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TORIC IDEALS AND THEIR CIRCUITS

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Dedicated to Professor Jürgen Herzog on his 70th birthday

ABSTRACT. In this paper, we study toric ideals generated by circuits. For toric ideals which have squarefree quadratic initial ideals, a sufficient condition to be generated by circuits is given. In particular, squarefree Veronese subrings, the second Veronese subrings and configurations arising from root systems satisfy the condition. In addition, we study toric ideals of finite graphs and characterize the graphs whose toric ideals are generated by circuits $u-v$ such that either u or v is squarefree. Several classes of graphs exist whose toric ideals satisfy this condition and whose toric rings are not normal.

1. Introduction. Let $\mathbf{Z}^{d\times n}$ be the set of all $d \times n$ integer matrices. A *configuration* of $\mathbf{R}^d$ is a matrix $A \in \mathbf{Z}^{d\times n}$, for which there exists a hyperplane $\mathcal{H} \subset \mathbf{R}^d$ not passing the origin of $\mathbf{R}^d$ such that each column vector of A lies on $\mathcal{H}$. Throughout this paper, we assume that the columns of A are pairwise distinct. Let K be a field and $K[T, T^{-1}] = K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$ the Laurent polynomial ring in d variables over K. Each column vector $\mathbf{a} = (a_1, \ldots, a_d)^\top \in \mathbf{Z}^d$ $(= \mathbf{Z}^{d\times 1})$, where $(a_1, \ldots, a_d)^\top$ is the transpose of $(a_1, \ldots, a_d)$, yields the Laurent monomial $T^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d}$. Let $A \in \mathbf{Z}^{d\times n}$ be a configuration of $\mathbf{R}^d$ with $\mathbf{a}_1, \ldots, \mathbf{a}_n$ its column vectors. The *toric ring* of A is the subalgebra $K[A]$ of $K[T, T^{-1}]$, which is generated by the Laurent monomials $T^{\mathbf{a}_1}, \ldots, T^{\mathbf{a}_n}$ over K. Let $K[X] = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K, and define the surjective ring homomorphism $\pi : K[X] \to K[A]$ by setting $\pi(x_i) = T^{\mathbf{a}_i}$ for $i = 1, \ldots, n$. We say that the kernel $I_A \subset K[X]$ of π is the *toric ideal* of A. It is known that, if $I_A \neq \{0\}$, then I_A is generated by homogeneous binomials of degree ≥ 2. More precisely,

$$I_A = \left\langle X^{\mathbf{u}^+} - X^{\mathbf{u}^-} \in K[X] \;\middle|\; \mathbf{u} \in \mathrm{Ker}_{\mathbf{Z}}(A) \right\rangle,$$

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where $\mathrm{Ker}_{\mathbf{Z}}(A) = \{\mathbf{u} \in \mathbf{Z}^n \mid A\mathbf{u} = \mathbf{0}\}$. Here $\mathbf{u}^+ \in \mathbf{Z}^n_{\geq 0}$ (respectively, $\mathbf{u}^- \in \mathbf{Z}^n_{\geq 0}$) is the positive part (respectively, negative part) of $\mathbf{u} \in \mathbf{Z}^n$. In particular, we have $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$. See [**13**] for details.

The support of a monomial u of $K[X]$ is $\mathrm{supp}\,(u) = \{x_i \mid x_i \text{ divides } u\}$, and the support of a binomial $f = u - v$ is $\mathrm{supp}\,(f) = \mathrm{supp}\,(u) \cup \mathrm{supp}\,(v)$. We say that an irreducible binomial $f \in I_A$ is a *circuit* of I_A if there is no binomial $g \in I_A$ such that $\mathrm{supp}\,(g) \subset \mathrm{supp}\,(f)$ and $\mathrm{supp}\,(g) \neq \mathrm{supp}\,(f)$. Note that a binomial $f \in I_A$ is a circuit of I_A if and only if $I_A \cap K[\{x_i \mid x_i \in \mathrm{supp}\,(f)\}]$ is generated by f. Let C_A be the set of circuits of I_A, and define its subsets C_A^{sf} and C_A^{sfsf} by

$$C_A^{\mathrm{sf}} = \{X^{\mathbf{u}} - X^{\mathbf{v}} \in C_A \mid \text{ either } X^{\mathbf{u}} \text{ or } X^{\mathbf{v}} \text{ is squarefree}\},$$
$$C_A^{\mathrm{sfsf}} = \{X^{\mathbf{u}} - X^{\mathbf{v}} \in C_A \mid \text{ both } X^{\mathbf{u}} \text{ and } X^{\mathbf{v}} \text{ are squarefree }\}.$$

It is known [**13**, Proposition 4.11] that $C_A \subset \mathcal{U}_A$ where $\mathcal{U}_A$ is the union of all reduced Gröbner bases of I_A. Since any Gröbner basis is a set of generators, we have $I_A = \langle \mathcal{U}_A \rangle$. Bogart, Jensen and Thomas [**1**] characterized the configuration A such that $I_A = \langle C_A \rangle$ in terms of polytopes. On the other hand, Martinez-Bernal and Villarreal [**5**] introduced the notion of "unbalanced circuits" and characterized the configuration A such that $I_A = \langle C_A \rangle$ in terms of unbalanced circuits when $K[A]$ is normal. Note that, if $K[A]$ is normal, then any binomial belonging to a minimal set of binomial generators of I_A has a squarefree monomial. (This fact appeared in many papers. See, e.g., [**11**, Lemma 6.1].)

One of the most important classes of toric ideals whose circuits are well-studied is toric ideals arising from finite graphs. Let G be a finite connected graph on the vertex set $[d] = \{1, 2, \ldots, d\}$ with the edge set $E(G) = \{e_1, \ldots, e_n\}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ stand for the canonical unit coordinate vector of $\mathbf{R}^d$. If $e = \{i, j\}$ is an edge of G, then the column vector $\rho(e) \in \mathbf{R}^d$ is defined by $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$. Let $A_G \in \mathbf{Z}^{d \times n}$ denote the matrix with column vectors $\rho(e_1), \ldots, \rho(e_n)$. Then A_G is a configuration of $\mathbf{R}^d$ which is the vertex-edge incidence matrix of G. Circuits of I_{A_G} are completely characterized in terms of graphs (Proposition 2.1). It is known that $K[A_G]$ is normal if and only if G satisfies "the odd cycle condition" (Proposition 2.3). In [**10**, Section 3], generators of I_{A_G} are studied when $K[A_G]$ is normal. It is essentially shown in [**10**, Proof of Lemma 3.2] that, if $K[A_G]$ is normal, then we have $I_{A_G} = \langle C_{A_G}^{\text{sf}} \rangle$. Martinez-Bernal and Villarreal [**5**, Theorem 3.2] also proved this fact and claimed that the converse is true. However, as they stated in [**5**, Note added in proof], the converse is false in general. Several classes of counterexamples are given in Section 2.

The content of this paper is as follows. In Section 1, we study toric ideals having squarefree quadratic initial ideals. For such configurations, a sufficient condition to be generated by circuits is given. In particular, squarefree Veronese subrings, the second Veronese subrings and configurations arising from root systems satisfy the condition. In Section 2, we study toric ideals of finite graphs. We characterize the graphs G whose toric ideals are generated by $C_{A_G}^{\text{sf}}$. A similar result is given for $C_{A_G}^{\text{sfsf}}$. By this characterization, we construct classes of graphs G such that $K[A_G]$ is nonnormal and that $I_{A_G} = \langle C_{A_G}^{\text{sf}} \rangle = \langle C_{A_G}^{\text{sfsf}} \rangle$.

1. Configurations with squarefree quadratic initial ideals. In this section, we study several classes of toric ideals with squarefree quadratic initial ideals. It is known [**13**, Proposition 13.15] that, if a toric ideal I_A has a squarefree initial ideal, then $K[A]$ is normal. First, we show a fundamental fact on quadratic binomials in toric ideals. (Since we assume the columns of A are pairwise distinct, I_A has no binomials of degree 1.)

Proposition 1.1. *Let $A = (a_{ij}) \in \mathbf{Z}^{d \times n}$ be a configuration. Suppose that, for each $1 \leq i \leq d$, there exists a $z_i \in \mathbf{Z}$ such that $z_i - 1 \leq a_{ij} \leq z_i + 1$ for all $1 \leq j \leq n$. Then, any quadratic binomial in I_A belongs to C_A^{sf}. Moreover, if, for each $1 \leq i \leq d$, there exists a $z_i \in \mathbf{Z}$ such that $z_i \leq a_{ij} \leq z_i + 1$ for all $1 \leq j \leq n$, then any quadratic binomial in I_A belongs to C_A^{sfsf}.*

Proof. Suppose that, for each $1 \leq i \leq d$, there exists a $z_i \in \mathbf{Z}$ such that $z_i - 1 \leq a_{ij} \leq z_i + 1$ for all $1 \leq j \leq n$. It is known [**13**, Lemma 4.14] that there exists a vector $\mathbf{w} \in \mathbf{R}^d$ such that $\mathbf{w} \cdot A = (1, 1, \dots, 1)$. Hence, by elementary row operations, we may assume that A is a $(0, \pm 1)$-configuration. Let $A = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbf{Z}^{d \times n}$, and let $f \in I_A$ be a quadratic binomial. Since the columns of A are pairwise distinct, f is either of the form $x_1 x_2 - x_3 x_4$ or $x_1 x_2 - x_3^2$. Note that $|\text{supp}\,(h)| \geq 3$ for any binomial $h \in I_A$. Hence, f is a circuit if $f = x_1 x_2 - x_3^2$.

Let $f = x_1 x_2 - x_3 x_4$, and suppose that $f \notin C_A$. By [**13**, Lemma 4.10], there exists a circuit $g = X^{\mathbf{u}} - X^{\mathbf{v}} \in C_A$ such that $\text{supp}\,(X^{\mathbf{u}}) \subset \{x_1, x_2\}$

and $\mathrm{supp}\,(X^{\mathbf{v}}) \subset \{x_3, x_4\}$. Since f is not a circuit, $|\mathrm{supp}\,(g)| < 4$. Hence, we have $|\mathrm{supp}\,(g)| = 3$. Thus, we may assume that $g = x_1^a x_2^b - x_3^c$ where $1 \leq a, b, c \in \mathbf{Z}$. Then, $a \cdot \mathbf{a}_1 + b \cdot \mathbf{a}_2 = c \cdot \mathbf{a}_3$ and $a + b = c$. Let $\mathbf{a}_k = (a_1^{(k)}, a_2^{(k)}, \ldots, a_d^{(k)})^\top$ for $k = 1, 2, 3$. Since $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are $(0, \pm 1)$-vectors, we have the following for each $1 \leq j \leq d$:

- If $a_j^{(3)} = 1$, then $a_j^{(1)} = a_j^{(2)} = 1$.
- If $a_j^{(3)} = -1$, then $a_j^{(1)} = a_j^{(2)} = -1$.

Since $\mathbf{a}_1$ and $\mathbf{a}_3$ are distinct, there exists $1 \leq k \leq d$ such that $a_k^{(3)} = 0$ and $a_k^{(1)} \neq 0$. Then $a \cdot a_k^{(1)} + b \cdot a_k^{(2)} = 0$, and hence $a = b$ and $a_k^{(2)} = -a_k^{(1)}$. Note that $g = x_1^a x_2^a - x_3^{2a}$ should be irreducible. It then follows that $a = 1$ and $g = x_1 x_2 - x_3^2$. Thus, $f - g = x_3^2 - x_3 x_4$ belongs to I_A, and hence $\mathbf{a}_3 = \mathbf{a}_4$, a contradiction. Therefore, $f \in C_A$.

Suppose that, for each $1 \leq i \leq d$, there exists a $z_i \in \mathbf{Z}$ such that $z_i \leq a_{ij} \leq z_i + 1$ for all $1 \leq j \leq n$. By elementary row operations, we may assume that A is a $(0, 1)$-configuration. Let $f = x_1 x_2 - x_3^2 \in I_A$. Then, $\mathbf{a}_1 + \mathbf{a}_2 = 2 \cdot \mathbf{a}_3$. Since $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are (0,1)-vectors, it follows that $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{a}_3$, a contradiction. □

By Proposition 1.1, we can prove that several classes of toric ideals are generated by circuits.

1.1. Veronese and squarefree Veronese configurations. Let $2 \leq d$, $r \in \mathbf{Z}$ and $V_d^{(r)} = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbf{Z}^{d \times n}$ be the matrix where

$$\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \left\{ (\alpha_1, \ldots, \alpha_d)^\top \in \mathbf{Z}^d \;\middle|\; \alpha_i \geq 0,\ \sum_{i=1}^{d} \alpha_i = r \right\}.$$

Then, $K[V_d^{(r)}]$ is called the *rth Veronese subring* of $K[t_1, \ldots, t_d]$. On the other hand, let $SV_d^{(r)} = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbf{Z}^{d \times n}$ be the matrix where

$$\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \left\{ (\alpha_1, \ldots, \alpha_d)^\top \in \{0, 1\}^d \;\middle|\; \sum_{i=1}^{d} \alpha_i = r \right\}.$$

Then, $K[SV_d^{(r)}]$ is called the *rth squarefree Veronese subring* of $K[t_1, \ldots, t_d]$. It is known (see, e.g., [**13**, Chapter 14]) that

Proposition 1.2. *Toric ideals* $I_{V_d^{(r)}}$ *and* $I_{SV_d^{(r)}}$ *have squarefree quadratic initial ideals, and hence* $K[V_d^{(r)}]$ *and* $K[SV_d^{(r)}]$ *are normal.*

We characterize such toric ideals that are generated by circuits.

Theorem 1.3. *Let* $2 \leq d$, $r \in \mathbf{Z}$. *Then, we have the following*:

(i) *For* $A = SV_d^{(r)}$, *the toric ideal* I_A *is generated by* C_A^{sfsf}.

(ii) *For* $A = V_d^{(r)}$, *the toric ideal* I_A *is generated by* C_A^{sf} *if and only if* $r = 2$.

Proof. First, by Propositions 1.1 and 1.2, (i) and the "if" part of (ii) hold.

Let $r \geq 3$. Since $K[V_2^{(r)}]$ is a combinatorial pure subring (see [**6**] for details) of $K[V_d^{(r)}]$ for all $d > 2$, it is sufficient to show that $I_{V_2^{(r)}}$ is not generated by circuits. Recall that the configuration $V_2^{(r)}$ is $\begin{pmatrix} r & r-1 & r-2 & r-3 & \cdots & 0 \\ 0 & 1 & 2 & 3 & \cdots & r \end{pmatrix}$. Then the binomial $x_1x_4 - x_2x_3 \in I_{V_2^{(r)}}$ is not a circuit since $x_2^2 - x_1x_3$ belongs to $I_{V_2^{(r)}}$. Suppose that $0 \neq x_1x_4 - x_ix_j$ belongs to $I_{V_2^{(r)}}$. Then $\mathbf{a}_i + \mathbf{a}_j = \mathbf{a}_1 + \mathbf{a}_4 = (2r-3, 3)^\top$. Since the last coordinate of $\mathbf{a}_i + \mathbf{a}_j$ is 3, it follows that $\{i, j\}$ is either $\{1, 4\}$ or $\{2, 3\}$. Hence, $x_1x_4 - x_ix_j = x_1x_4 - x_2x_3$. Thus, $x_1x_4 - x_2x_3$ is not generated by other binomials in $I_{V_2^{(r)}}$, as desired. $\square$

1.2. Configurations arising from root systems. For an integer $d \geq 2$, let $\Phi \subset \mathbf{Z}^d$ be one of the classical irreducible root systems $\mathbf{A}_{d-1}$, $\mathbf{B}_d$, $\mathbf{C}_d$ and $\mathbf{D}_d$ ([**4**, pages 64–65]) and write $\Phi^{(+)}$ for the set consisting of the origin of $\mathbf{R}^d$ together with all positive roots of Φ. More explicitly,

$$\begin{aligned}
\mathbf{A}_{d-1}^{(+)} &= \{\mathbf{0}\} \cup \{\mathbf{e}_i - \mathbf{e}_j \mid 1 \leq i < j \leq d\} \\
\mathbf{B}_d^{(+)} &= \mathbf{A}_{d-1}^{(+)} \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_d\} \cup \{\mathbf{e}_i + \mathbf{e}_j \mid 1 \leq i < j \leq d\} \\
\mathbf{C}_d^{(+)} &= \mathbf{A}_{d-1}^{(+)} \cup \{\mathbf{e}_i + \mathbf{e}_j \mid 1 \leq i \leq j \leq d\} \\
\mathbf{D}_d^{(+)} &= \mathbf{A}_{d-1}^{(+)} \cup \{\mathbf{e}_i + \mathbf{e}_j \mid 1 \leq i < j \leq d\},
\end{aligned}$$

where $\mathbf{e}_i$ is the ith unit coordinate vector of $\mathbf{R}^d$ and $\mathbf{0}$ is the origin of

$\mathbf{R}^d$. For each $\Phi^{(+)} \in \{\mathbf{A}_{d-1}^{(+)}, \mathbf{B}_d^{(+)}, \mathbf{C}_d^{(+)}, \mathbf{D}_d^{(+)}\}$, we identify $\Phi^{(+)}$ with the matrix whose columns are $\Phi^{(+)}$ and associate the configuration

$$\widetilde{\Phi}^{(+)} = \left(\begin{array}{ccc} & \Phi^{(+)} & \\ \hline 1 & \cdots & 1 \end{array} \right).$$

Proposition 1.4 [**2, 9**]. *Working with the same notation as above, the toric ideal $I_{\widetilde{\Phi}^{(+)}}$ has a squarefree quadratic initial ideal, and hence $K[\widetilde{\Phi}^{(+)}]$ is normal.*

By Proposition 1.1, we have the following.

Corollary 1.5. *If $A \in \{\widetilde{\mathbf{A}}_{d-1}^{(+)}, \widetilde{\mathbf{B}}_d^{(+)}, \widetilde{\mathbf{C}}_d^{(+)}, \widetilde{\mathbf{D}}_d^{(+)}\}$, then I_A is generated by quadratic binomials in C_A^{sf}.*

Proof. Since $\widetilde{\mathbf{A}}_{d-1}^{(+)}$, $\widetilde{\mathbf{B}}_d^{(+)}$ and $\widetilde{\mathbf{D}}_d^{(+)}$ are $(0, \pm 1)$ configurations, by Propositions 1.1 and 1.4, I_A is generated by quadratic binomials in C_A^{sf} if $A \in \{\widetilde{\mathbf{A}}_{d-1}^{(+)}, \widetilde{\mathbf{B}}_d^{(+)}, \widetilde{\mathbf{D}}_d^{(+)}\}$.

Let $A = \widetilde{\mathbf{C}}_d^{(+)}$. By elementary row operations, one can transform the matrix A as follows:

$$A \longrightarrow \left(\begin{array}{ccc|ccc} & \mathbf{A}_{d-1}^{(+)} & & & P & \\ \hline 1 & \cdots & 1 & 0 & \cdots & 0 \end{array} \right)$$
$$\longrightarrow \left(\begin{array}{ccc|ccc} & \mathbf{A}_{d-1}^{(+)} + \mathbf{1} & & & P & \\ \hline 1 & \cdots & 1 & 0 & \cdots & 0 \end{array} \right) = Q,$$

where $\mathbf{1}$ is the matrix with all entries equal to one and P is the matrix whose columns are $\{\mathbf{e}_i + \mathbf{e}_j \mid 1 \leq i \leq j \leq d\}$. Since Q is a $(0, 1, 2)$-configuration, $I_Q = I_A$ is generated by quadratic binomials in C_A^{sf} by Propositions 1.1 and 1.4. $\square$

2. Configurations arising from graphs. In this section, we study toric ideals arising from graphs. First, we introduce some graph terminology. A *walk* of G of length q is a sequence $\Gamma = (e_{i_1}, e_{i_2}, \dots, e_{i_q})$ of edges of G, where $e_{i_k} = \{u_k, v_k\}$ for $k = 1, \dots, q$, such that $v_k = u_{k+1}$ for $k = 1, \dots, q-1$. Then,

- A walk Γ is called a *path* if $|\{u_1, \dots, u_q, v_q\}| = q + 1$.
- A walk Γ is called a *closed walk* if $v_q = u_1$.
- A walk Γ is called a *cycle* if $v_q = u_1$, $q \geq 3$ and $|\{u_1, \dots, u_q\}| = q$.

For a cycle $\Gamma = (e_{i_1}, e_{i_2}, \dots, e_{i_q})$, an edge $e = \{s, t\}$ of G is called a *chord* of Γ if s and t are vertices of Γ and if $e \notin \{e_{i_1}, e_{i_2}, \dots, e_{i_q}\}$. A cycle Γ is called *minimal* if Γ has no chord. If $\Gamma = (\{u_1, v_1\}, \{u_2, v_2\}, \dots, \{u_{2q}, v_{2q}\})$ is an even closed walk of G, then it is easy to see that the binomial

$$f_\Gamma = \prod_{\ell=1}^{q} x_{u_{2\ell-1}v_{2\ell-1}} - \prod_{\ell=1}^{q} x_{u_{2\ell}v_{2\ell}}$$

belongs to I_{A_G}. Circuits of I_{A_G} are characterized in terms of graphs (see, e.g., [**13**, Lemma 9.8]).

Proposition 2.1. *Let G be a finite connected graph. Then, $f \in C_{A_G}$ if and only if $f = f_\Gamma$ for some even closed walk Γ which is one of the following even closed walks*:

(i) Γ *is an even cycle of* G;

(ii) $\Gamma = (C_1, C_2)$, *where C_1 and C_2 are odd cycles of G having exactly one common vertex*;

(iii) $\Gamma = (C_1, e_1, \dots, e_r, C_2, e_r, \dots, e_1)$, *where C_1 and C_2 are odd cycles of G having no common vertex and where $(e_1, \dots, e_r)$ is a path of G which combines a vertex of C_1 and a vertex of C_2.*

In particular, $f \notin C_{A_G}^{\mathrm{sf}}$ if and only if Γ satisfies (iii) *and $r > 1$.*

Moreover, it is known [**8**, Lemma 3.2] that

Proposition 2.2. *Let G be a finite connected graph. Then I_{A_G} is generated by all f_Γ where Γ is one of the following even closed walks*:

(i) Γ *is an even cycle of* G;

(ii) $\Gamma = (C_1, C_2)$, *where C_1 and C_2 are odd cycles of G having exactly one common vertex*;

(iii) $\Gamma = (C_1, \Gamma_1, C_2, \Gamma_2)$, *where* C_1 *and* C_2 *are odd cycles of* G *having no common vertex and where* Γ_1 *and* Γ_2 *are walks of* G *both of which combine a vertex* v_1 *of* C_1 *and a vertex* v_2 *of* C_2.

See also [**12**] for a characterization of generators of I_{A_G}. The normality of $K[A_G]$ is characterized in terms of graphs.

Proposition 2.3 [**7**]. *Let* G *be a finite connected graph. Then* $K[A_G]$ *is normal if and only if* G *satisfies the odd cycle condition, i.e., for an arbitrary two odd cycles* C_1 *and* C_2 *in* G *without common vertex, there exists an edge of* G *joining a vertex of* C_1 *with a vertex of* C_2.

Let $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbf{Z}^{d \times n}$ be a configuration. Given binomial $f = u - v \in I_A$, we write T_f for the set of those variables t_i such that t_i divides $\pi(u)(= \pi(v))$. Let $K[T_f] = K[\{t_i \mid t_i \in T_f\}]$, and let A_f be the matrix whose columns are $\{\mathbf{a}_i \mid T^{\mathbf{a}_i} \in K[T_f]\}$. The toric ideal I_{A_f} coincides with $I_A \cap K[\{x_i \mid \pi(x_i) \in K[T_f]\}]$. A binomial $f \in I_A$ is called *fundamental* if I_{A_f} is generated by f. A binomial $f \in I_A$ is called *indispensable* if, for any system of binomial generators F of I_A, either f or $-f$ belongs to F. A binomial $f \in I_A$ is called *not redundant* if f belongs to a minimal system of binomial generators of I_A. Given binomial $f \in I_A$, it is known [**11**] that

- f is fundamental $\Rightarrow$ f is a circuit
- f is fundamental $\Rightarrow$ f is indispensable $\Rightarrow$ f is not redundant

hold in general.

We give a characterization of toric ideals of graphs generated by $C_{A_G}^{\mathrm{sf}}$.

Theorem 2.4. *Let* G *be a finite connected graph. Then the following conditions are equivalent*:

(i) $I_{A_G} = \langle C_{A_G}^{\mathrm{sf}} \rangle$;

(ii) *Any circuit in* $C_{A_G} \setminus C_{A_G}^{\mathrm{sf}}$ *is redundant*;

(iii) *Any circuit in* $C_{A_G} \setminus C_{A_G}^{\mathrm{sf}}$ *is not indispensable*;

(iv) *Any circuit in* $C_{A_G} \setminus C_{A_G}^{\mathrm{sf}}$ *is not fundamental*;

(v) *There exists no induced subgraph of* G *consisting of two odd cycles* C_1, C_2 *having no common vertex and a path of length* ≥ 2,

which connects a vertex of C_1 *and a vertex of* C_2*.*
In particular, if G *satisfies the odd cycle condition, then* G *satisfies* (v)*.*

In order to prove Theorem 2.4, we need the following lemma:

Lemma 2.5. *Let* G *be a finite connected graph which satisfies condition* (v) *in Theorem* 2.4*. Let* C *and* C' *be two odd cycles of* G *having no common vertex, and let* Γ *be a path of* G *which combines a vertex* v *of* C *and a vertex* v' *of* C'*. Then, at least one of the following holds*:

(a) *There exists an edge of* G *joining a vertex of* C *with a vertex of* C'*.*

(b) *There exists an edge of* G *joining a vertex* p *of* C *with a vertex* q *of* Γ *where* $q \neq v$ *and* $\{p, q\} \notin \Gamma$*.*

(c) *There exists an edge of* G *joining a vertex* p *of* C' *with a vertex* q *of* Γ *where* $q \neq v'$ *and* $\{p, q\} \notin \Gamma$*.*

Proof. The proof is by induction on the sum of the length of C and C'.

(Step 1) Suppose that C and C' are cycles of length 3. Then, C and C' are minimal. If C, C' and Γ satisfy none of (a), (b) nor (c), then, by condition (v), it follows that Γ is not an induced subgraph of G. Then there exists a path Γ' which combines v and v', whose vertex set is a proper subset of the vertex set of Γ. By repeating the same argument, we may assume that the path Γ' is an induced subgraph of G. This contradicts condition (v).

(Step 2) Let C and C' be odd cycles of G having no common vertex, and let Γ be a path of G which combines a vertex v of C and a vertex v' of C'. If both C and C' are minimal, then one of (a), (b) or (c) follows from the same argument in Step 1. Suppose that C is not minimal, i.e., there exists a chord e of C. It is easy to see that there exists a unique odd cycle C_e such that $e \in E(C_e) \subset E(C) \cup \{e\}$. Note that the length of C_e is less than the length of C.

If v is a vertex of C_e, then C_e, C' and Γ satisfy one of (a), (b) or (c) by the hypothesis of induction. Thus, C, C' and Γ satisfy the same condition.

Suppose that v is not a vertex of C_e for any chord e of C. Then C_e, C' and a path $\Gamma' = (e_{i_1}, \dots, e_{i_s}, \Gamma)$ where $(e_{i_1}, \dots, e_{i_s})$ is a part of C satisfy one of (a), (b) or (c) by the hypothesis of induction. We may assume that s (≥ 1) is minimal. If C, C' and Γ satisfy none of (a), (b) nor (c), then C_e, C and Γ' satisfy condition (b) where q is not a vertex of Γ. This contradicts the minimality of s. $\square$

Proof of Theorem 2.4. In general, (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) hold. Moreover, by Proposition 2.1, (iv) $\Rightarrow$ (v) holds.

(v) $\Rightarrow$ (i). Suppose that G satisfies condition (v). Let $f = f_\Gamma \notin C_{A_G}^{\mathrm{sf}}$ where Γ is an even closed walk satisfying condition (iii) in Proposition 2.2, i.e., $\Gamma = (C_1, \Gamma_1, C_2, \Gamma_2)$, where C_1 and C_2 are odd cycles of G having no common vertex, and Γ_1 and Γ_2 are walks of G, both of which combine a vertex v_1 of C_1 and a vertex v_2 of C_2. By Propositions 2.1 and 2.2, it is sufficient to show that f is redundant. Since f does not belong to $C_{A_G}^{\mathrm{sf}}$, at least one of Γ_i is of length > 1. We may assume that, except for starting and ending vertices, each Γ_i does not contain the vertices of two odd cycles. (Otherwise, Γ separates into two even closed walks, and hence f is redundant.)

If there exists an edge of G joining a vertex of C_1 with a vertex of C_2, then f is redundant by [**10**, Proof of Lemma 3.2]. Suppose that no such edge exists. (Then, in particular, the length of Γ_i is greater than 1 for $i = 1, 2$.) By Lemma 2.5, there exists an edge of G joining a vertex p of C_1 with a vertex $q\,(\neq v_1)$ of Γ_1 and $\{p, q\}$ does not belong to Γ. Let $C_1 = (V_1, V_2)$ and $\Gamma_1 = (W_1, W_2)$, where

- V_1 and V_2 are paths joining v_1 and p;
- W_i is a walk joining v_i and q for $i = 1, 2$.

Since the length of C_1 is odd, we may assume that the length of the walk (V_1, W_1) is odd. Note that both $\Gamma_3 = (V_1, W_1, \{q, p\})$ and $\Gamma_4 = (V_2, \Gamma_2, C_2, W_2, \{q, p\})$ are even closed walks. It then follows that $f \in \langle f_{\Gamma_3}, f_{\Gamma_4} \rangle$ and $\deg(f_{\Gamma_3}), \deg(f_{\Gamma_4}) < \deg(f)$.

Thus, f is redundant, and hence G satisfies condition (i).

(v) $\Rightarrow$ (ii). Suppose that G satisfies condition (v). Let $f = f_\Gamma \in C_{A_G} \setminus C_{A_G}^{\mathrm{sf}}$, where $\Gamma = (C_1, e_1, \dots, e_r, C_2, e_r, \dots, e_1)$ $(r > 1)$ is an even closed walk satisfying condition (iii) in Proposition 2.1. Then f is redundant by the same argument above. Thus, G satisfies condition (ii). $\square$

A similar theorem holds for $C^{\mathrm{sfsf}}_{A_G}$.

Theorem 2.6. *Let G be a finite connected graph. Then the following conditions are equivalent*:

(i) $I_{A_G} = \langle C^{\mathrm{sfsf}}_{A_G} \rangle$;

(ii) *Any circuit in* $C_{A_G} \setminus C^{\mathrm{sfsf}}_{A_G}$ *is redundant*;

(iii) *Any circuit in* $C_{A_G} \setminus C^{\mathrm{sfsf}}_{A_G}$ *is not indispensable*;

(iv) *Any circuit in* $C_{A_G} \setminus C^{\mathrm{sfsf}}_{A_G}$ *is not fundamental*;

(v) *No induced subgraph of G exists consisting of two odd cycles C_1, C_2 having no common vertex and a path of length ≥ 1 which connects a vertex of C_1 and a vertex of C_2.*

Proof. As stated in the Proof of Theorem 2.4, it is sufficient to show "(v) $\Rightarrow$ (i)" and "(v) $\Rightarrow$ (ii)." Suppose that G satisfies (v). By Theorem 2.4, $I_{A_G} = \langle C^{\mathrm{sf}}_{A_G} \rangle$ and any circuit in $C_{A_G} \setminus C^{\mathrm{sf}}_{A_G}$ is redundant. Thus, in order to prove (i) and (ii), it is sufficient to show that any circuit in $C^{\mathrm{sf}}_{A_G} \setminus C^{\mathrm{sfsf}}_{A_G}$ is redundant. Let f be a binomial in $C^{\mathrm{sf}}_{A_G} \setminus C^{\mathrm{sfsf}}_{A_G}$. By Proposition 2.1, $f = f_\Gamma$ where Γ is an even closed walk which consists of two odd cycles C_1 and C_2 having no common vertex and an edge e_0 of G which combines a vertex v of C_1 and a vertex of C_2. Since G satisfies condition (v), Γ is not an induced subgraph of G. If there exists an edge $e'(\neq e_0)$ of G joining a vertex of C_1 with a vertex of C_2, then f is redundant by [**10**, Proof of Lemma 3.2]. Suppose that no such edge exists. Since Γ is not an induced subgraph of G, we may assume that C_1 is not minimal. Then there exist a chord e of C_1 and an odd cycle C_e such that $e \in E(C_e) \subset E(C_1) \cup \{e\}$. If v is a vertex of C_e, then f is redundant by [**10**, Proof of Lemma 3.2]. Suppose that v is not a vertex of C_e for any chord e of C. Note that C_e, C_2 and $\Gamma' = (e_{i_1}, \dots, e_{i_s}, e_0)$ where $(e_{i_1}, \dots, e_{i_s})$ is a part of C_1 satisfy one of (a), (b) or (c) in Lemma 2.5. Suppose that s is minimal. Since no edge $e'(\neq e_0)$ of G exists joining a vertex of C_1 with a vertex of C_2, C_e, C_2 and Γ' satisfy condition (b). This contradicts the minimality of s. $\square$

Using Theorems 2.4 and 2.6, we give several classes of graphs G such that $I_{A_G} = \langle C^{\mathrm{sf}}_{A_G} \rangle$ and $K[A_G]$ is nonnormal.

Example 2.7. Let G be the graph whose vertex-edge incidence matrix is

$$A_G = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Then, I_{A_G} is generated by the circuits $x_1x_3 - x_2x_4$, $x_3x_4x_6x_9 - x_5^2x_7x_8$ ([**10**, Example 3.5]). Since G does not satisfy the odd cycle condition, $K[A_G]$ is not normal.

Example 2.8. Let G be the graph whose vertex-edge incidence matrix is

$$A_G = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

Then, I_{A_G} is generated by the circuits $x_5x_7 - x_6x_8$, $x_1x_3 - x_2x_4$, $x_3x_4x_{10} - x_5x_8x_9$. Since G does not satisfy the odd cycle condition, $K[A_G]$ is not normal.

Example 2.8 is the most simple nonnormal example whose toric ideal is generated by circuits $u - v$ such that the two monomials u and v are squarefree. In fact,

Proposition 2.9. *If I_A is generated by binomials $f_1 = X^{\mathbf{u}^+} - X^{\mathbf{u}^-}$, $f_2 = X^{\mathbf{v}^+} - X^{\mathbf{v}^-}$ such that $X^{\mathbf{u}^+}$, $X^{\mathbf{u}^-}$, $X^{\mathbf{v}^+}$ and $X^{\mathbf{v}^-}$ are squarefree, then there exists a monomial order such that $\{f_1, f_2\}$ is a Gröbner basis of I_A and hence $K[A]$ is normal.*

Proof. Suppose that $x_i \in \mathrm{supp}\,(X^{\mathbf{u}^+}) \cap \mathrm{supp}\,(X^{\mathbf{v}^-})$ and $x_j \in \mathrm{supp}\,(X^{\mathbf{u}^-}) \cap \mathrm{supp}\,(X^{\mathbf{v}^+})$. Let $\mathbf{w} = \mathbf{u} + \mathbf{v} \in \mathrm{Ker}_{\mathbf{Z}}(A)$ and $g =$

$X^{\mathbf{w}^+} - X^{\mathbf{w}^-}$. Then g belongs to I_A. Since x_i belongs to $\operatorname{supp}(X^{\mathbf{u}^+}) \cap \operatorname{supp}(X^{\mathbf{v}^-})$, $\operatorname{supp}(g)$ does not contain x_i. Similarly, since x_j belongs to $\operatorname{supp}(X^{\mathbf{u}^-}) \cap \operatorname{supp}(X^{\mathbf{v}^+})$, $\operatorname{supp}(g)$ does not contain x_j. Hence, g is not generated by f_1 and f_2. This contradicts that $g \in I_A$. Thus, we may assume that $\operatorname{supp}(X^{\mathbf{u}^+}) \cap \operatorname{supp}(X^{\mathbf{v}^-}) = \varnothing$ and $\operatorname{supp}(X^{\mathbf{u}^+}) \cap \operatorname{supp}(X^{\mathbf{v}^+}) = \varnothing$. Let $<$ be a lexicographic order induced by the ordering

$$\operatorname{supp}(X^{\mathbf{u}^+}) > \operatorname{supp}(X^{\mathbf{v}^+}) > \text{ other variables.}$$

Then, $\operatorname{in}_<(f_1) = X^{\mathbf{u}^+}$ and $\operatorname{in}_<(f_2) = X^{\mathbf{v}^+}$ are relatively prime. Hence, $\{f_1, f_2\}$ is a Gröbner basis of I_A. Since both $\operatorname{in}_<(f_1)$ and $\operatorname{in}_<(f_2)$ are squarefree, $K[A]$ is normal. $\square$

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $V_1 \cap V_2$ is a clique of both graphs. The new graph $G = G_1 \sharp G_2$ with the vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$ is called the *clique sum* of G_1 and G_2 along $V_1 \cap V_2$. If the cardinality of $V_1 \cap V_2$ is $k+1$, this operation is called a *k-sum* of the graphs.

Example 2.10. Let G be the 0-sum of two complete graphs having at least 4 vertices. Then, G satisfies condition (v) in Theorem 2.6, and hence I_{A_G} is generated by $C_{A_G}^{\mathrm{sfsf}}$. Since G does not satisfy the odd cycle condition, $K[A_G]$ is not normal. On the other hand, by the criterion [**3**, Theorem 2.1], it follows that $K[A_G]$ does not satisfy Serre's condition (R_1).

Example 2.11. Let G be the 1-sum of two complete graphs having at least 5 vertices. Then, G satisfies condition (v) in Theorem 2.6, and hence I_{A_G} is generated by $C_{A_G}^{\mathrm{sfsf}}$. Since G does not satisfy the odd cycle condition, $K[A_G]$ is not normal. On the other hand, by the criterion [**3**, Theorem 2.1], it follows that $K[A_G]$ satisfies Serre's condition (R_1).

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