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MONOMIAL RESOLUTIONS SUPPORTED BY SIMPLICIAL TREES

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ABSTRACT. We explore resolutions of monomial ideals supported by simplicial trees. We argue that, since simplicial trees are acyclic, the criterion of Bayer, Peeva and Sturmfels for checking whether a simplicial complex supports a free resolution of a monomial ideal reduces to checking that certain induced subcomplexes are connected. We then use results of Peeva and Velasco to show that every simplicial tree appears as the Scarf complex of a monomial ideal and hence supports a minimal resolution. We also provide a way to construct smaller Scarf ideals than those constructed by Peeva and Velasco.

1. Introduction. Simplicial trees [**5**] were first introduced as a generalization of graph trees and in the context of studying normal ideals. It became increasingly clear that the combinatorics of simplicial trees mimics that of graph trees quite nicely. As a result, many Cohen-Macaulay type properties known for edge ideals of graphs were generalized for facet ideals of trees ([**6, 7**]), and a "cycle theory" for simplicial trees was developed ([**3, 4**]).

The purpose of this paper is to demonstrate that simplicial trees have the potential to be used as an effective tool in resolutions of monomial ideals. As first noted by Taylor [**11**], given an ideal I in a polynomial ring R minimally generated by monomials $m_1, \ldots, m_q$, a free resolution of I can be given by the simplicial chain complex of a simplex with q vertices. Most often, Taylor's resolution is not minimal. Bayer, Peeva and Sturmfels [**1**] refined Taylor's construction: they provided a criterion to check whether the simplicial chain complex of any simplicial complex on q vertices is a (minimal) free resolution of I (Theorem 3.1).

If Δ is a simplicial complex with q vertices, the criterion of Bayer, Peeva and Sturmfels determines if Δ supports a free resolution of I

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based on whether certain subcomplexes of Δ are acyclic. The goal of this note is to point out that if the simplicial complex Δ being considered is a simplicial tree (Definition 2.3), then all that needs to be checked is that these subcomplexes are connected. We accomplish this by proving that simplicial trees are acyclic (Theorem 2.9), and every induced subcomplex of a simplicial tree is a simplicial forest (Theorem 2.5).

We then use a result of Peeva and Velasco **[9]** to conclude that every simplicial tree supports a minimal resolution of a monomial ideal. Peeva and Velasco's result is that every simplicial complex (other than the boundary of a simplex) is the Scarf complex of some monomial ideal, and they give a specific method to build such an ideal. We refine their result to describe ideals minimally resolved by a Scarf complex, and therefore by a given simplicial tree, and compare them to such ideals described by Phan **[10]**.

2. Simplicial trees and some of their properties.

Definition 2.1 (Simplicial complex)**.** A *simplicial complex* Δ over a set of vertices $V = \{v_1, \dots, v_n\}$ is a collection of subsets of V, with the property that $\{v_i\} \in \Delta$ for all i, and if $F \in \Delta$ then all subsets of F are also in Δ. An element of Δ is called a *face* of Δ, and the *dimension* of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively, and $\dim \emptyset = -1$. The maximal faces of Δ under inclusion are called *facets* of Δ. The dimension of the simplicial complex Δ is the maximal dimension of its facets. A *subcollection* of Δ is a simplicial complex whose facets are also facets of Δ; in other words a simplicial complex generated by a subset of the set of facets of Δ.

Suppose Δ is a simplicial complex with facets $F_1, \dots, F_q$. The simplicial complex obtained by *removing the facet* F_i from Δ is the simplicial complex

$$\Delta \setminus \langle F_i \rangle = \langle F_1, \dots, \widehat{F_i}, \dots, F_q \rangle.$$

Definition 2.2 (**[5]** leaf, joint)**.** A facet F of a simplicial complex is called a *leaf* if either F is the only facet of Δ or for some facet

$G \in \Delta \setminus \langle F \rangle$ we have

$$F \cap (\Delta \setminus \langle F \rangle) \subseteq G.$$

Such a facet G is called a *joint* of F.

Equivalently, a facet F is a leaf of Δ if $F \cap (\Delta \setminus \langle F \rangle)$ is a face of $\Delta \setminus \langle F \rangle$.

Note that it follows immediately from the definition above that a leaf F must contain at least one *free vertex*; namely, a vertex that belongs to no other facet of Δ but F.

Definition 2.3 ([**5**] tree, forest)**.** A connected simplicial complex Δ is a *tree* if every nonempty subcollection of Δ has a leaf. If Δ is not necessarily connected, but every subcollection has a leaf, then Δ is called a *forest*.

Definition 2.4 (induced subcomplex)**.** Suppose Δ is a simplicial complex over a vertex set V, and let $\mathcal{X} \subseteq V$. The *induced subcomplex on* $\mathcal{X}$, denoted by $\Delta_{\mathcal{X}}$, is defined as

$$\Delta_{\mathcal{X}} = \{F \in \Delta \mid F \subseteq \mathcal{X}\}.$$

Theorem 2.5. *An induced subcomplex of a simplicial tree is a simplicial forest.*

Proof. Let $\Delta = \langle F_1, \ldots, F_q \rangle$ be a simplicial tree, and suppose $\mathcal{X} = \{x_1, \ldots, x_s\}$ is a subset of the vertex set of Δ. We would like to show that $\Delta_{\mathcal{X}}$ is a forest. The facets of $\Delta_{\mathcal{X}}$ are clearly a subset of $\{F_1 \cap \mathcal{X}, \ldots, F_q \cap \mathcal{X}\}$. Let Γ be a subcollection of $\Delta_{\mathcal{X}}$ consisting of facets $F_{\alpha_1} \cap \mathcal{X}, \ldots, F_{\alpha_r} \cap \mathcal{X}$. We need to show Γ has a leaf. Since Δ is a tree, the corresponding subcollection $F_{\alpha_1}, \ldots, F_{\alpha_r}$ of Δ has a leaf F_{α_i} with joint F_{α_j}. So, for every $h \in \{1, \ldots, r\} \setminus \{i\}$, we have

$$F_{\alpha_i} \cap F_{\alpha_h} \subseteq F_{\alpha_j}$$

which implies that

$$(F_{\alpha_i} \cap \mathcal{X}) \cap (F_{\alpha_h} \cap \mathcal{X}) \subseteq (F_{\alpha_j} \cap \mathcal{X}).$$

So $F_{\alpha_i} \cap \mathcal{X}$ is a leaf of Γ, and therefore $\Delta_{\mathcal{X}}$ is a forest. $\square$

One property of simplicial trees that we will need is that they are acyclic. While this can be shown via a direct calculation of homological cycles and boundaries, we show more: simplicial trees are collapsible, hence contractible, and therefore acyclic. We refer the reader to [**2**] for more details on these concepts.

Definition 2.6 (Collapsible simplicial complex)**.** Let Δ be a simplicial complex and F' a maximal proper face of exactly one facet F of Δ. The complex $\Gamma = \Delta \setminus \{F, F'\}$ is said to be obtained from Δ using an *elementary collapse*. If a sequence of elementary collapses reduces Δ to a single point, then Δ is called *collapsible*.

Below we use the phrase "Δ collapses to Δ'" to imply that the complex Δ' can be obtained from Δ via a sequence of elementary collapses.

Proposition 2.7. *Let Δ be a simplex with facet F, and let F' be a proper nonempty face of F. Then Δ collapses to $\langle F' \rangle$. In particular, every simplex is collapsible.*

Proof. Suppose $F = \{x_1, \ldots, x_n\}$. We use induction on n. The case $n = 2$ is clear, since F' would be a point, say $\{x_1\}$, and the edge $\{x_1, x_2\}$ clearly collapses to $\{x_1\}$.

Suppose $n > 2$, and let $F_1, \ldots, F_n$ be the maximal proper faces of F where, for each i, $F_i = F \setminus \{x_i\}$. Suppose, without loss of generality, $F' \subset F_n$. We perform the following elementary collapse on Δ:

$$\Delta \setminus \{F, F_1\} = \langle F_2, \ldots, F_n \rangle. \qquad \square$$

Claim 2.8. For $i \geq 2$, there is a series of elementary collapses taking the complex $\langle F_i, \ldots, F_n \rangle$ to the complex $\langle F_{i+1}, \ldots, F_n \rangle$.

Proof of Claim 2.8. If $i = 2$, then the complex $\Delta_2 = \langle F_2, \ldots, F_n \rangle$ has $F_2 \cap F_1$ as a maximal proper face of F_2 (note that $F_2 \cap F_1 = \{x_3, \ldots, x_n\} \not\subset F_i$ if $i > 2$). Now we do the elementary collapse

$$\Delta_2 \setminus \{F_2, F_1 \cap F_2\} = \langle F_3, \ldots, F_n \rangle,$$

and we are done.

Now suppose that we have arrived at $\Delta_i = \langle F_i, \ldots, F_n \rangle$. In what follows we will repeatedly use two basic observations.

(i) The maximal proper subfaces of a face $F_{i_1,\ldots,i_k} = F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_k}$ are of the form

$$F_{i_1,\ldots,i_k,j} = F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_k} \cap F_j$$

where $j \notin \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.

(ii) Suppose $n \geq i_1 > i_2 > \cdots > i_s \geq 1$ and $n \geq j_1 > j_2 > \cdots > j_t \geq 1$. Then we have

$$F_{i_1,\ldots,i_s} \subseteq F_{j_1,\ldots,j_t} \iff \{i_1, \ldots, i_s\} \supseteq \{j_1, \ldots, j_t\}.$$

So the maximal proper faces of F_i that are not contained in any of $F_{i+1}, \ldots, F_n$ are

$$F_{1,i}, F_{2,i}, \ldots, F_{i-1,i}.$$

Let $\Delta_{i+1} = \langle F_{i+1}, \ldots, F_n \rangle$. Using (i) and (ii) above we perform the repeated elementary collapses

$$\begin{aligned}
\Delta_{i,1} &= \Delta_i \setminus \{F_i, F_{i,1}\} = \langle F_{i,2}, \ldots, F_{i,i-1} \rangle \cup \Delta_{i+1} \\
\Delta_{i,2,1} &= \Delta_{i,1} \setminus \{F_{i,2}, F_{i,2,1}\} = \langle F_{i,3}, \ldots, F_{i,i-1} \rangle \cup \Delta_{i+1} \\
\Delta_{i,3,1} &= \Delta_{i,2,1} \setminus \{F_{i,3}, F_{i,3,1}\} = \langle F_{i,3,2}, F_{i,4}, \ldots, F_{i,i-1} \rangle \cup \Delta_{i+1} \\
\Delta_{i,3,2,1} &= \Delta_{i,3,1} \setminus \{F_{i,3,2}, F_{i,3,2,1}\} = \langle F_{i,4}, \ldots, F_{i,i-1} \rangle \cup \Delta_{i+1} \\
\Delta_{i,4,1} &= \Delta_{i,3,2,1} \setminus \{F_{i,4}, F_{i,4,1}\} = \langle F_{i,4,2}, F_{i,4,3}, F_{i,5}, \ldots, F_{i,i-1} \rangle \cup \Delta_{i+1} \\
&\vdots \\
\Delta_{i,\ldots,1} &= \Delta_{i,i-1,i-2,\ldots,3,1} \setminus \{F_{i,\ldots,2}, F_{i,\ldots,1}\} = \Delta_{i+1}. \quad \square
\end{aligned}$$

It now follows from repeated applications of Claim 2.8 that Δ collapses to $\Delta_n = \langle F_n \rangle$, which is a simplex on $n-1$ vertices. If $F' = F_n$, we are done, and if not, the induction hypothesis implies that Δ_n collapses to $\langle F' \rangle$ via a series of elementary collapses.

Theorem 2.9. *Simplicial trees are collapsible, and therefore contractible and acyclic.*

Proof. We prove this by induction on the number q of facets of a simplicial tree Δ. If $q = 1$, the statement follows from Proposition 2.7. Suppose $q > 1$, and let F be a leaf of Δ with joint G. Let $F' = F \cap G$. We know by Proposition 2.7 that $\langle F \rangle$ reduces to $\langle F' \rangle$ via a series of elementary collapses. Moreover, the faces being eliminated in each of the collapses are not faces of $\Delta \setminus \langle F \rangle$, since they are not faces of $F' = F \cap \Delta \setminus \langle F \rangle$. Therefore, all the elementary collapses that reduce $\langle F \rangle$ to $\langle F' \rangle$ are elementary collapses on Δ that reduce Δ to $\Delta \setminus \langle F \rangle$. The latter is a tree with $q-1$ facets and hence collapsible by the induction hypothesis.

All collapsible complexes are contractible so the rest of the statement follows directly. $\square$

3. Resolutions by trees. We now review monomial resolutions as described by Bayer, Peeva and Sturmfels [**1**] and show how simplicial trees fit in that picture. The construction in [**1**] considers a monomial ideal I in a polynomial ring S over a field, where I is minimally generated by monomials $m_1, \ldots, m_t$. If Δ is a simplicial complex on t vertices, one can label each vertex of Δ with one of the generators of $m_1, \ldots, m_t$ and each face with the least common multiple of the labels of its vertices. If m is a monomial in S, let Δ_m be the subcomplex of Δ induced on the vertices of Δ whose labels divide m.

Theorem 3.1 (Bayer, Peeva, Sturmfels [**1**]). *Let Δ be a simplicial complex labeled by monomials $m_1, \ldots, m_t \in S$, and let $I = (m_1, \ldots, m_t)$ be the ideal in S generated by the vertex labels. The chain complex $\mathcal{C}(\Delta) = \mathcal{C}(\Delta; S)$ of Δ is a free resolution of S/I if and only if the induced subcomplex Δ_m is empty or acyclic for every monomial $m \in S$. Moreover, $\mathcal{C}(\Delta)$ is a minimal free resolution if and only if $m_A \neq m_{A'}$ for every proper subface A' of a face A.*

Note that we can determine whether $\mathcal{C}(\Delta)$ is a resolution just by checking the vanishing condition for monomials that are least common multiples of sets of vertex labels.

Combinatorially, what Theorem 3.1 is saying is that the Betti vector of S/I is bounded by the f-vector of an eligible Δ:

$$\begin{aligned} \beta(S/I) &= (\beta_0(S/I), \ldots, \beta_q(S/I)) \\ &\leq (f_0(\Delta), \ldots, f_q(\Delta)) = \mathbf{f}(\Delta) \end{aligned} \tag{3.1}$$

with equality holding if some extra conditions are satisfied.

We now turn our attention back to simplicial trees. If the Δ under consideration in Theorem 3.1 is a tree, then we can show the following.

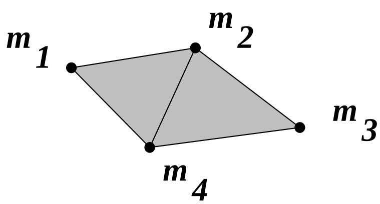
Theorem 3.2 (Resolutions via simplicial trees)**.** *Let Δ be a simplicial tree labeled by monomials $m_1, \ldots, m_t \in S$, and let $I = (m_1, \ldots, m_t)$ be the ideal in S generated by the vertex labels. The chain complex $\mathcal{C}(\Delta) = \mathcal{C}(\Delta; S)$ is a free resolution of S/I if and only if the induced subcomplex Δ_m is connected for every monomial m.*

Proof. By Theorem 2.5, every induced subcomplex of Δ is a forest. By Theorem 2.9, forests are acyclic in all but possibly the 0th reduced homology, that is, they may not be connected. This proves the theorem. $\square$

The strength of Theorem 3.2 is in that it reduces the question of whether a simplicial complex resolves an ideal to checking whether some of its induced subcomplexes are connected.

One type of question one could then ask is, given a tree Δ, what ideals could it resolve? Our first example displays this line of questioning.

Example 3.3. Let Δ be the simplicial tree below on 4 vertices, which we have labeled with monomials $m_1, \ldots, m_4$.

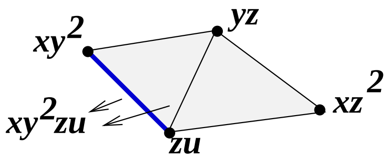


The only induced subcomplex of Δ that is not connected is the one induced on the vertices labeled m_1 and m_3, so by Theorem 3.2, for $I = (m_1, m_2, m_3, m_4)$ to be resolved by Δ, we need to have

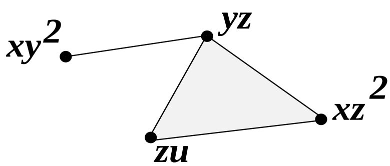
$$m_2 \mid \operatorname{lcm}(m_1, m_3) \quad \text{or} \quad m_4 \mid \operatorname{lcm}(m_1, m_3).$$

A more concrete example using the same complex comes next.

Example 3.4. The ideal $I = (xy^2, yz, xz^2, zu)$ can be resolved by Δ.



However, $\beta(S/I) = (4,4,1) \lneq (4,5,2) = \mathbf{f}(\Delta)$, so the resolution is not minimal. We try to make it minimal by removing the faces with equal labels.



Note that the resulting complex is also a simplicial tree satisfying the conditions of Theorem 3.2 and whose f-vector is $(4,4,1)$. It therefore minimally resolves S/I.

4. Scarf complexes and Scarf ideals. We now come to the question of which monomial ideals can be (minimally) resolved by a simplicial tree. It is known from work of Velasco **[12]** that there are classes of monomial ideals whose minimal resolutions are not supported by any simplicial complex. However, most simplicial complexes, and all simplicial trees do appear as *Scarf complexes* of some monomial ideal. Given a monomial ideal, its Scarf complex is a subcomplex of its Taylor complex with the same labeling and with the added condition that, if a face has the same label as another face, neither face can appear in the Scarf complex. The last simplicial complex appearing in Example 3.4 is a Scarf complex of the ideal I in that example.

By construction, if the Scarf complex resolves an ideal, it does so minimally. Moreover most simplicial complexes appear as the Scarf complex of some monomial ideal.

Theorem 4.1 (**[9]**, **[10]**). *Let Δ be a simplicial complex on r vertices.*

(i) *Δ is the Scarf complex of a monomial ideal if and only if Δ is not the boundary of a simplex on r vertices.*

(ii) Δ *minimally resolves a monomial ideal if and only if* Δ *is acyclic.*

Since simplicial trees are acyclic, it immediately follows that:

Corollary 4.2. *Every simplicial tree is the Scarf complex of a monomial ideal I and supports a minimal resolution of I.*

An ideal (minimally) resolved by its Scarf complex is called a *Scarf ideal*. Given an eligible simplicial complex Δ with vertices labeled $1, \dots, n$, Peeva and Velasco in **[9]** build a Scarf ideal J_Δ using the following steps. Define a variable x_σ corresponding to each face σ of Δ. In the polynomial ring generated by all these variables, define the ideal J_Δ whose generators are enumerated by the vertices of Δ, and for every given vertex v of Δ, the corresponding monomial generator is the product of all x_σ where $v \notin \sigma$. In short,

$$J_\Delta = \left(\prod_{\substack{\sigma \in \Delta \\ v \notin \sigma}} x_\sigma \mid v = 1, \dots, n \right) = (m_1, \dots, m_n). \tag{4.1}$$

The ideal J_Δ defined above is generated by rather large monomials. In what follows, we will demonstrate that one can shave off some variables in each monomial to reduce the size of the generator and still have a Scarf ideal of Δ.

Suppose Δ is a simplicial complex with vertices labeled $1, \dots, n$. And, for each vertex v, let $A_\Delta(v)$ be the set of facets of Δ that do not contain v, and let $B_\Delta(v)$ be the set of facets of Δ that do contain v. With variables labeled as

$$J'_\Delta = (m'_1, \dots, m'_n)$$

where

$$m'_v = \sqrt{\prod_{G \in B_\Delta(v)} x_{G \setminus \{v\}} \prod_{F \in A_\Delta(v)} \left(x_F \prod_{\substack{\sigma \subset F \\ |\sigma| = |F|-1}} x_\sigma \right)} \tag{4.2}$$
$$\text{for } v = 1, \dots, n,$$

and for a monomial m by $\sqrt{m}$, we mean the square-free monomial which is the product of all variables dividing m. It is clear that the $m'_v \mid m_v$ for all v.

Proposition 4.3. *Let Δ be a simplicial complex which is not the boundary of an n-simplex, and let J'_Δ be the ideal described in* (4.2).

(i) *Δ is the Scarf complex for J'_Δ.*
(ii) *If Δ is acyclic* (*and in particular if Δ is a simplicial tree*), *then J'_Δ is a Scarf ideal.*

Proof. We first show that J'_Δ has no redundant generators. Suppose that we have $m'_i \mid m'_j$ and $i \neq j$.

Clearly, $A_\Delta(i) \subseteq A_\Delta(j)$. If $G \in B_\Delta(i)$, then $G \setminus \{i\}$ can only be a maximal proper face of a facet in $A_\Delta(j)$; otherwise, $H = \{j\} \cup G \setminus \{i\} \in B_\Delta(j)$ and $i \notin H$. Therefore, $H \in A_\Delta(i) \subseteq A_\Delta(j)$, which is a contradiction since $j \in H$. In particular, $G \in A_\Delta(j)$.

We have shown that

$$A_\Delta(i) \cup B_\Delta(i) \subseteq A_\Delta(j).$$

This implies that all facets of Δ belong to $A_\Delta(j)$, and hence j is not in any facet of Δ; a contradiction.

So we can label the vertices of Δ with the monomials $m'_1, \ldots, m'_n$, where the labeling is consistent with $m_1, \ldots, m_n$ as in (4.1). Next, we have to make sure that Δ is a Scarf complex of J'_Δ. For this purpose and what follows, the next claim will be useful.

Claim 4.4. Suppose $\sigma = \{u_1, \ldots, u_s\}$ and $\tau = \{v_1, \ldots, v_t\}$ are two faces of the simplex on $\{1, \ldots, n\}$. Then

$$\begin{aligned} \operatorname{lcm}(m'_{u_1}, \ldots, m'_{u_s}) = \operatorname{lcm}(m'_{v_1}, \ldots, m'_{v_t}) \iff \operatorname{lcm}(m_{u_1}, \ldots, m_{u_s}) \\ = \operatorname{lcm}(m_{v_1}, \ldots, m_{v_t}). \end{aligned}$$

Proof of Claim 4.4. For ease of argument, we label the above lcms from the left to the right with the symbols M'_σ, M'_τ, M_σ and M_τ, respectively. Now suppose $M'_\sigma = M'_\tau$. Then it follows directly that $M_\sigma = M_\tau$. Conversely, suppose $M_\sigma = M_\tau$. Then, in particular, we

have
$$\bigcup_{i=1}^{s} A_\Delta(u_i) = \bigcup_{i=1}^{t} A_\Delta(v_i)$$
so all the factors x_F where F is a facet of Δ are the same in both monomials M'_σ and M'_τ, as well as all x_σ for maximal proper faces σ of such F. So we only have to worry about terms of the form $x_{G\setminus\{j\}}$ for a facet G of Δ that contains j. Suppose $x_{G\setminus\{u_h\}} \mid M'_\sigma$. If x_G appears in M'_σ, we are done, as $G \setminus \{u_h\}$ is a maximal proper face of G which appears as a label in M'_τ as well. If not, we conclude that $u_1, \ldots, u_s, v_1, \ldots, v_t \in G$, which means that σ and τ are both faces of Δ with the same lcms; a contradiction, as Δ is a Scarf complex of J_Δ. □

The statement we just proved implies that Δ is the Scarf complex of J'_Δ, as it is the Scarf complex of J_Δ.

We now show that, if Δ is acyclic, then it supports a (minimal) resolution of J'_Δ. So we need to show that, for any set of vertices $u_1, \ldots, u_s$ of Δ, the induced subcomplex on the vertex set
$$\mathcal{X} = \{j \mid m'_j \mid \operatorname{lcm}(m'_{u_1}, \ldots, m'_{u_s})\}$$
is acyclic. Notice that
$$\operatorname{lcm}(m'_j \mid j \in \mathcal{X}) = \operatorname{lcm}(m'_{u_1}, \ldots, m'_{u_s})$$
which, by Claim 4.4, is equivalent to
$$\operatorname{lcm}(m_j \mid j \in \mathcal{X}) = \operatorname{lcm}(m_{u_1}, \ldots, m_{u_s})$$
and
$$\mathcal{X} = \{j \mid m_j \mid \operatorname{lcm}(m_{u_1}, \ldots, m_{u_s})\}.$$
So the induced subcomplex $\Delta_{\mathcal{X}}$ is the same under both labelings (by J_Δ and J'_Δ), and is therefore acyclic. □

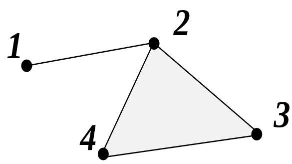
Note that Claim 4.4 shows that J_Δ and J'_Δ have isomorphic lcm lattices, which in fact by itself implies that the two ideals have isomorphic minimal resolutions [**8**], giving an alternate proof.

Remark 4.5. Using the lattice coming from the face poset of a simplicial complex Δ, Phan [**10**] described a "minimal" monomial ideal

$\mathcal{P}_\Delta$ resolved by Δ. The generators of $\mathcal{P}_\Delta$ are monomials labeled with "meet irreducible" elements of this lattice, that is, all faces of Δ that are not intersections of two other faces of Δ. It is straightforward to show that every such face is a facet or a maximal proper face of a facet. It follows that the minimal generators of $\mathcal{P}_\Delta$ divide the m'_v described in (4.2).

We demonstrate all this via an example.

Example 4.6. For the complex Δ below, $\beta(J_\Delta) = (4,4,1) = \beta(J'_\Delta) = \mathbf{f}(\Delta)$.



$$J_\Delta = (\mathbf{x_2}x_3x_4\mathbf{x_{23}}\mathbf{x_{24}}\mathbf{x_{34}}\mathbf{x_{234}}, \mathbf{x_1}x_3x_4\mathbf{x_{34}}, \mathbf{x_1}\mathbf{x_2}x_4\mathbf{x_{12}}\mathbf{x_{24}}, \mathbf{x_1}\mathbf{x_2}x_3\mathbf{x_{12}}\mathbf{x_{23}})$$
$$\downarrow$$
$$J'_\Delta = (\underline{x_2}x_{23}x_{24}x_{34}x_{234}, x_1x_{34}, x_1\underline{x_2}x_{12}x_{24}, x_1\underline{x_2}x_{12}x_{23})$$
$$\downarrow$$
$$\mathcal{P}_\Delta = (x_{23}x_{24}x_{34}x_{234}, x_1x_{34}, x_1x_{12}x_{24}, x_1x_{12}x_{23})$$

Computational evidence has shown that many ideals "in-between" J_Δ and J'_Δ can be resolved by Δ, though not all of them, as indicated in Example 4.8. Given a vertex v of Δ, we know that

$$m_v = \prod_{\substack{\sigma\in\Delta \\ v\notin\sigma}} x_\sigma = m''_v m'_v, \tag{4.3}$$

where, by m''_v, we are denoting the product of the x_σ that does not appear in m'_v. We show below that the Scarf complexes of these in-between ideals contain Δ as a subcomplex, but our discussion further down demonstrates that these Scarf complexes could be larger than Δ.

Proposition 4.7. *Let Δ be a simplicial complex on n vertices which is not the boundary of a simplex. For a vertex v of Δ, let the monomials*

m_v, m'_v and m''_v be as defined in (4.1), (4.2) *and* (4.3)*, respectively, and suppose h_v is a monomial such that $h_v \mid m''_v$. Let I be the monomial ideal*

$$I = (h_1 m'_1, \ldots, h_n m'_n).$$

Then the Scarf complex Γ of I has n vertices and contains Δ as a subcomplex.

Proof. First, we have to show that I has no redundant generators. Consider two monomials $h_i m'_i$ and $h_j m'_j$ for some $i \neq j$. We have proved before that $m'_i \nmid m'_j$, so there are two possibilities:

(i) There is $F \in A_\Delta(i)$ such that $F \notin A_\Delta(j)$ (therefore $j \in F$), in which case $x_F \nmid m_j$, and therefore $h_i m'_i \nmid h_j m'_j$; a contradiction.
(ii) $A_\Delta(i) \subseteq A_\Delta(j)$, in which case there is $G \in B_\Delta(i)$ such $x_{G\setminus\{i\}} \nmid m'_j$, so $G \notin A_\Delta(j)$, and therefore $j \in G$, which implies that $j \in G \setminus \{i\}$. So $x_{G\setminus\{i\}} \nmid m_j$, and therefore $h_i m'_i \nmid h_j m'_j$.

This shows that $h_1 m'_1, \ldots, h_n m'_n$ is a minimal generating set for I.

Let Γ be the Scarf complex of I, and suppose $\sigma = \{u_1, \ldots, u_s\}$ and $\tau = \{v_1, \ldots, v_t\}$ are two faces of the simplex on $\{1, \ldots, n\}$ with the same labels:

$$\operatorname{lcm}(h_{u_1} m'_{u_1}, \ldots, h_{u_s} m'_{u_s}) = \operatorname{lcm}(h_{v_1} m'_{v_1}, \ldots, h_{v_t} m'_{v_t}).$$

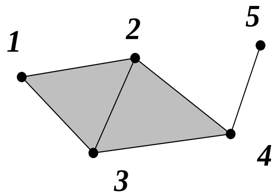
Suppose $u_i \notin \{v_1, \ldots, v_t\}$ for some i. Then we have $h_{u_i} m'_{u_i} \mid \operatorname{lcm}(h_{v_1} m'_{v_1}, \ldots, h_{v_t} m'_{v_t})$. So all variables labeled by facets in $A_\Delta(u_i)$, their maximal proper faces, and by $G \setminus \{u_i\}$ for $G \in B_\Delta(u_i)$ already appear in $\operatorname{lcm}(h_{v_1} m'_{v_1}, \ldots, h_{v_t} m'_{v_t}) \mid \operatorname{lcm}(m_{v_1}, \ldots, m_{v_t})$. Therefore,

$$\begin{aligned} m_{u_i} \mid \operatorname{lcm}(m_{v_1}, \ldots, m_{v_t}) \Longrightarrow \operatorname{lcm}(m_{u_i}, m_{v_1}, \ldots, m_{v_t}) \\ = \operatorname{lcm}(m_{v_1}, \ldots, m_{v_t}). \end{aligned}$$

Since Δ is the Scarf complex for J_Δ, this implies that $\tau \notin \Delta$. Similarly, we have $\sigma \notin \Delta$. This proves that the Scarf complex Γ of I contains Δ. $\square$

Below is an example demonstrating that Γ may not be equal to Δ, even though they are quite often equal.

Example 4.8. For the complex Δ below,

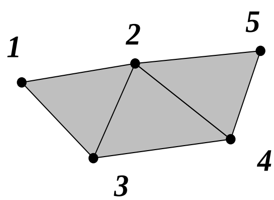


we have $J_\Delta = (m_1, \ldots, m_5)$ and $J'_\Delta = (m'_1, \ldots, m'_5)$, where

$$\begin{array}{llll} m_1 = & x_2x_3m'_1 & m'_1 = & x_{23}x_{24}x_{34}x_{234}x_4x_5x_{45} \\ m_2 = & x_1x_3m'_2 & m'_2 = & x_{13}x_{34}x_4x_5x_{45} \\ m_3 = & x_1x_2m'_3 & m'_3 = & x_{12}x_{24}x_4x_5x_{45} \\ m_4 = & x_1x_2x_3m'_4 & m'_4 = & x_{12}x_{13}x_{23}x_{123}x_5 \\ m_5 = & x_1x_2x_3m'_5 & m'_5 = & x_{12}x_{13}x_{23}x_{123}x_{24}x_{34}x_{234}x_4 \end{array}$$

In this case, $\beta(S/J_\Delta) = \beta(S/J'_\Delta) = \mathbf{f}(\Delta) = (5, 6, 2)$ as expected (though J_Δ and J'_Δ have different graded Betti numbers).

Now consider the ideal $I = (m'_1, m'_2, m'_3, x_1m'_4, m'_5)$. We have $\beta(S/I) = (5, 7, 3)$ and the (acyclic) Scarf complex of I is



which contains Δ as a subcomplex.

It is worth noting that only very low degree choices of h_v will give strictly larger Scarf complexes. That is, given an acyclic simplicial complex, one can find a whole class of Scarf ideals for it by making appropriate (large enough) choices for the monomials h_v.

There are many questions that naturally follow from this work, answers to which would greatly contribute to understanding monomial resolutions. For example, can one describe classes of monomial ideals resolved by a given tree? What roles do localization, removal of facets and other such operations that preserve forests play on Scarf ideals? Can one describe classes of complexes (trees) resolving a given monomial ideal?

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