$ is a regular sequence on $R_{m(P_m)}$. Thus, $\underline{y}$ is a regular sequence on R_P .

In view of [**5,** Theorem 3.3, Lemma 3.5], $R_{\mathfrak{p}}/\underline{y}R_{\mathfrak{p}}$ is Cohen-Macaulay in the sense of ideals. It follows from [**5,** Lemma 3.9] that

$$\mathrm{wAss}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\underline{y}R_{\mathfrak{p}}) = \mathrm{Min}(R_{\mathfrak{p}}/\underline{y}R_{\mathfrak{p}}),$$

and so $\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})$. $\square$

Remark 3.5.

(i) As Remark 3.3 states, Cohen-Macaulay in the sense of ideals implies weak Bourbaki unmixedness when the base ring is coherent. Note that, in Theorem 3.4, R is not necessarily coherent, see [**8,** Example 2].

(ii) It may be worth noting that one can construct a direct system of Noetherian Cohen-Macaulay rings such that its direct limit is not Cohen-Macaulay, see [**4,** Example 4.7].

We are now ready to prove:

Corollary 3.6. *Let k be a field and R a pure k-subalgebra of $S := k[X_1, \ldots]$ generated by monomials. Then R is Cohen-Macaulay in the sense of each part of Definition* 3.2.

Proof. There is a natural projection $\pi_n : k[X_1, \ldots] \to k[X_1, \ldots, X_n]$ defined by evaluation: for each $f \in k[X_1, \ldots]$, $\pi_n(f)$ is given by the substitution $X_{n+i} = 0$ for all $i \geq 1$. Set $R_n := R \cap k[X_1, \ldots, X_n]$ for all $n \in \mathbb{N}$.

We claim that

$$\pi_n(R) \subseteq R_n.$$

To see this, let $r \in R$. As R is generated by monomials, $r = a_0 + \cdots + a_m$ where $a_i \in R$ is a monomial. Note that either $\pi_n(a_i) = 0$ or $\pi_n(a_i) = a_i$. In both cases, $\pi_n(r) \in R$, as claimed.

Hence, we can define $\pi_n : R \to R_n$, and π_n provides a retraction for the natural inclusion $R_n \to R$. Thus, R_n is a direct summand of R as an R_n-module for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. It follows from the commutative diagram

$$\begin{tikzcd} R_n \arrow[r] \arrow[d] & k[X_1, \ldots, X_n] \arrow[d] \\ R \arrow[r] & k[X_1, \ldots], \end{tikzcd}$$

that the ring homomorphism $R_n \hookrightarrow k[X_1, \ldots, X_n]$ is pure. Now, the claim is an immediate consequence of Theorem 3.4. $\square$

In the following, we cite some aspects of invariant theory that we need in the sequel. We refer the reader to **[6, 14]** for more details.

Remark 3.7. Let k be an algebraically closed field.

(i) Recall that a *linear algebraic group* over k is a Zariski closed subgroup of some $\mathrm{GL}(V) := \mathrm{Aut}_k(V)$, where V is a finite-dimensional k-vector space. By a homomorphism of linear algebraic groups we mean a group homomorphism which is a morphism of varieties.

(ii) Let G be a linear algebraic group. Then G acts *k-rationally* on a finite-dimensional k-vector space V if the map $\Phi : G \to \mathrm{GL}(V)$ defining the action is a homomorphism of the linear algebraic groups. If V is infinite-dimensional, G acts k-rationally on V if the action is such that V is a union of finite-dimensional G-stable subspaces W such that G acts k-rationally on W in the sense above.

If R is a k-algebra, G acts on R to mean that G acts k-rationally on the k-vector space R by k-algebra automorphism. In invariant theory, it is commonly accepted that "an action of an algebraic group on a k-algebra" means a rational one. So, in the sequel we treat only with rational actions on k-algebras.

When G acts k-rationally on a k-vector space V, we shall say that V is a *G-module.* Recall that $U \subset V$ is said to be G-submodule, if it is a vector subspace of V and $g(u) \in U$ for all $g \in G$ and $u \in U$. Also, U is called *irreducible* if it has no nontrivial G-submodule.

(iii) Let G be a linear algebraic group. Then G is called *linearly reductive*, if every G-module V is a direct sum of irreducible G-submodules. An equivalent condition is that every G-submodule W of V has a G-stable complement L, i.e., $V = W \oplus L$ as G-modules, see [**14,** page 170].

The most classical examples of linearly reductive groups are finite groups G whose order is not divisible by $\mathrm{char}(k)$. In characteristic 0, the groups $\mathrm{GL}(n,k)$ and $\mathrm{SL}(n,k)$ are linearly reductive, and so are the orthogonal and symplectic groups. The tori $\mathrm{GL}(1,k)^m$ are linearly reductive independently of $\mathrm{char}(k)$, see [**6,** page 292].

(iv) Let G be a linearly reductive group and V be a G-module. Let V^G be the subspace of invariants, i.e.,

$$V^G := \{v \in V : \text{for all } g \in G,\ g(v) = v\}.$$

Then V^G is the largest G-submodule of V on which G acts trivially. Let W be the sum of all irreducible G-subspaces of V on which G acts non-trivially. Then $V = V^G \oplus W$, and W is the unique complementary G-subspace of V, see [**14,** page 170].

(v) Let G be a linearly reductive group and $R := k[X_1, \ldots, X_n]$. Denote the graded component containing homogenous elements of degree i of R by R_i. Suppose G acts on R by degree-preserving k-algebra homomorphisms. This means that $g(R_n) \subseteq R_n$ for all

$g \in G$ and $n \in \mathbb{N}$. Then

$$R^G := \{f \in R : g(f) = f \text{ for all } g \in G\}$$

is the ring of invariants. There exists a finite-dimensional representation $\varphi_i : G \to \mathrm{GL}_k(R_i)$ for each i. By (iv), $R_i = R_i^G \oplus W_i$ for each i. Then

$$R = \oplus_{i\geq 0} R_i \cong (\oplus_{i\geq 0} R_i^G) \oplus (\oplus_{i\geq 0} W_i).$$

Keep in mind that the action is degree preserving. Then we have $R^G = (\oplus_{i\geq 0} R_i^G)$. Set $W = (\oplus_{i\geq 0} W_i)$. We show that W is an R^G-module. Consequently, R^G is a direct summand of R as an R^G-module.

Let $r \in R^G$ and $a \in W$. Then $r = r_1 + \cdots + r_t$ and $a = a_1 + \cdots + a_t$ where $r_i \in R_i^G$ and $a_i \in W_i$. For each a_j, there exists an irreducible G-subspace U of W_j such that $a_j \in U$. Consider the G-homomorphism $r_i : U \to r_i U$. This map is zero or one-to-one. If the map is zero, then $r_i U = 0 \subseteq W_{i+j}$. If the map is one-to-one, then $U \simeq r_i U$ as G-spaces. It follows that G acts nontrivially on the irreducible G-space $r_i U$. Since $r_i U \subseteq R_{i+j}$, one has $r_i U \subseteq W_{i+j}$. So, $ra \in W$ and W is an R^G-module.

Now, we are ready to prove the following result.

Corollary 3.8. *Let k be an algebraically closed field and $A := k[X_1, \ldots]$. Suppose G is a linearly reductive group over k acting on A by degree-preserving k-algebra automorphisms. Then A^G is Cohen-Macaulay in the sense of each part of Definition* 3.2.

Proof. Indeed, for simplicity, assume that each X_i is of degree 1. Set

$$V := \bigoplus_{i=1}^{\infty} kX_i.$$

It is easy to see that V is a G-module. Then, by Remark 3.7 (iii), there is a decomposition $V = \bigoplus V_i$ with each V_i a finite-dimensional G-submodule of V. Now, set $b_0 = 0$, and

$$b_i = \sum_{j=1}^{i} \dim_k V_j,$$

and take a k-basis $\{Y_{b_{i-1}+1}, \ldots, Y_{b_i}\}$ of V_i for each $i \geq 1$. The notation $\mathrm{Sym}_k(W)$ stands for the *symmetric* algebra of a k-vector space W. Recall from [**9,** subsections 8.3.3, 8.3.5] the following two items:

(i) $\mathrm{Sym}_k(V) = \mathrm{Sym}(\bigoplus V_i) \simeq \bigcup \mathrm{Sym}_k(V_i) = \bigcup k[Y_{b_{i-1}+1}, \ldots, Y_{b_i}]$,
(ii) $\mathrm{Sym}_k(V) = \mathrm{Sym}(\bigoplus_n (\bigoplus_{i=1}^n kX_i)) \simeq \bigcup \mathrm{Sym}_k(\bigoplus_{i=1}^n kX_i) = \bigcup k[X_1, \ldots, X_n]$.

Then, without loss of the generality, one can replace $\{X_1, \ldots\}$ by the new variables $\{Y_i\}$, that is, there is a strictly increasing infinite sequence $\{b_n\}_{n\in\mathbb{N}}$ of positive integers such that

$$k[Y_1, \ldots, Y_{b_n}] \text{ is a } G\text{-submodule of } A \text{ for all } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, set $A_n := k[Y_1, \ldots, Y_{b_n}]$. Then, G acts on A_n by degree-preserving k-algebra automorphisms. By Remark 3.7 (v), A_n^G is a direct summand of A_n as an A_n^G-module. Hence, $A_n^G \to A_n$ is pure. By applying Theorem 3.4, A^G is Cohen-Macaulay in the sense of each part of Definition 3.2. $\square$

Also, Question 1.3 has an affirmative answer in the following case:

Remark 3.9. Let $R := k[x_1, \ldots]$ be an infinite-dimensional polynomial ring over a field k and G a finite group of automorphisms of R such that the order of G is a unit in R. Recall from [**5,** Theorem 4.1] that R is Cohen-Macaulay in the sense of each part of Definition 3.2. By [**5,** Theorem 5.6 and Proposition 5.7], R^G is Cohen-Macaulay in the sense of each part of Definition 3.2.

In the next section, we give several examples in the context of Corollary 3.8. As a special case, the next result provides more evidence for an affirmative answer for Question 1.3.

Example 3.10. Let k be a field with $\mathrm{char}(k) \neq 2$ and $S := k[X_1, \ldots]$. The assignments $X_{2i+1} \mapsto X_{2i+2}$ and $X_{2i} \mapsto X_{2i-1}$ define an automorphism $g : S \to S$. Let G be the group generated by g. Then

(i) the ring $k[X_1, \ldots, X_n]$ is not G-submodule of S for all $n \in \mathbb{N}$.
(ii) The ring $R := S^G$ can not be generated by monomials.
(iii) The ring R is Cohen-Macaulay in the sense of each part of Definition 3.2.

Proof. Note that the order of g is 2. So, $G = \{1, g\}$.

(i) This is trivial.
(ii) It is clear that $X_1 + X_2$ is invariant by G. If R were generated by monomials, then X_1 and X_2 should be invariant, which is impossible.
(iii) The order of G is invertible in S, and S is Cohen-Macaulay in the sense of each part of Definition 3.2. To conclude the argument, see Remark 3.9. $\square$

4. Examples. Next, we present several examples of non Noetherian Cohen-Macaulay rings, as an application of our main result. The following gives Cohen-Macaulayness of infinite-dimensional determinantal rings.

Example 4.1. Let $\{z_{ij} : i, j \in \mathbb{N}\}$ be a family of variables over an algebraically closed field k of characteristic 0. Let $Z := (z_{ij})$ be a matrix. We denote the polynomial ring $k[z_{ij} : i, j \in \mathbb{N}]$ by $k[Z]$. Let $I_n(Z)$ be the ideal of $k[Z]$ generated by the n-minors of Z. Then $k[Z]/I_{n+1}(Z)$ is Cohen-Macaulay in the sense of each part of Definition 3.2.

Proof. First note that, by an n-minor of Z, we mean the determinant of an $n \times n$ submatrix of Z. Let $\{x_{ij} : i \in \mathbb{N}, 1 \leq j \leq n\}$ and $\{y_{jk} : k \in \mathbb{N}, 1 \leq j \leq n\}$ be two families of variables over k. Define the matrices $X := (x_{ij})$ and $Y := (y_{jk})$. Look at the polynomial ring $R := k[X, Y]$. First, we show that $k[XY] \cong k[Z]/I_{n+1}(Z)$.

Consider the matrices

$$X_m := (x_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \quad \text{and} \quad Y_m := (y_{jk})_{\substack{1 \leq j \leq n \\ 1 \leq k \leq m}},$$

where m is an integer greater than $n+1$. Let $Z_m := (z_{ij})_{1 \leq i \leq m, 1 \leq j \leq m}$ be an $m \times m$ submatrix of Z. Then there exists a homomorphism of k-algebras $\varphi_m : k[Z_m]/I_{n+1}(Z_m) \to k[X_m, Y_m]$ such that $Z_m + I_{n+1}(Z_m) \to X_m Y_m$. By [**7**, Theorem 7.2], φ_m is an embedding. So the induced homomorphism $\widehat{\varphi_m} : k[Z_m]/I_{n+1}(Z_m) \to k[X_m Y_m]$ is an isomorphism. For each m, l such that $n + 1 \leq m \leq l$, let

$$\pi_{ml} : k[Z_m]/I_{n+1}(Z_m) \longrightarrow k[Z_l]/I_{n+1}(Z_l)$$

and

$$\lambda_{ml} : k[X_m Y_m] \longrightarrow k[X_l Y_l]$$

be the natural homomorphism of k-algebras. Then

$$\{\widehat{\varphi_m}\}_{m\geq n+1} : (k[Z_m]/I_{n+1}(Z_m), \pi_{ml}) \longrightarrow (k[X_m Y_m], \lambda_{ml})$$

is an isomorphism of direct systems. On the other hand,

$$\varinjlim_{m\geq n+1} k[Z_m]/I_{n+1}(Z_m) \cong k[Z]/I_{n+1}(Z)$$

and

$$\varinjlim_{m\geq n+1} k[X_m Y_m] = k[XY].$$

Hence, $k[XY] \cong k[Z]/I_{n+1}(Z)$.

Let $G := \mathrm{GL}_n(k)$ be the general linear group. By Remark 3.7 (iii), G is linearly reductive. For $M \in G$ and a polynomial $f(X,Y) \in k[X,Y]$ one puts

$$M(f) := f(XM^{-1}, MY).$$

As M runs through G, this defines an action of G on $R := k[X,Y]$ as a group of k-algebra automorphisms. Denote the polynomial ring $k[X_m, Y_m]$ by R_m for all $m \in \mathbb{N}$. Then G acts on R_m, likewise R, i.e., R_m is G-stable. By Corollary 3.8, R^G is Cohen-Macaulay in the sense of ideals. In order to show $k[Z]/I_{n+1}(Z)$ is Cohen-Macaulay, it is enough to show that $R^G = k[XY]$. In the light of **[7,** Proposition 7.4, Theorem 7.6], $R_m^G = k[X_m Y_m]$. Also, we have $R^G = \cup_{m\in\mathbb{N}} R_m^G$ and $k[XY] = \cup_{m\in\mathbb{N}} k[X_m Y_m]$. Therefore, $R^G = k[XY]$. □

The following gives Cohen-Macaulayness of infinite-dimensional Grassmanian rings.

Example 4.2. Let $\{x_{ij} : j \in \mathbb{N}, 1 \leq i \leq m\}$ be a family of variables over an algebraically closed field k of characteristic 0, and let $X := (x_{ij})$ be the corresponding matrix. Set $R := k[X]$. Let $\mathrm{Gr}_{m\infty}(k)$ be the k-subalgebra of R generated by the m-minors of X. Then $\mathrm{Gr}_{m\infty}(k)$ is Cohen-Macaulay in the sense of each part of Definition 3.2.

Proof. For each $n \in \mathbb{N}$, set $X_n := \{x_{ij} : 1 \leq j \leq n, 1 \leq i \leq m\}$ and $R_n := k[X_n]$. Suppose $n \geq m$, and denote the k-subalgebra of R_n

generated by the m-minors of X_n by $\mathrm{Gr}_{mn}(k)$. Clearly, $\mathrm{Gr}_{m\infty}(k) = \cup_{n\geq m} \mathrm{Gr}_{mn}(k)$.

Let $G := \mathrm{SL}_m(k)$. By Remark 3.7 (iii), G is linearly reductive. G acts on R via the assignment $X \mapsto TX$ for all $T \in G$. Also, G acts on R_n, likewise R, for all $n \in \mathbb{N}$. By [**7,** Corollary 7.7], $\mathrm{Gr}_{mn}(k) = R_n^G$. So

$$\mathrm{Gr}_{m\infty}(k) = \cup_{n\geq m} \mathrm{Gr}_{mn}(k) = \cup_{n\geq m} R_n^G = R^G.$$

Now, it follows from Corollary 3.8 that $\mathrm{Gr}_{m\infty}(k)$ is Cohen-Macaulay in the sense of each part of Definition 3.2. □

The following extends [**5,** Corollary 5.8] to a more general situation.

Example 4.3. Let k be a field and $A := k[X_1, \ldots]$. We recall the definition of Veronese rings. Let $f := X_{i_1}^{j_1} \cdots X_{i_\ell}^{j_\ell}$ be a monomial in A. The degree of f is defined by $d(f) := \sum_{k=1}^{\ell} j_k$. Let d be a positive integer. We call the k-algebra $A^{(d)}$, generated by all monomials of degree d, the dth Veronese subring of A. Then $A^{(d)}$ is Cohen-Macaulay in the sense of each part of Definition 3.2.

Proof. Denote the Veronese subring of $A_n := k[X_1, \ldots, X_n]$ by $A_n^{(d)}$. Recall that $A_n^{(d)}$ is the k-subspace of A_n generated by

$$\{X_1^{v_1} \cdots X_n^{v_n} \mid v_1, \ldots, v_n \in \mathbb{N}_0, v_1 + \cdots + v_n \equiv 0 \pmod d\}.$$

Define $\rho : A_n \to A_n^{(d)}$ such that ρ maps each monomial $r \in A_n \setminus A_n^{(d)}$ to 0 and each monomial $r \in A_n^{(d)}$ to itself. Extend ρ linearly to A_n. One can easily see that ρ is a retraction of $A_n^{(d)}$ to A_n. So, $A_n^{(d)}$ is a direct summand of A_n. It turns out that the ring extension $A_n^{(d)} \to A_n$ is pure. On the other hand, $A^{(d)} \cap A_n = A_n^{(d)}$. By applying Theorem 3.4, $A^{(d)}$ is Cohen-Macaulay in the sense of each part of Definition 3.2. □

Example 4.4. Here, we give a natural extension of Example 4.3. Let $\{X_j : j \in \mathbb{N}\}$ be a family of variables over a field k and $A := k[X_1, \ldots]$. Fix $s, t \in \mathbb{N}$, and choose integers $k_{1,j}, \ldots, k_{s,j} \in \mathbb{Z}$ for each $j \in \mathbb{N}$. Let H be the submonoid of $\mathbb{N}^\infty := \cup_{n\in\mathbb{N}} \mathbb{N}^n$ consisting of the solutions of the homogeneous linear equations

$$\sum_{1\leq j\leq n} k_{i,j}X_j = 0, 1 \leq i \leq s$$

for all $n \geq t$. Then H is a full subsemigroup of $\mathbb{N}^\infty$,that is, for each $\alpha, \beta \in H$ with $\alpha - \beta \in \mathbb{N}^\infty$, one has $\alpha - \beta \in H$. Let W be the k-span of the monomials $X_1^{a_1} \cdots X_n^{a_n}$ such that $(a_1, \ldots, a_n, 0, \ldots) \in \mathbb{N}^\infty \setminus H$. If $\beta \in \mathbb{N}^\infty \setminus H$ and $\alpha \in H$, then $\alpha + \beta \in \mathbb{N}^\infty \setminus H$. Hence, W is a $k[H]$-module, and $k[H]$ is direct summand of A. Since $k[H]$ is a k-subalgebra of A, generated by monomials, then, by Corollary 3.6, $k[H]$ is Cohen-Macaulay in the sense of each part of Definition 3.2.

Remark 4.5. In view of **[13]**, the ring $k[H]$ of Example 4.4 appears in the following way. Let $G = \mathrm{GL}(1, k)^s$ and $\gamma = (\gamma_1, \ldots, \gamma_s) \in G$. The assignments $X_j \mapsto \gamma_1^{k_{1,j}} \ldots \gamma_s^{k_{s,j}} X_j$ define an action of G on A. For any monomial $\lambda = X_1^{a_1} \cdots X_n^{a_n}$ and, for each $\gamma = (\gamma_1, \ldots, \gamma_s) \in G$, γ sends λ to $(\prod_{1\leq i\leq s}(\gamma_i^{k_{i,1}a_1+\cdots+k_{i,n}a_n}))\lambda$. It is well known that the ring of invariants is spanned over k by all monomials $x_1^{a_1} \cdots x_n^{a_n}$, where $t \leq n$ and the equations

$$\sum_{1\leq j\leq n} k_{i,j}X_j = 0, \quad 1 \leq i \leq s,$$

are solved by $(a_1, \ldots, a_n)$. This means that $A^G = k[H]$.

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