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ON THE NONNEGATIVITY OF NORMAL HILBERT COEFFICIENTS OF TWO IDEALS

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Dedicated to Professor Jürgen Herzog on the occasion of his 70th birthday

ABSTRACT. Let $(R,\mathfrak{m})$ be an analytically unramified local ring of dimension $d \geq 1$, and let I, J be $\mathfrak{m}$-primary ideals. Let $\overline{e}_{(i,j)}(I,J)$ be the coefficient of $(-1)^{d-(i+j)}\binom{x+i-1}{i}\binom{y+j-1}{j}$ of the normal Hilbert polynomial of I and J. In this paper we prove that $\overline{e}_{(i,j)}(I,J)$ are nonnegative for $i+j \geq d-3$ in Cohen-Macaulay local rings. We also prove that, if $i+j = d-1$, then $\overline{e}_{(i,j)}(I,J)$ are nonnegative in unmixed local rings.

1. Introduction. Let R be a commutative ring and I an ideal of R. We say that $x \in R$ is integral over I if x satisfies

$$x^n + a_1x^{n-1} + \cdots + a_n = 0$$

for some $a_i \in I^i, i = 1, 2, \ldots, n$. The set $\overline{I}$ of elements that are integral over I is an ideal, called the *integral closure* of I. A Noetherian local ring $(R,\mathfrak{m})$ is called *analytically unramified* if its $\mathfrak{m}$-adic completion is reduced. For an $\mathfrak{m}$-primary ideal I in an analytically unramified local ring R of dimension d, there exist uniquely determined integers $\overline{e}_0(I), \ldots, \overline{e}_d(I)$ such that, for large n,

$$\lambda(R/\overline{I^{n+1}}) = \overline{e}_0(I)\binom{n+d}{d} - \overline{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d\overline{e}_d(I),$$

where λ denotes length [**10**, Theorem 1.4] and [**11**, Theorem 1.1]. Bhattacharya [**1**, Theorem 8] showed that, for $\mathfrak{m}$-primary ideals I and J in a Noetherian local ring $(R,\mathfrak{m})$ of dimension d, there exist integers $e_{(i,j)}(I,J)$ such that, for large r, s,

$$\lambda(R/I^rJ^s) = \sum_{i+j\leq d}(-1)^{d-(i+j)}e_{(i,j)}(I,J)\binom{r+i-1}{i}\binom{s+j-1}{j}.$$

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Rees [**13**] proved that there exists a polynomial $\overline{P}_{I,J}(x,y) \in \mathbf{Q}[x,y]$ of total degree d such that $\overline{P}_{I,J}(r,s) = \overline{H}_{I,J}(r,s) := \lambda(R/\overline{I^rJ^s})$ for $r,s \gg 0$ in an analytically unramified local ring.

We write

$$\overline{P}_{I,J}(x,y) = \sum_{i+j \le d} (-1)^{d-(i+j)} \overline{e}_{(i,j)}(I,J) \binom{x+i-1}{i} \binom{y+j-1}{j}$$

for some integers $\overline{e}_{(i,j)}(I,J)$. The coefficients $e_{(i,j)}(I,J)$ such that $i+j=d$ are called *mixed multiplicities* of I and J.

In Section 2, we prove that $\overline{e}_{(i,j)}(I,J) = e_{(i,j)}(I,J)$ for $i+j=d$, and hence these coefficients are positive [**5**, Corollary 17.4.7]. Marley proved that $\overline{e}_1(I), \overline{e}_2(I)$ are nonnegative if R is Cohen-Macaulay [**7**, Lemma 3.19 and Proposition 3.23]. Itoh proved nonnegativity of $\overline{e}_3(I)$ in Cohen-Macaulay local rings [**6**, Theorem 3]. D'Cruz and Guerrieri proved that $e_{(i,j)}(I,J)$ are nonnegative for $i+j \ge d-2$ in Cohen-Macaulay local rings [**2**, Theorem 4.2].

In Section 3, we prove nonnegativity of $\overline{e}_{(i,j)}(I,J)$ for $i+j \ge d-3$ in Cohen-Macaulay local rings. Vasconcelos [**14**] conjectured that $\overline{e}_1(I)$ is nonnegative in any analytically unramified local ring of positive dimension. Goto, Hong and Mandal in [**3**] proved nonnegativity of $\overline{e}_1(I)$ in an analytically unramified unmixed local ring of positive dimension d.

In Section 4, we prove that $\overline{e}_{(i,j)}(I,J)$ are nonnegative for $i+j=d-1$ in analytically unramified unmixed local rings.

In the rest of the paper we use the following notation for various Rees algebras and associated graded rings.

$$\text{Rees ring of } I \text{ and } J = \mathcal{R}(I,J) = \bigoplus_{r,s \ge 0} I^rJ^st_1^rt_2^s,$$

$$\text{Extended Rees ring of } I \text{ and } J = \mathcal{R}'(I,J) = \bigoplus_{r,s \in \mathbf{Z}} I^rJ^st_1^rt_2^s,$$

$$\text{Rees ring of } \mathcal{I} = \{\overline{I^rJ^s}\} = \overline{\mathcal{R}}(I,J) = \bigoplus_{r,s \ge 0} \overline{I^rJ^s}t_1^rt_2^s,$$

$$\text{Extended Rees ring of } \mathcal{I} = \{\overline{I^rJ^s}\} = \overline{\mathcal{R}'}(I,J) = \bigoplus_{r,s \in \mathbf{Z}} \overline{I^rJ^s}t_1^rt_2^s.$$

Associated graded rings of

$$I, J \text{ with respect to } I \text{ as } \mathcal{R}(I, J \mid I) = \bigoplus_{r,s\geq 0} \frac{I^r J^s}{I^{r+1} J^s},$$

$$I, J \text{ with respect to } J \text{ as } \mathcal{R}(I, J \mid J) = \bigoplus_{r,s\geq 0} \frac{I^r J^s}{I^r J^{s+1}},$$

$$\mathcal{I} = \{\overline{I^r J^s}\} \text{ with respect to } I \text{ as } \overline{\mathcal{R}}(I, J \mid I) = \bigoplus_{r,s\geq 0} \frac{\overline{I^r J^s}}{\overline{I^{r+1} J^s}},$$

$$\mathcal{I} = \{\overline{I^r J^s}\} \text{ with respect to } J \text{ as } \overline{\mathcal{R}}(I, J \mid J) = \bigoplus_{r,s\geq 0} \frac{\overline{I^r J^s}}{\overline{I^r J^{s+1}}}.$$

2. Preliminary results. Rees [**13**] sketched a proof for the existence of integers $\overline{e}_{(i,j)}(I, J)$ such that for $r, s \gg 0$,

$$\lambda(R/\overline{I^r J^s}) = \sum_{i+j\leq d} (-1)^{d-(i+j)} \overline{e}_{(i,j)}(I, J) \binom{r+i-1}{i} \binom{s+j-1}{j}.$$

For the sake of completeness we give a complete proof in this section. We prove that $\overline{e}_{(i,j)}(I, J) = e_{(i,j)}(I, J)$ for $i + j = d$, and hence these are positive. If $(R, \mathfrak{m})$ is a complete normal domain or complete reduced Cohen-Macaulay local ring of dimension $d \geq 2$, we prove that quotients of a general extension of R by general elements are analytically unramified.

For a numerical function $f : \mathbf{Z}^2 \to \mathbf{Z}$ let $\Delta_{\mathbf{e_1}} f, \Delta_{\mathbf{e_2}} f : \mathbf{Z}^2 \to \mathbf{Z}$ be defined as

$$\Delta_{\mathbf{e_1}} f(r, s) := f(r, s) - f(r-1, s) \text{ and } \Delta_{\mathbf{e_2}} f(r, s) := f(r, s) - f(r, s-1).$$

Lemma 2.1. *Let $f : \mathbf{N}^2 \to \mathbf{N}$ be a function such that there exist polynomials $p(x, y), q(x, y) \in \mathbf{Q}[x, y]$ of total degree d such that $\Delta_{\mathbf{e_1}} f(r, s) = p(r, s)$ and $\Delta_{\mathbf{e_2}} f(r, s) = q(r, s)$ for $r, s \gg 0$. Then there exists a polynomial $F(x, y) \in \mathbf{Q}[x, y]$ of total degree $d + 1$ such that $f(r, s) = F(r, s)$ for $r, s \gg 0$.*

Proof. We write

$$p(x,y)=\sum_{i+j\le d} p_{(i,j)}\binom{x+i-1}{i}\binom{y+j-1}{j},$$
$$q(x,y)=\sum_{i+j\le d} q_{(i,j)}\binom{x+i-1}{i}\binom{y+j-1}{j}.$$

Let $\Delta_{\mathbf{e_1}}f(r,s)=p(r,s)$ and $\Delta_{\mathbf{e_2}}f(r,s)=q(r,s)$ for $r\ge r_0, s\ge s_0$. Define

$$P(x,y)=\sum_{i+j\le d} p_{(i,j)}\binom{x+i}{i+1}\binom{y+j-1}{j}.$$

Since $P(x,y)-P(x-1,y)=p(x,y)$, for $r\ge r_0, s\ge s_0$, we have

$$f(r,s)=f(r_0-1,s)+\sum_{k=r_0}^{r} p(k,s)=f(r_0-1,s)-P(r_0-1,s)+P(r,s).$$

Thus, $f(r_0,s)=f(r_0-1,s)-P(r_0-1,s)+P(r_0,s)$. Since $\Delta_{\mathbf{e_2}}f=q(r,s)$ for $r\ge r_0$, $s\ge s_0$, there exists a polynomial $Q(x)\in\mathbf{Q}[x]$ such that, for $s\ge s_0$,

$$\Delta_{\mathbf{e_2}}f(r_0-1,s):=f(r_0-1,s)-f(r_0-1,s-1)=Q(s).$$

Thus,

$$f(r_0-1,s)=f(r_0-1,s_0-1)+\sum_{k=s_0}^{s}Q(k).$$

Hence, for $r\ge r_0$ and $s\ge s_0$,

$$f(r,s)=f(r_0-1,s_0-1)-P(r_0-1,s)+\sum_{k=s_0}^{s}Q(k)+P(r,s). \qquad \square$$

A proof of the next theorem is sketched in [**13**]. We provide a detailed proof.

Theorem 2.2. *Let $(R,\mathfrak{m})$ be an analytically unramified local ring of dimension d. Let I and J be $\mathfrak{m}$-primary ideals. Then there exists a* *polynomial* $\overline{P}_{I,J}(x,y) \in \mathbf{Q}[x,y]$ *of total degree* d *such that* $\lambda(R/\overline{I^rJ^s}) = \overline{P}_{I,J}(r,s)$ *for* $r,s \gg 0$.

Proof. Since there exists an $h \geq 0$ such that $\overline{I^rJ^s} \subseteq I^{r-h}J^{s-h}$ for all r,s, by [**10**, Theorem 1.4], $\overline{\mathcal{R}'}(I,J) \subseteq \mathcal{R}'(I,J)t_1^ht_2^h$. Thus, $\overline{\mathcal{R}'}(I,J)$ is a finite graded $\mathcal{R}'(I,J)$-module. Let $\overline{\mathcal{R}'}(I,J)$ be generated by elements of the form $b_{(r,s)}t_1^rt_2^s$, where $b_{(r,s)} \in \overline{I^rJ^s}$ and $r,s \leq N$. Then $\overline{\mathcal{R}}(I,J)$ is generated by homogeneous elements of degree (r,s), where $r,s \leq N$, and hence $\overline{\mathcal{R}}(I,J)$ is a finite $\mathcal{R}(I,J)$-module. Let $\mathfrak{a}$ be the ideal of $\overline{\mathcal{R}}(I,J)$ consisting of all polynomials $\sum_{r,s} c_{(r,s)}t_1^rt_2^s$ with $c_{(r,s)} \in \overline{I^{r+1}J^s}$, and let $\mathfrak{b}$ be the ideal consisting of all polynomials $\sum_{r,s} c_{(r,s)}t_1^rt_2^s$ with $c_{(r,s)} \in \overline{I^rJ^{s+1}}$. Then $\overline{\mathcal{R}}(I,J)/\mathfrak{a}$ is a finite $\mathcal{R}(I,J \mid I)$-module and $\overline{\mathcal{R}}(I,J)/\mathfrak{b}$ is a finite $\mathcal{R}(I,J \mid J)$-module. Now, by [**4**, Theorem 4.1], there exist polynomials $p(x,y), q(x,y) \in \mathbf{Q}[x,y]$ of total degree $d-1$ such that $\lambda(\overline{I^rJ^s}/\overline{I^{r+1}J^s}) = p(r,s)$ and $\lambda(\overline{I^rJ^s}/\overline{I^rJ^{s+1}}) = q(r,s)$ for $r,s \gg 0$. Since

$$\Delta_{\mathbf{e_1}}\overline{H}_{I,J}(r,s) = \lambda\left(\frac{\overline{I^{r-1}J^s}}{\overline{I^rJ^s}}\right) \quad \text{and} \quad \Delta_{\mathbf{e_2}}\overline{H}_{I,J}(r,s) = \lambda\left(\frac{\overline{I^rJ^{s-1}}}{\overline{I^rJ^s}}\right),$$

by Lemma 2.1, the result follows. $\square$

In order to prove $\overline{e}_{(i,j)}(I,J) = e_{(i,j)}(I,J)$ for $i,j \in \{0,1,\dots,d\}$ so that $i+j=d$, we need the following lemma.

Lemma 2.3. *Let*

$$p(x,y) = \sum_{i+j\leq d} (-1)^{d-(i+j)} p_{(i,j)} \binom{x+i-1}{i}\binom{y+j-1}{j}$$

and

$$q(x,y) = \sum_{i+j\leq d} (-1)^{d-(i+j)} q_{(i,j)} \binom{x+i-1}{i}\binom{y+j-1}{j}$$

be polynomials of total degree d *in* $\mathbf{Q}[x,y]$. *If there exist integers* g,h *such that*

$$p(r-g,s-g) \leq q(r,s) \leq p(r-h,s-h) \quad \textit{for } r,s \gg 0, \tag{2.4}$$

then $p_{(i,j)} = q_{(i,j)}$ *for all* $i, j \in \{0, 1, \ldots, d\}$ *such that* $i + j = d$.

Proof. By equation (2.4), we get

$$\begin{aligned}\lim_{r\to\infty} \frac{1}{r^d} p(r-g, s-g) &\leq \lim_{r\to\infty} \frac{1}{r^d} q(r,s) \\ &\leq \lim_{r\to\infty} \frac{1}{r^d} p(r-h, s-h).\end{aligned}$$

Hence, $p_{(d,0)} = q_{(d,0)}$. By symmetry, $p_{(0,d)} = q_{(0,d)}$. Therefore, for $r, s \gg 0$, we get

$$\begin{aligned}p(r-g, s-g) - p_{(d,0)} r^d &\leq q(r,s) - q_{(d,0)} r^d \\ &\leq p(r-h, s-h) - p_{(d,0)} r^d.\end{aligned}$$

By taking $r = n^d$ and $s = n^{d-1}$, we get

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n^{d^2-1}} [p(n^d - g, n^{d-1} - g) - p_{(d,0)} n^{d^2}] \\ &\qquad \leq \lim_{n\to\infty} \frac{1}{n^{d^2-1}} [q(n^d, n^{d-1}) - q_{(d,0)} n^{d^2}] \\ &\qquad \leq \lim_{n\to\infty} \frac{1}{n^{d^2-1}} [p(n^d - h, n^{d-1} - h) - p_{(d,0)} n^{d^2}].\end{aligned}$$

Hence, $p_{(d-1,1)} = q_{(d-1,1)}$. Continuing, we get $p_{(i,j)} = q_{(i,j)}$ for all $i, j \in \{0, 1, \ldots, d\}$ such that $i + j = d$. □

Theorem 2.5. *Let* $(R, \mathfrak{m})$ *be an analytically unramified local ring of dimension* d. *Let* I *and* J *be* $\mathfrak{m}$*-primary ideals in* R. *Then, for* $i, j \in \{0, 1, \ldots, d\}$ *such that* $i + j = d$,

$$\overline{e}_{(i,j)}(I, J) = e_{(i,j)}(I, J).$$

Proof. There exists an $n \geq 0$ such that $I^r J^s \subseteq \overline{I^r J^s} \subseteq I^{r-n} J^{s-n}$ for $r, s \geq n$, [**10**, Theorem 1.4]. Therefore,

$$\lambda(R/I^{r-n} J^{s-n}) \leq \lambda(R/\overline{I^r J^s}) \leq \lambda(R/I^r J^s).$$

Thus, by Lemma 2.3, $\overline{e}_{(i,j)}(I, J) = e_{(i,j)}(I, J)$ for $i + j = d$. □

Lemma 2.6. *Let $(R, \mathfrak{m})$ be an analytically unramified local ring of dimension d. Then*

$$\overline{e}_{(0,0)}(I, J) = \overline{e}_d(IJ).$$

Proof. We have

$$\lambda(R/\overline{I^r J^s}) = \sum_{i+j \leq d} (-1)^{d-(i+j)} \overline{e}_{(i,j)}(I, J) \binom{r+i-1}{i} \binom{s+j-1}{j}$$

for $r, s \gg 0$. Taking $r = s = n \gg 0$,

$$\lambda(R/\overline{(IJ)^n}) = \sum_{i+j \leq d} (-1)^{d-(i+j)} \overline{e}_{(i,j)}(I, J) \binom{n+i-1}{i} \binom{n+j-1}{j}.$$

Therefore, $\overline{e}_d(IJ) = \overline{e}_{(0,0)}(I, J)$. □

We shall prove a few preliminary results about the behavior of normal Hilbert functions under general extension of local rings. These will be used to apply induction on the dimension of R. To study $e_{(i,j)}(I, J)$, one can use superficial elements and use induction on d. But, in general, $\overline{I} + (x)/(x) \neq \overline{I + (x)/(x)}$ in $R/(x)$. To overcome this, one uses general elements. Let $I = (a_1, \ldots, a_p)$ and $J = (b_1, \ldots, b_q)$. Let $T = R[\mathbf{X}, \mathbf{Y}]$, where $\mathbf{X}$ denotes the sequence of indeterminates $\{X_1, \ldots, X_p\}$ and $\mathbf{Y}$ denotes the sequence of indeterminates $\{Y_1, \ldots, Y_q\}$. Let $\mathfrak{q} = \mathfrak{m}T$, $R' = T_{\mathfrak{q}}$. First we prove that the normal Hilbert function of I, J in R and the normal Hilbert function of IR', JR' in R' are equal.

Proposition 2.7. *Under the notation as above,* $\dim R' = \dim R$ *and, for all integers $r, s \geq 0$,*

$$\overline{H}_{I,J}(r, s) = \overline{H}_{IR',JR'}(r, s).$$

Proof. Let $\mathfrak{a} = (x_1, \ldots, x_d)$ be a system of parameters in R. As R' is a flat extension of R and $\mathfrak{m}R'$ is the maximal ideal of R', we have

$$\lambda_{R'}(R'/\mathfrak{a}R') = \lambda_R(R/\mathfrak{a}) < \infty.$$

Hence, $\dim R' \leq d$. Let $\wp_0 \subsetneq \wp_1 \subsetneq \cdots \subsetneq \wp_d = \mathfrak{m}$ be a maximal chain of prime ideals in R. Then

$$\wp_0 R' \subsetneq \wp_1 R' \subsetneq \cdots \subsetneq \wp_d R' = \mathfrak{m} R'.$$

Hence, $\dim R' \geq d$. Thus, $\dim R = \dim R'$. To prove the second assertion first we show that, for any ideal $\mathfrak{a}$ in R, $\overline{\mathfrak{a}T} = \overline{\mathfrak{a}}T$. It is enough to prove it for $T = R[z]$. Clearly, $\overline{\mathfrak{a}}T \subseteq \overline{\mathfrak{a}T}$. Let $\alpha \in \overline{\mathfrak{a}T}$. Since $\mathfrak{a}T$ is a homogeneous ideal in T, $\overline{\mathfrak{a}T}$ is a homogeneous ideal in T by [**7**, Lemma A.1.5]. Let $\alpha = az^n$, where $a \in R$. Then α satisfies an equation

$$\alpha^m + r_1\alpha^{m-1} + \cdots + r_m = 0,$$

where $r_i \in \mathfrak{a}^i T$ for $i = 1, \ldots, m$. Comparing the coefficient of z^{nm}, we get an equation of the form

$$a^m + s_1 a^{m-1} + \cdots + s_m = 0,$$

where $s_i \in \mathfrak{a}^i$. Therefore, $a \in \overline{\mathfrak{a}}$ and $\alpha \in \overline{\mathfrak{a}}T$. Hence, $\overline{\mathfrak{a}T} = \overline{\mathfrak{a}}T$.

Therefore, $\overline{I^r J^s R'} = \overline{I^r J^s} R'$ for all integers $r, s \geq 0$. Hence,

$$\lambda_R(R/\overline{I^r J^s}) = \lambda_{R'}(R'/\overline{I^r J^s R'})$$

for all r, s. Thus, $\overline{H}_{I,J}(r,s) = \overline{H}_{IR',JR'}(r,s)$ for all integers $r, s \geq 0$. $\square$

Let $x = a_1X_1 + \cdots + a_pX_p$ and $y = b_1Y_1 + \cdots + b_qY_q$. Let $D' = R'/xR'$ and $E' = R'/yR'$. Next, we prove that, if $(R, \mathfrak{m})$ is a complete analytically unramified Cohen-Macaulay local ring of dimension $d \geq 2$, then D', E' are analytically unramified. We also prove D', E' are analytically unramified if $(R, \mathfrak{m})$ is a complete normal local domain of dimension $d \geq 2$. In order to prove this we recall some definitions and results about Nagata rings. Let R be an integral domain with quotient field K. We say that R is N-2 if, for any finite field extension L of K, the integral closure of R in L is a finite R-module. We say a ring R is a *Nagata* ring if it is Noetherian and R/P is N-2 for every $P \in \operatorname{Spec} R$.

Proposition 2.8. (1) [**8**, Theorem 69]. *A Noetherian complete local ring is a Nagata ring.*

(2) [**8**, Theorem 72]. *An affine algebra over a Nagata ring is Nagata.*

(3) [**8**, Theorem 70]. *A Noetherian semi-local Nagata domain is analytically unramified.*

(4) [**5**, Proposition 9.1.3]. *Let* $(R, \mathfrak{m})$ *be a reduced local ring. If* R/P *is analytically unramified for each minimal prime* P *of* R*, then* R *is analytically unramified.*

Proposition 2.9. *Let* $(R, \mathfrak{m})$ *be a local ring satisfying any one of the following conditions*:

(1) $(R, \mathfrak{m})$ *is a complete Cohen-Macaulay analytically unramified local ring of dimension* $d \geq 2$*,*

(2) $(R, \mathfrak{m})$ *is a complete normal local domain of dimension* $d \geq 2$*.*

Then D' *and* E' *are analytically unramified.*

Proof. Since R is complete, by Proposition 2.8 (1), R is a Nagata ring. Since localization of a Nagata ring is Nagata, by Proposition 2.8 (2), D' is a Nagata ring. Hence D'/P is a Nagata domain for every minimal prime P of D'. In order to prove D' is analytically unramified, by Proposition 2.8 (3), (4), it is enough to prove D' is reduced. We show $D = T/xT$ is reduced.

(1) First we prove that x is a nonzerodivisor in R'. It is enough to prove that x is a nonzerodivisor in T. Let $\deg X_i = (1, 0)$ and $\deg Y_i = (0, 1)$. Then T is an $\mathbf{N}^2$-graded ring. Since x is a bihomogeneous element of T, $\operatorname{Ann} x$ is a bihomogeneous ideal. Suppose $b\mathbf{X}^\alpha \mathbf{Y}^\beta x = 0$, where $\alpha = (\alpha_1, \dots, \alpha_p)$, $\beta = (\beta_1, \dots, \beta_q)$ and $\mathbf{X}^\alpha = X_1^{\alpha_1} \cdots X_p^{\alpha_p}$, $\mathbf{Y}^\beta = Y_1^{\beta_1} \cdots Y_q^{\beta_q}$. Then $ba_i = 0$ for all i. Hence, $bI = 0$. Since R is analytically unramified and I is $\mathfrak{m}$-primary, $\operatorname{grade}(I, R) = \operatorname{depth} R$. As R is reduced and $\dim R \geq 2$, $\operatorname{depth} R \geq 1$. Hence, $b = 0$.

Since T is Cohen-Macaulay, (x) is an unmixed ideal. Hence, $\operatorname{height} P = 1$ for all $P \in \operatorname{Ass}_T D$. Let $\mathcal{P} = P \cap R$. Since T is a flat extension of R, by the going-down theorem, $\operatorname{height} \mathcal{P} \leq 1$. Hence, $I \not\subseteq \mathcal{P}$. Without loss of generality, we may assume that $a_p \notin \mathcal{P}$. Then

$$\begin{aligned} \frac{T_{\mathcal{P}}}{xT_{\mathcal{P}}} &\cong \frac{R_{\mathcal{P}}[X_1, \dots, X_p, \mathbf{Y}]}{(\sum_{i=1}^{p-1} (a_i X_i / a_p) + X_p)} \\ &\cong R_{\mathcal{P}}[X_1, \dots, X_{p-1}, \mathbf{Y}]. \end{aligned}$$

Since R is analytically unramified and complete, R is reduced. Thus, $D_{\mathcal{P}}$ is reduced. Hence, D_P is reduced.

Let $(0) = q_1 \cap \cdots \cap q_s$ be a minimal primary decomposition of (0) in D and $P_i = \sqrt{q_i}$. Since D_{P_i} is reduced, $\operatorname{Ass} D_{P_i} = \{P_i D_{P_i}\}$ for every $1 \le i \le s$. Hence, $q_i = P_i$ as $q_i D_{P_i} = P_i D_{P_i}$. Therefore,

$$(0) = P_1 \cap \cdots \cap P_s,$$

and hence D is reduced. A similar argument shows that E' is analytically unramified.

(2) In order to prove D is reduced we show that $D_P = T_P/xT_P$ is an integral domain for all $P \in \operatorname{Ass}_T D$. Let $\mathcal{P} = P \cap R$. Since T is a normal domain, xT is an unmixed ideal. Thus, height $P = 1$. Hence, height $\mathcal{P} \le 1$. Therefore, $I \nsubseteq \mathcal{P}$. Without loss of generality, we may assume that $a_p \notin \mathcal{P}$. Then

$$\begin{aligned}\frac{T_{\mathcal{P}}}{xT_{\mathcal{P}}} &\cong \frac{R_{\mathcal{P}}[X_1, \ldots, X_p, \mathbf{Y}]}{(\sum_{i=1}^{p-1}(a_i X_i/a_p) + X_p)} \\ &\cong R_{\mathcal{P}}[X_1, \ldots, X_{p-1}, \mathbf{Y}].\end{aligned}$$

Since R is a domain, $D_{\mathcal{P}}$ is a domain. Thus, D_P is a domain. A similar argument shows that E' is analytically unramified. □

3. The nonnegativity of $\overline{e}_{(i,j)}(I, J)$ for $i + j \ge d - 3$ in Cohen-Macaulay local rings. Throughout this section, let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension d. Marley [**7**, Lemma 3.19, Proposition 3.23] proved that $\overline{e}_1(I)$ and $\overline{e}_2(I) \ge 0$. Itoh [**6**, Theorem 3 (1)] proved that $\overline{e}_3(I) \ge 0$. In this section, we prove that $\overline{e}_{(i,j)}(I, J) \ge 0$ for $i + j \ge d - 3$. Itoh [**6**] used general elements in order to prove nonnegativity of normal Hilbert coefficients. More precisely,

Theorem 3.1 [**6**, Theorem 1]. *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d \ge 2$, and let I be a parameter ideal in R. Let $T = (T_1, \ldots, T_d)$ be d indeterminates and $R(T) = R[T]_{\mathfrak{m}[T]}$. There exist generators $x_1, \ldots, x_d$ of I such that, if we put $C = R(T)/(\sum_i x_i T_i)$ and $J = IC$, then $\overline{J^n} = \overline{I^n}C$ for large n.*

We prove an analogous result in the bigraded case using techniques in [**3, 6**].

Theorem 3.2. *Let* $(R, \mathfrak{m})$ *be either a complete Cohen-Macaulay analytically unramified local ring or a normal complete local domain of dimension* $d \geq 2$*. Let* I *and* J *be* $\mathfrak{m}-$*primary ideals of* R*. Then* $\overline{I^rJ^sD'} = \overline{I^rJ^s}D'$ *and* $\overline{I^rJ^sE'} = \overline{I^rJ^s}E'$ *for* $r, s \gg 0$.

Proof. Let $D = T/xT$, $\mathcal{A} = \mathcal{R}(IT, JT)$, $\overline{\mathcal{A}} = \overline{\mathcal{R}}(IT, JT)$, $\mathcal{T} = \mathcal{R}(ID, JD)$ and $\overline{\mathcal{T}} = \overline{\mathcal{R}}(ID, JD)$. The natural map $T[t_1, t_2] \to D[t_1, t_2]$ induces an $\mathcal{A}$-linear map $\phi : \overline{\mathcal{A}}/xt_1\overline{\mathcal{A}} \to \overline{\mathcal{T}}$. Put $N = (IJTt_1t_2)\mathcal{A}$. We prove that, for all $P \in \operatorname{Spec} \mathcal{A} \setminus V(N)$, ϕ_P is an isomorphism. Let $P \in \operatorname{Spec} \mathcal{A} \setminus V(N)$. Then $ITt_1 \nsubseteq P$. Without loss of generality, we may assume that $a_pt_1 \notin P$. We have the isomorphisms:

$$\begin{aligned}[\overline{\mathcal{A}}/xt_1\overline{\mathcal{A}}]_{a_pt_1} &\cong \frac{\overline{\mathcal{R}}(I, J)_{a_pt_1}[\mathbf{X}, \mathbf{Y}]}{((1/a_pt_1)(a_1X_1t_1 + \cdots + a_pX_pt_1))} \\ &\cong \overline{\mathcal{R}}(I, J)_{a_pt_1}[X_1, \dots, X_{p-1}, \mathbf{Y}],\end{aligned}$$

$$\begin{aligned}D[t_1, t_2]_{a_pt_1} &\cong \frac{T[t_1, t_2]_{a_pt_1}}{xT[t_1, t_2]_{a_pt_1}} \cong \frac{R[t_1, t_2]_{a_pt_1}[\mathbf{X}, \mathbf{Y}]}{(\sum_{i=1}^{p-1}(a_iX_i/a_p) + X_p)} \\ &\cong R[t_1, t_2]_{a_pt_1}[X_1, \dots, X_{p-1}, \mathbf{Y}].\end{aligned}$$

The above isomorphisms give us the commutative diagram:

$$\begin{array}{ccccc} [\overline{\mathcal{A}}/xt_1\overline{\mathcal{A}}]_{a_pt_1} & \xrightarrow{\phi_{a_pt_1}} & \overline{\mathcal{T}}_{a_pt_1} & \longrightarrow & D[t_1, t_2]_{a_pt_1} \\ \downarrow \cong & & & & \downarrow \cong \\ \overline{\mathcal{R}}(I, J)_{a_pt_1}[X_1, \dots, X_{p-1}, \mathbf{Y}] & & \longrightarrow & & R[t_1, t_2]_{a_pt_1}[X_1, \dots, X_{p-1}, \mathbf{Y}]. \end{array}$$

Hence, $\phi_{a_pt_1}$ is injective. As $\overline{\mathcal{R}}(I, J)$ is integrally closed in $R[t_1, t_2]$, $\overline{\mathcal{R}}(I, J)_{a_pt_1}[X_1, \dots, X_{p-1}, \mathbf{Y}]$ is integrally closed in $R[t_1, t_2]_{a_pt_1}[X_1, \dots, X_{p-1}, \mathbf{Y}]$. Hence, $[\overline{\mathcal{A}}/xt_1\overline{\mathcal{A}}]_{a_pt_1}$ is integrally closed in $D[t_1, t_2]_{a_pt_1}$. But $\mathcal{T}_{a_pt_1} \subseteq \operatorname{image} \phi_{a_pt_1} \subseteq \overline{\mathcal{T}}_{a_pt_1}$ implies that $[\overline{\mathcal{A}}/xt_1\overline{\mathcal{A}}]_{a_pt_1} \cong \operatorname{image} \phi_{a_pt_1} \subseteq \overline{\mathcal{T}}_{a_pt_1}$ is an integral extension. Thus, $\operatorname{image} \phi_{a_pt_1} = \overline{\mathcal{T}}_{a_pt_1}$. Therefore, ϕ_P is an isomorphism.

Therefore, $\operatorname{supp}(\operatorname{coker}\phi_{\mathfrak{q}}) \subseteq V(N\mathcal{A}_{\mathfrak{q}})$. Since D' is analytically unramified, $\overline{\mathcal{T}}_{\mathfrak{q}} = \overline{\mathcal{R}}(ID', JD')$ is a finite $\mathcal{T}_{\mathfrak{q}} = \mathcal{R}(ID', JD')$-module. Thus, $\overline{\mathcal{T}}_{\mathfrak{q}}$ is a finite $[\overline{\mathcal{A}}/xt_1\overline{\mathcal{A}}]_{\mathfrak{q}}$-module. Since $[\overline{\mathcal{A}}/xt_1\overline{\mathcal{A}}]_{\mathfrak{q}}$ is a finite $\mathcal{A}_{\mathfrak{q}}$-module, $\overline{\mathcal{T}}_{\mathfrak{q}}$ is a finite $\mathcal{A}_{\mathfrak{q}}$-module, and hence $\operatorname{coker}\phi_{\mathfrak{q}}$ is a finite $\mathcal{A}_{\mathfrak{q}}$-module. Thus, $(N\mathcal{A}_{\mathfrak{q}})^k(\operatorname{coker}\phi_{\mathfrak{q}}) = 0$ for some k. Let $\operatorname{coker}\phi_{\mathfrak{q}}$ be generated by $u_1, \ldots, u_g$, where $\deg u_i = (r_i, s_i)$ and $r_i, s_i \leq n_0$, say. Then $[\operatorname{coker}\phi_{\mathfrak{q}}]_{(r,s)} = 0$ for $r, s \geq n_0 + k$. But

$$[\operatorname{coker}\phi_{\mathfrak{q}}]_{(r,s)} = \frac{\overline{I^rJ^sD'}}{\overline{I^rJ^sR'} + (x)/(x)} = \frac{\overline{I^rJ^sD'}}{\overline{I^rJ^s}D'}.$$

Therefore, for large r, s, we have $\overline{I^rJ^sD'} = \overline{I^rJ^s}D'$. A similar argument gives $\overline{I^rJ^sE'} = \overline{I^rJ^s}E'$ for large r, s. $\square$

Now we relate $\overline{P}_{IR',JR'}$ and $\overline{P}_{ID',JD'}$.

Rees [**12**, Lemma 2.4] proved that the general elements x and y satisfy:

$$(xR') \cap I^rJ^sR' = xI^{r-1}J^sR' \text{ for all } r \gg 0 \text{ and } s \geq 0,$$
$$(yR') \cap I^rJ^sR' = yI^rJ^{s-1}R' \text{ for all } r \geq 0 \text{ and } s \gg 0.$$

The same proof shows the following.

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then*

$$(xR') \cap \overline{I^rJ^sR'} = x\overline{I^{r-1}J^sR'} \text{ for all } r \gg 0 \text{ and } s \geq 0,$$
$$(yR') \cap \overline{I^rJ^sR'} = y\overline{I^rJ^{s-1}R'} \text{ for all } r \geq 0 \text{ and } s \gg 0.$$

Proposition 3.4. *Let $(R, \mathfrak{m})$ be an analytically unramified local ring. Then, for all integers r, s,*

$$\overline{P}_{ID',JD'}(r,s) = \Delta_{\mathbf{e_1}}\overline{P}_{IR',JR'}(r,s)$$

and

$$\overline{P}_{IE',JE'}(r,s) = \Delta_{\mathbf{e_2}}\overline{P}_{IR',JR'}(r,s).$$

Proof. Since x is a nonzerodivisor for large r, s, we have

$$\begin{aligned}\lambda(D'/\overline{I^rJ^sD'}) &= \lambda(D'/\overline{I^rJ^s}D') \\ &= \lambda(R'/\overline{I^rJ^s}R' + xR') \\ &= \lambda(R'/\overline{I^rJ^s}R') - \lambda(\overline{I^rJ^s}R' + xR'/\overline{I^rJ^s}R') \\ &= \lambda(R'/\overline{I^rJ^s}R') - \lambda(xR'/xR' \cap \overline{I^rJ^s}R') \\ &= \lambda(R'/\overline{I^rJ^s}R') - \lambda(xR'/x\overline{I^{r-1}J^s}R') \\ &= \lambda(R'/\overline{I^rJ^s}R') - \lambda(R'/\overline{I^{r-1}J^s}R').\end{aligned}$$

Therefore, for all integers r, s, $\overline{P}_{ID',JD'}(r, s) = \Delta_{\mathbf{e_1}}\overline{P}_{IR',JR'}(r, s)$. By a similar argument, one can show $\overline{P}_{IE',JE'}(r, s) = \Delta_{\mathbf{e_2}}\overline{P}_{IR',JR'}(r, s)$. ◻

Theorem 3.5. *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d \geq 1$. Let I and J be $\mathfrak{m}$-primary ideals. Then*

(1) $\overline{e}_{(i,j)}(I, J) \geq 0$ *for* $i + j = d - 1$ *if* $d \geq 1$.

(2) $\overline{e}_{(i,j)}(I, J) \geq 0$ *for* $i + j = d - 2$ *if* $d \geq 2$.

(3) $\overline{e}_{(i,j)}(I, J) \geq 0$ *for* $i + j = d - 3$ *if* $d \geq 3$.

Proof. (1) Let $\widehat{R}$ be the $\mathfrak{m}-$adic completion of R. Since $\overline{I}\widehat{R} = \overline{I\widehat{R}}$ for every $\mathfrak{m}$-primary ideal in R, $\overline{e}_{(i,j)}(I, J) = \overline{e}_{(i,j)}(I\widehat{R}, J\widehat{R})$. Hence, we may assume that R is complete.

Induct on d. For $d = 1$, $\overline{e}_{(0,0)} = \overline{e}_1(IJ) \geq 0$ by [**7**, Lemma 3.19]. Let $d \geq 2$. Let $I = (a_1, \dots, a_p)$ and $J = (b_1, \dots, b_q)$. Let $T = R[\mathbf{X}, \mathbf{Y}]$, where $\mathbf{X}$ denotes the sequence $\{X_1, \dots, X_p\}$ and $\mathbf{Y}$ denotes the sequence $\{Y_1, \dots, Y_q\}$. Let $\mathfrak{q} = \mathfrak{m}T$, $R' = T_{\mathfrak{q}}$. Let $x = a_1X_1 + \cdots + a_pX_p$ and $y = b_1Y_1 + \cdots + b_qY_q$. Let $D' = R'/xR'$ and $E' = R'/yR'$. Then by Propositions 2.9 and 3.4, D' is a Cohen-Macaulay analytically unramified local ring and

$$\overline{e}_{(i,j)}(I, J) = \overline{e}_{(i,j)}(IR', JR') = \overline{e}_{(i-1,j)}(ID', JD').$$

Therefore, by induction, $\overline{e}_{(i,j)}(I, J) \geq 0$ for $i \neq 0$. Similarly, for $j \neq 0$,

$$\overline{e}_{(i,j)}(I, J) = \overline{e}_{(i,j-1)}(IE', JE') \geq 0.$$

Thus, $\overline{e}_{(i,j)}(I, J) \geq 0$ for $i + j = d - 1$.

(2) Let $d = 2$. Then $\overline{e}_{(0,0)}(I, J) = \overline{e}_2(IJ) \geq 0$ by [**7**, Proposition 3.23]. Let $d > 2$. Then a similar argument as above shows that $\overline{e}_{(i,j)}(I, J) \geq 0$ for $i + j = d - 2$.

(3) Let $d = 3$. Then $\overline{e}_{(0,0)}(I, J) = \overline{e}_3(IJ) \geq 0$ by [**6**, Theorem 3 (1)]. Hence, the result follows by induction as above. ◻

4. The nonnegativity of $\overline{e}_{(i,j)}(I, J)$ for $i + j = d - 1$ in unmixed local rings. In this section, we prove that $\overline{e}_{(i,j)}(I, J) \geq 0$ for $i + j = d - 1$ in analytically unramified unmixed local rings of positive dimension. Recall that a local ring $(R, \mathfrak{m})$ is called *unmixed* if $\dim \widehat{R}/P = \dim \widehat{R}$ for all $P \in \operatorname{Ass} \widehat{R}$. We say R is *equidimensional* if $\dim R/P = \dim R$ for every minimal prime P of R. In order to prove D', E' are unmixed we recall the following result.

Lemma 4.1 [**9**, Corollary of Theorem 31.5]. *Let $(R, \mathfrak{m})$ be a quotient of a regular local ring. If R is equidimensional, then so is $\widehat{R}$.*

Proposition 4.2. *Let D', E' be as before. Let $(R, \mathfrak{m})$ be complete normal local domain of dimension $d \geq 2$. Then D' and E' are unmixed.*

Proof. Since R is complete, by the Cohen structure theorem R is a quotient of a regular local ring and so is D'. By [**8**, Proposition 17.B] R' is a normal domain. Hence, by [**8**, Theorem 38], xR' is unmixed. Thus, $\operatorname{height} P = 1$ for all $P \in \operatorname{Ass}_{R'} D'$. Since R' is catenary, $\dim R'/P = d - 1$. Thus, $\dim D'/P = \dim D'$ for all $P \in \operatorname{Ass}_{D'} D'$. Therefore, D' is equidimensional. By Lemma 4.1, $\widehat{D'}$ is equidimensional. By Proposition 2.9, all associated primes of $\widehat{D'}$ are minimal. Therefore, D' is unmixed. Similarly, E' is unmixed. ◻

Next we prove that $\overline{e}_{(i,j)}(I, J) \geq 0$ for $i + j = d - 1$ in analytically unramified unmixed local rings.

Theorem 4.3. *Let $(R, \mathfrak{m})$ be an analytically unramified unmixed local ring of dimension $d > 0$. Let I and J be $\mathfrak{m}$-primary ideals in R. Then, for $i + j = d - 1$,*

$$\overline{e}_{(i,j)}(I, J) \geq 0.$$

Proof. Apply induction on d. Let $d=1$. Then R is a Cohen-Macaulay local ring. Since $\overline{e}_{(0,0)}(I,J)=\overline{e}_1(IJ)$, by [**7**, Lemma 3.19], the result follows. Let $d\geq 2$. Since $\overline{I}\widehat{R}=\overline{I\widehat{R}}$ for every $\mathfrak{m}$-primary ideal in R, $\overline{e}_{(i,j)}(I,J)=\overline{e}_{(i,j)}(I\widehat{R},J\widehat{R})$. Hence, we may assume that R is complete.

Let S be the integral closure of R in its total quotient ring and, for each $\wp\in \operatorname{Ass} R$, let $S(\wp)$ be the integral closure of $R/\wp$ in its quotient field. Then

$$S=\prod_{\wp\in \operatorname{Ass} R} S(\wp).$$

Since $\overline{I^rJ^sS}\cap R=\overline{I^rJ^s}$ for all $r,s\geq 0$, the natural map

$$\eta_{(r,s)}: R/\overline{I^rJ^s}\longrightarrow S/\overline{I^rJ^sS}$$

is injective. Note that S is a finite R-module by [**5**, Corollary 4.6.2]. Therefore,

$$\begin{aligned}\lambda_R(R/\overline{I^rJ^s}) &\leq \lambda_R(S/\overline{I^rJ^sS})\\ &=\sum_{\wp\in \operatorname{Ass} R}\lambda_R(S(\wp)/\overline{I^rJ^sS(\wp)})\\ &=\sum_{\wp\in \operatorname{Ass} R}\lambda_{S(\wp)}(S(\wp)/\overline{I^rJ^sS(\wp)})\lambda_R(S(\wp)/\mathfrak{m}_{S(\wp)}),\end{aligned}$$

where $\mathfrak{m}_{S(\wp)}$ denotes the maximal ideal of local ring $S(\wp)$. For any finite R-module M, there exist integers $e_{(i,j)}(I,J;M)$ such that

$$\begin{aligned}&\lambda(M/I^rJ^sM)\\ &=\sum_{i+j\leq \dim M}(-1)^{\dim M-(i+j)}e_{(i,j)}(I,J;M)\binom{r+i-1}{i}\binom{s+j-1}{j}\end{aligned}$$

for $r,s\gg 0$ [**5**, Theorem 17.4.2]. Since R is unmixed $\dim S(\wp)=d$ for all $\wp\in \operatorname{Ass} R$. Therefore, for $i+j=d$,

$$\begin{aligned}e_{(i,j)}(I,J;S) &=\sum_{\wp\in \operatorname{Ass} R}\lambda_R(S(\wp)/\mathfrak{m}_{S(\wp)})e_{(i,j)}(IS(\wp),JS(\wp))\\ &=\sum_{\wp\in \operatorname{Ass} R}\lambda_R(S(\wp)/\mathfrak{m}_{S(\wp)})\overline{e}_{(i,j)}(IS(\wp),JS(\wp)).\end{aligned}$$

But, by the associativity formula [**5**, Theorem 17.4.8] for $i+j=d$,

$$e_{(i,j)}(I,J)=\sum_{\wp}\lambda(R_{\wp})e_{(i,j)}(IR/\wp,JR/\wp)=\sum_{\wp}e_{(i,j)}(IR/\wp,JR/\wp),$$

where $\wp$ varies over minimal primes of R. Similarly,

$$e_{(i,j)}(I,J;S)=\sum_{\wp}\lambda(S_{\wp})e_{(i,j)}(IR/\wp,JR/\wp)=\sum_{\wp}e_{(i,j)}(IR/\wp,JR/\wp),$$

where $\wp$ varies over minimal primes of R. Hence, $e_{(i,j)}(I,J) = e_{(i,j)}(I,J;S)$ for $i+j=d$. Hence, for $i+j=d$, by Theorem 2.5,

$$\overline{e}_{(i,j)}(I,J)=e_{(i,j)}(I,J)=\sum_{\wp\in \operatorname{Ass} R}\lambda_R(S(\wp)/\mathfrak{m}_{S(\wp)})\overline{e}_{(i,j)}(IS(\wp),JS(\wp)).$$

Thus,

$$\begin{aligned}0&\le \lambda_R(S/\overline{I^rJ^sS})-\lambda_R(R/\overline{I^rJ^s})\\ &=\sum_{i+j=d-1}\Big[\overline{e}_{(i,j)}(I,J)-\sum_{\wp\in\operatorname{Ass} R}\lambda_R(S(\wp)/\mathfrak{m}_{\wp})\overline{e}_{(i,j)}(IS(\wp),JS(\wp))\Big]\\ &\quad\times\binom{r+i-1}{i}\binom{s+j-1}{j}\\ &\quad+\text{terms of lower degree.}\end{aligned}$$

Let $C_{(r,s)}=\lambda_R(\operatorname{coker}\eta_{(r,s)})$. Then

$$\begin{aligned}\Delta_{\mathbf{e_1}}(C(r+1,s))&=C_{(r+1,s)}-C_{(r,s)}\\ &=\lambda_R\bigg(\frac{\overline{I^rJ^sS}}{\overline{I^{r+1}J^sS}}\bigg)-\lambda_R\bigg(\frac{\overline{I^rJ^s}}{\overline{I^{r+1}J^s}}\bigg).\end{aligned}$$

Note that the natural map

$$\tau_{(r,s)}:\frac{\overline{I^rJ^s}}{\overline{I^{r+1}J^s}}\longrightarrow\frac{\overline{I^rJ^sS}}{\overline{I^{r+1}J^sS}}$$

is injective. Let $L_{(r,s)}=\operatorname{coker}\tau_{(r,s)}, L=\bigoplus_{r,s\ge 0}L_{(r,s)}$. Then

$$0\longrightarrow\overline{\mathcal{R}}(I,J\mid I)\longrightarrow\overline{\mathcal{R}}(IS,JS\mid IS)\longrightarrow L\longrightarrow 0,$$

is an exact sequence of bigraded $\mathcal{R}(I, J \mid I)$-modules. Since S is a finite R-module, for every ideal K of $S[\mathbf{X}, \mathbf{Y}]$, $S[\mathbf{X}, \mathbf{Y}]/K$ is a finite $R[\mathbf{X}, \mathbf{Y}]/K \cap R[\mathbf{X}, \mathbf{Y}]$-module. Since $\mathcal{R}(IS, JS) \cong S[\mathbf{X}, \mathbf{Y}]/K$ for some ideal K of $S[\mathbf{X}, \mathbf{Y}]$ and $\mathcal{R}(I, J) \cong R[\mathbf{X}, \mathbf{Y}]/K'$ for some ideal K' of $R[\mathbf{X}, \mathbf{Y}]$ and $K' \subseteq K \cap R[\mathbf{X}, \mathbf{Y}]$, $\mathcal{R}(IS, JS)$ is a finite $\mathcal{R}(I, J)$-module. Since $R/\wp$ is a complete local domain, by [**5**, Theorem 4.3.4], $S(\wp)$ is a complete local domain for all $\wp \in \operatorname{Ass} R$. Hence, there exists an integer $h \geq 0$ such that $\overline{I^r J^s S} \subseteq I^{r-h} J^{s-h} S$ for all r, s. Therefore $\overline{\mathcal{R}}(IS, JS)$ is a finite $\mathcal{R}(IS, JS)$-module. Hence, $\overline{\mathcal{R}}(IS, JS)$ is a finite $\mathcal{R}(I, J)$-module. Let $\mathfrak{b} = \bigoplus_{r,s \geq 0} \overline{I^{r+1} J^s S}$. Then $\overline{\mathcal{R}}(IS, JS|IS) = \overline{\mathcal{R}}(IS, JS)/\mathfrak{b}$ is finite $\mathcal{R}(I, J)/\mathfrak{b} \cap \mathcal{R}(I, J)$-module. Since $I^{r+1} J^s \subseteq \overline{I^{r+1} J^s S} \cap I^r J^s$, $\overline{\mathcal{R}}(IS, JS \mid IS)$ is a finite $\mathcal{R}(I, J \mid I)$-module. Thus, L is a finitely generated bigraded module over the standard bigraded algebra $\mathcal{R}(I, J \mid I)$. Therefore, by [**4**, Theorem 4.1], the coefficients of highest degree monomials in the polynomial associated with $\lambda(L_{(r,s)}) = \Delta_{\mathbf{e_1}}(C(r+1, s))$ are nonnegative. Similarly, the coefficients of highest degree monomials in the polynomial associated with $\Delta_{\mathbf{e_2}}(C(r+1, s))$ are nonnegative. Therefore, for $i + j = d - 1$,

$$\overline{e}_{(i,j)}(I, J) \geq \sum_{\wp \in \operatorname{Ass} R} \lambda_R(S(\wp)/\mathfrak{m}_\wp) \overline{e}_{(i,j)}(IS(\wp), JS(\wp)).$$

Hence, we may assume that R is a complete normal local domain. Therefore, by Proposition 2.9 and Lemma 4.2, D' is analytically unramified and unmixed. By Theorem 3.2 and Proposition 3.4, $\overline{e}_{(i,j)}(IR', JR') = \overline{e}_{(i-1,j)}(ID', JD')$. Since $\overline{e}_{(i,j)}(I, J) = \overline{e}_{(i,j)}(IR', JR')$, by induction, for $i \neq 0$ and $i + j = d - 1$,

$$\overline{e}_{(i,j)}(I, J) \geq 0.$$

Now, let $E' = R'/yR'$. Then $\overline{e}_{(0,d-1)}(I, J) = \overline{e}_{(0,d-2)}(IE', JE') \geq 0$, by induction. $\square$

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