GORENSTEIN HILBERT COEFFICIENTS

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Dedicated to Jürgen Herzog on his 70th birthday

ABSTRACT. We prove upper and lower bounds for all the coefficients in the Hilbert polynomial of a graded Gorenstein algebra $S = R/I$ with a quasi-pure resolution over R. The bounds are in terms of the minimal and the maximal shifts in the resolution of R. These bounds are analogous to the bounds for the multiplicity found in [9] and are stronger than the bounds for the Cohen Macaulay algebras found in [5].

1. Introduction. Let $S = \bigoplus S_i$ be a standard graded k-algebra of dimension d, finitely generated in degree one. $H(S, i) = \dim_k S_i$ is the Hilbert function of S. It is well known that $H(S, i)$, for $i \gg 0$, is a polynomial $P_S(x)$, called the *Hilbert polynomial* of S. $P_S(x)$ has degree $d - 1$. If we write

$$P_S(x) = \sum_{i=0}^{d-1} (-1)^i e_i \binom{x + d - 1 - i}{x} = \frac{e_0}{(d-1)!} x^{d-1} + \cdots + (-1)^{d-1} e_{d-1},$$

then the coefficients e_i are called the *Hilbert coefficients* of S. The first one, e_0 called the multiplicity, is the most studied and is denoted by e.

If we write $S = R/I$, where R is the polynomial ring in n variables and I is a homogeneous ideal of R, then all these coefficients can be computed from the shifts in the minimal homogeneous free R-resolution **F** of S given as follows:

$$0 \longrightarrow \bigoplus_{j=m_s}^{M_s} R(-j)^{\beta_{sj}} \xrightarrow{\delta_s} \cdots$$

$$\longrightarrow \bigoplus_{j=m_i}^{M_i} R(-j)^{\beta_{ij}} \xrightarrow{\delta_i} \cdots$$

$$\longrightarrow \bigoplus_{j=m_1}^{M_1} R(-j)^{\beta_{1j}} \xrightarrow{\delta_1} R \longrightarrow R/I \longrightarrow 0.$$

Let h be the height of I, so that $h \leq s$. In 1995, Herzog and Srinivasan [4] proved that if this resolution is quasi-pure, i.e., if $m_i \geq M_{i-1}$, then

$$\frac{\prod_{i=1}^{s} m_i}{s!} \leq e(S) \leq \frac{\prod_{i=1}^{s} M_i}{s!}, \quad \text{if } h = s$$

and $e(S) \leq \prod_{i=1}^{h} M_i / h!$ if $h < s$.

Further, Herzog, Huneke and Srinivasan conjectured this to hold for all homogeneous algebras S which came to be known as the *multiplicity conjecture*.

When S is Gorenstein, Srinivasan established stronger bounds for the multiplicity.

Theorem. (Srinivasan [9]). *If S is a homogeneous Gorenstein algebra with quasi-pure resolution of length $s = 2k$ or $2k + 1$, then*

$$\frac{m_1 \cdots m_k M_{k+1} \cdots M_s}{s!} \leq e(S) \leq \frac{M_1 \cdots M_k m_{k+1} \cdots m_s}{s!}.$$

In this paper, we establish bounds for all the remaining Hilbert coefficients of Gorenstein algebras with quasi-pure resolutions analogous to the above bounds for the multiplicity. We prove this in Theorem 4.2.

Theorem 4.2. *If S is a homogeneous Gorenstein algebra with quasi-pure resolution of length $s = 2k$ or $2k + 1$, then, for $0 \leq l < \dim S$,*

$$f_l(m_1, \ldots, m_k, M_{k+1}, \ldots, M_s) \frac{m_1 \cdots m_k M_{k+1} \cdots M_s}{(s+l)!} \leq e_l(S)$$

$$\leq f_l(M_1, \ldots, M_k, m_{k+1}, \ldots, m_s) \frac{M_1 \cdots M_k m_{k+1} \cdots m_s}{(s+l)!}$$

with

$$f_l(a_1, \ldots, a_s) = \sum_{1 \le i_1 \le \cdots i_l \le s} \prod_{t=1}^{l} (a_{i_t} - (i_t + t - 1)) \quad and \quad f_0 = 1.$$

Boij and Söderberg [1] conjectured that Betti sequences of all graded algebras can be written (uniquely) as sums of positive rational multiples of Betti sequences of pure algebras which in turn implied the multiplicity conjecture. In 2008, these conjectures were proved by Eisenbud and Schreyer [3] for C-M modules in characteristic zero and extended to non C-M modules by Boij and Söderberg [1].

Using these results, Herzog and Zheng [5] showed that if S is Cohen-Macaulay of codimension s, then all Hilbert coefficients satisfy

$$f_i(m_1, \ldots, m_s) \frac{m_1 m_2 \cdots m_s}{(s+i)!} \le e_i(S) \le \frac{M_1 M_2 \cdots M_s}{(s+i)!} f_i(M_1, \ldots, M_s)$$

with

$$f_i(d_1, \ldots, d_s) = \sum_{1 \le j_1 \le \cdots j_i \le s} \prod_{k=1}^{i} (d_{j_k} - (j_k + k - 1)) \text{ and } f_0(d_1, \ldots, d_s) = 1.$$

Our results extend those of Srinivasan [9] as well as the above result [5] to all coefficients of Gorenstein algebras with quasi-pure resolutions.

In Section 3 we give an explicit formula of the Hilbert coefficients as a function of the shifts of the minimal free resolution of a Gorenstein algebra. These expressions depend on whether the projective dimension is even or odd.

In Section 4, we establish stronger bounds for the higher Hilbert coefficients when the algebra has a quasi-pure resolution.

2. Preliminaries and notations. Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables, I a homogeneous ideal contained in $(x_1, x_2, \ldots, x_n)$ and $S = R/I$. Let **F** be the minimal homogeneous free

resolution of S over R given by:

$$0 \longrightarrow \bigoplus_{j=1}^{b_s} R(-d_{sj}) \xrightarrow{\delta_s} \cdots$$

$$\longrightarrow \bigoplus_{j=1}^{b_i} R(-d_{ij}) \xrightarrow{\delta_i} \cdots$$

$$\longrightarrow \bigoplus_{j=1}^{b_1} R(-d_{1j}) \xrightarrow{\delta_1} R \longrightarrow R/I \longrightarrow 0.$$

Definition 2.1. A resolution is called quasi-pure if $d_{ij} \geq d_{i-1,l}$ for all j and l, that is, if $m_i \geq M_{i-1}$ for all i.

Suppose S is Gorenstein. Then, by duality of the resolution, the resolution of S can be written as follows.

If I is of height $2k+1$, then

$$0 \longrightarrow R(-c) \longrightarrow \sum_{j=1}^{b_1} R(-(c-a_{1j})) \longrightarrow \cdots$$

(1) $$\longrightarrow \sum_{j=1}^{b_k} R(-(c-a_{kj})) \longrightarrow \sum_{j=1}^{b_k} R(-a_{kj}) \longrightarrow \cdots$$

$$\longrightarrow \sum_{j=1}^{b_1} R(-a_{1j}) \longrightarrow R,$$

and if I is of height $2k$, then

$$0 \longrightarrow R(-c) \longrightarrow \sum_{j=1}^{b_1} R(-(c-a_{1j})) \longrightarrow \cdots$$

(2) $$\longrightarrow \sum_{j=1}^{b_k/2=r_k} R(-(c-a_{kj})) \oplus \sum_{j=1}^{b_k/2=r_k} R(-a_{kj}) \longrightarrow \cdots$$

$$\longrightarrow \sum_{j=1}^{b_1} R(-a_{1j}) \longrightarrow R.$$

Remark 2.2. (1) The minimal shifts in the resolution are:

$$m_i = \min_j a_{ij} \quad 1 \leq i \leq k,$$
$$= c - \max_j a_{s-i,j} \quad k+1 \leq i < s,$$
$$= c \quad i = s.$$

The maximal shifts in the resolution are:

$$M_i = \max_j a_{ij} \quad 1 \leq i \leq k,$$
$$= c - \min_j a_{s-i,j} \quad k+1 \leq i < s,$$
$$= c \quad i = s,$$

and $M_s = m_s = c$.

(2) Let $\alpha_{ij} = a_{ij}(c - a_{ij})$ for $i \leq k$ with $p_i = \min_j \alpha_{ij} = m_i M_{s-i}$ and $P_i = \max_j \alpha_{ij} = M_i m_{s-i}$.

Definition 2.3. Given $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ a sequence of real numbers, we denote the following Vandermonde determinants by

$$V_t = V_t(\alpha_1, \alpha_2, \ldots, \alpha_k) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_k^2 \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_1^{k-2} & \alpha_2^{k-2} & \cdots & \alpha_k^{k-2} \\ \alpha_1^{k-1+t} & \alpha_2^{k-1+t} & \cdots & \alpha_k^{k-1+t} \end{vmatrix}$$
$$= \prod_{1 \leq j < i \leq k} (\alpha_i - \alpha_j) + \sum_{\beta_1 + \beta_2 + \cdots \beta_k = t} (\alpha_1^{\beta_1} \alpha_2^{\beta_2} \cdots \alpha_k^{\beta_k}).$$

Remark 2.4. $V_t(\alpha_1, \alpha_2, \ldots, \alpha_k) \geq 0$ if the sequence is in ascending order.

As a convention, for any non-negative integers n, p, we set the binomial coefficient $\binom{n}{p} = 0$ if $n < p$.

The following binomial identities are essential to our theorems. In [9, Lemmas 2 and 3], Srinivasan showed

Lemma 2.5. *For all $n \geq 0$:*

$$(c-a)^n - a^n = \sum_{t=0}^{[n/2]} (-1)^t \binom{n-t-1}{t} a^t (c-a)^t (c-2a) c^{n-2t-1}$$

$$(c-a)^n + a^n = \sum_{t=0}^{[n/2]} (-1)^t \binom{n-t}{t} a^t (c-a)^t c^{n-2t}$$

$$+ \sum_{t=1}^{[n/2]} (-1)^t \binom{n-t-1}{t-1} a^t (c-a)^t c^{n-2t}.$$

3. Hilbert coefficients of Gorenstein algebras. Let $R = K[x_1, \ldots, x_n]$, and let I be a graded ideal. Let **F** be the minimal free resolution of $S = R/I$,

$$0 \longrightarrow \bigoplus_{j=1}^{b_s} R(-d_{sj}) \xrightarrow{\delta_s} \cdots$$

$$\longrightarrow \bigoplus_{j=1}^{b_i} R(-d_{ij}) \xrightarrow{\delta_i} \cdots$$

$$\longrightarrow \bigoplus_{j=1}^{b_1} R(-d_{1j}) \xrightarrow{\delta_1} R \longrightarrow R/I \longrightarrow 0.$$

Theorem 3.1 [8]. *Suppose S is C-M. Then these shifts d_{ij} are known to satisfy* [1, 8]:

$$\sum_{i=1}^{s} (-1)^i \sum_{j=1}^{b_i} d_{ij}^k = \begin{cases} -1 & k = 0 \\ 0 & 1 \leq k < s \\ (-1)^s s! e & k = s. \end{cases}$$

These equations can be thought of as defining the multiplicity, $e = e_0(S)$. In fact, the higher Hilbert coefficients can also be expressed in

terms of the shifts in the resolution [**7**]. We include a simple proof for the sake of completeness.

Theorem 3.2.

$$(-1)^s(s+l)!e_l = \sum_{r=0}^{l}(-1)^{l-r}\nu_{l-r}\sum_{i=0}^{s}(-1)^i\sum_{j=1}^{b_i}d_{ij}^{s+r}$$

with

$$\nu_{l-r} = \sum_{1\le\xi_1<\xi_2<\cdots<\xi_{l-r}\le s+l-1}\xi_1\xi_2\cdots\xi_{l-r} \quad and \quad \nu_0 = 1.$$

Proof. We know that the Hilbert function of R/I is

$$\frac{\sum_{i=0}^{s}(-1)^i\sum_{j=1}^{b_i}t^{d_{ij}}}{(1-t)^n} = \frac{Q(t)}{(1-t)^d},$$

where $d = \dim R/I = n - s$ and $Q^{(i)}(1)/i! = e_i$. We get

$$(3) \qquad \sum_{i=0}^{s}(-1)^i\sum_{j=1}^{b_i}t^{d_{ij}} = Q(t)(1-t)^s.$$

We denote these two quantities by $S_{R/I}(t)$. We differentiate both sides $l+s$ times and evaluate them at $t = 1$. We first start by the right hand side

$$S_{R/I}^{(s+l)}(t) = (-1)^s\binom{s+l}{l}s!Q^{(l)}(t) + (1-t)P(t),$$

where $P(t)$ is a polynomial in t. Evaluating at $t = 1$:

$$S_{R/I}^{(s+l)}(1) = (-1)^s\binom{s+l}{l}s!Q^{(l)}(1) + 0$$
$$= (-1)^s(s+l)!e_l.$$

On the other hand, $S_{R/I}(t) = \sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} t^{d_{ij}}$. So

$$S_{R/I}^{(l)}(1) = \sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} \binom{d_{ij}}{l} l!$$

$$= \sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} \prod_{r=0}^{l-1}(d_{ij} - r)$$

$$= \sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} \sum_{r=1}^{l}(-1)^{l-r} \nu_{l-r} d_{ij}^r,$$

with

$$\nu_{l-r} = \sum_{1 \le \xi_1 < \xi_2 < \cdots < \xi_{l-r} < l-1} \xi_1 \xi_2 \cdots \xi_{l-r} \text{ and } \nu_0 = 1.$$

$$S_{R/I}^{(s+l)}(1) = \sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} \sum_{r=1}^{s+l}(-1)^{s+l-r} \nu_{s+l-r} d_{ij}^r$$

$$= \sum_{r=1}^{s+l}(-1)^{s+l-r} \nu_{s+l-r} \sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} d_{ij}^r$$

with

$$\nu_{s+l-r} = \sum_{1 \le \xi_1 < \xi_2 < \cdots < \xi_{s+l-r} \le s+l-1} \xi_1 \xi_2 \cdots \xi_{s+l-r} \text{ and } \nu_0 = 1.$$

Note that

$$\sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} d_{ij}^r = 0$$

when $r < s$, and

$$S_{R/I}^{(s+l)}(1) = \sum_{r=s}^{s+l}(-1)^{s+l-r} \nu_{s+l-r} \sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} d_{ij}^r$$

$$= \sum_{r=0}^{l}(-1)^{l-r} \nu_{l-r} \sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} d_{ij}^{s+r},$$

and hence the result.   ◻

Let I be Gorenstein. The minimal free resolution of I is written as in (1) and (2) depending on whether the projective dimension of I is even or odd. In [**9**], Srinivasan gave a more simplified expression for the multiplicity in both cases. She proved:

Theorem 3.3 [9]. *Let I be Gorenstein of grade $s = 2k + 1$, and let the minimal graded free resolution of $S = R/I$ be as in* (1). *Then:*

$$\sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^t (c - a_{ij})^t (c - 2a_{ij})$$

$$= \begin{cases} 0 & \text{if } 1 \leq t < k \\ (-1)^k (2k+1)! e(S) & \text{if } t = k \\ -c & \text{if } t = 0. \end{cases}$$

Theorem 3.4 [9]. *Let I be Gorenstein of grade $s = 2k$, and let the minimal graded free resolution of $S = R/I$ be as in* (2). *Then:*

$$\sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^t (c - a_{ij})^t = \begin{cases} 0 & \text{if } 1 \leq t < k \\ (-1)^k ((2k)!/2) e(S) & \text{if } t = k \\ -1 & \text{if } t = 0. \end{cases}$$

We extend these results to all coefficients, and we show:

Theorem 3.5. *Let I be Gorenstein of grade $s = 2k + 1$, and let the minimal free resolution of $S = R/I$ be as in* (1). *Then $(-1)^k (s+l)! e_l$ is equal to:*

$$\sum_{0 \leq r \leq l}(-1)^{l-r}\nu_{l-r}\sum_{t=0}^{[r/2]}(-1)^t \binom{k+r-t}{k+t}c^{r-2t}$$

$$\cdot \sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^{k+t}(c - a_{ij})^{k+t}(c - 2a_{ij}).$$

Proof. Following the result of Theorem 3.2, it suffices to show that

$$\sum_{i=0}^{s}(-1)^i\sum_{j=1}^{b_i}d_{ij}^{s+r} = -\sum_{i=1}^{k}(-1)^{k+i}\sum_{j=1}^{b_i}\sum_{t=0}^{[r/2]}(-1)^t$$
$$\times\binom{k+r-t}{k+t}c^{r-2t}a_{ij}^{k+t}(c-a_{ij})^{k+t}(c-2a_{ij}).$$

We have that

$$\sum_{i=0}^{s}(-1)^i\sum_{j=1}^{b_i}d_{ij}^{s+r}$$

$$= -c^{s+r}+\sum_{i=1}^{k}\sum_{j=1}^{b_j}a_{ij}^{s+r}(-1)^i$$

$$+\sum_{i=1}^{k}\sum_{j=1}^{b_j}(-1)^{2k+1-i}(c-a_{ij})^{s+r}$$

$$= -c^{s+r}-\sum_{i,j}(-1)^i[(c-a_{ij})^{s+r}-a_{ij}^{s+r}]$$

$$= -c^{s+r}-\sum_{i,j}(-1)^i\sum_{t=0}^{[s+r/2]}(-1)^t$$
$$\times\binom{s+r-1-t}{t}a_{ij}^t(c-a_{ij})^t(c-2a_{ij})c^{s+r-1-2t}\quad\text{by Lemma 2.5}$$

$$= -c^{s+r}-\sum_{i,j}(-1)^i\sum_{t=0}^{k+[r+1/2]}(-1)^t$$
$$\times\binom{2k+r-t}{t}a_{ij}^t(c-a_{ij})^t(c-2a_{ij})c^{2k+r-2t}.$$

By Theorem 3.3, the only remaining terms in the sum are $t = 0$ and $t \geq k$, so

$$\sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} d_{ij}^{s+r} = -\sum_{i,j}(-1)^i \sum_{t=k}^{k+[r+1/2]}(-1)^t$$

$$\times \binom{2k+r-t}{t}a_{ij}^t(c-a_{ij})^t(c-2a_{ij})c^{2k+r-2t}$$

$$= -\sum_{i,j}(-1)^i \sum_{t=0}^{[r/2]}(-1)^{t+k}$$

$$\times \binom{k+r-t}{k+t}a_{ij}^{k+t}(c-a_{ij})^{k+t}(c-2a_{ij})c^{r-2t}. \qquad \square$$

Example 3.6. $(-1)^k(s+1)!e_1 = [-\nu_1+\binom{k+1}{k}c]\sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^k(c-a_{ij})^k(c-2a_{ij}).$

$$(-1)^k(s+2)!e_2 = \left[\nu_2 - \nu_1\binom{k+1}{k}c + \binom{k+2}{k}c^2\right]$$

$$\cdot \sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^k(c-a_{ij})^k(c-2a_{ij})$$

$$- \sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^{k+1}(c-a_{ij})^{k+1}(c-2a_{ij}).$$

We now consider the case when s is even.

Theorem 3.7. *Let I be Gorenstein of grade $s = 2k$, and let the minimal free resolution of $S = R/I$ be as in (1). Then $(-1)^k(s+l)!e_l$ is equal to*:

$$\sum_{\substack{t=0 \\ 0\le r\le l}}^{[r/2]}(-1)^{l-r+t}\nu_{l-r}\left[\binom{k+r-t}{k+t} + \binom{k+r-t-1}{k+t-1}\right]c^{r-2t}$$

$$\cdot \sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^{k+t}(c-a_{ij})^{k+t}.$$

In the summation j runs from 1 to b_i if $i < k$ and from 1 to $b_k/2$ if $i = k$.

Proof. We proceed the same way as in the proof of Theorem 3.5. We have

$$\sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} d_{ij}^{s+r}$$

$$= c^{s+r} + \sum_{i=1}^{k}\sum_{j=1}^{b_j} a_{ij}^{s+r}(-1)^i$$

$$+ \sum(-1)^{2k-i}(c - a_{ij})^{s+r}$$

$$= c^{s+r} + \sum_{i,j}(-1)^i \left[(c - a_{ij})^{s+r} + a_{ij}^{s+r}\right]$$

$$= c^{2k+r} + \sum_{i,j}(-1)^i \left[\sum_{t=0}^{k+[r/2]}(-1)^t \binom{2k + r - t}{t} a_{ij}^t(c - a_{ij})^t c^{2k+r-2t} \right.$$

$$\left. + \sum_{t=1}^{k+[r/2]}(-1)^t \binom{2k + r - 1 - t}{t - 1} a_{ij}^t(c - a_{ij})^t c^{2k+r-2t} \right]$$

<div align="right">by Lemma 2.5.</div>

By Theorem 3.4, the only remaining terms in the sum are $t = 0$ and $t \geq k$; we obtain that

$$\sum_{i=0}^{s}(-1)^i \sum_{j=1}^{b_i} d_{ij}^{s+r}$$

$$= \sum_{i,j}(-1)^i \left[\sum_{t=0}^{[r/2]}(-1)^{t+k} \binom{k + r - t}{t + k} a_{ij}^{t+k}(c - a_{ij})^{t+k} c^{r-2t} \right.$$

$$\left. + \sum_{t=0}^{[r/2]}(-1)^{t+k} \binom{k + r - 1 - t}{t + k - 1} a_{ij}^{t+k}(c - a_{ij})^{t+k} c^{r-2t} \right]. \qquad \square$$

Example 3.8.

$$(-1)^k(s+1)!e_1 = \left[-\nu_1\left(\binom{k}{k} + \binom{k-1}{k-1}\right) + \left(\binom{k+1}{k} + \binom{k}{k-1}\right)c\right]$$

$$\cdot \sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^k(c-a_{ij})^k$$

$$(-1)^k(s+2)!e_2 = \left[\nu_2\left(\binom{k}{k} + \binom{k-1}{k-1}\right) - \nu_1\left(\binom{k+1}{k} + \binom{k}{k-1}\right)c\right.$$

$$\left. + \left(\binom{k+2}{k} + \binom{k+1}{k-1}\right)c^2\right]$$

$$\cdot \sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^k(c-a_{ij})^k$$

$$- \left[\binom{k+1}{k+1} + \binom{k}{k}\right]\sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^{k+1}(c-a_{ij})^{k+1}.$$

4. Bounds for the coefficients with quasi-pure resolutions.

Definition 4.1. For any ordered s-tuple of positive integers $(y_1, \ldots, y_s)$, define

$$f_l(y_1, \ldots, y_s) = \sum_{1 \le i_1 \le \cdots i_l \le s}\left(\prod_{t=1}^{l} y_{i_t} - (i_t + t - 1)\right), \quad 1 \le l \le s,$$

and $f_0 = 1$.

In this section, we prove:

Theorem 4.2. *If S is a homogeneous Gorenstein algebra with quasi-pure resolution of length $s = 2k$ or $2k+1$, then, for $0 \le l < \dim S$,*

$$f_l(m_1, \ldots, m_k, M_{k+1}, \ldots, M_s)\frac{m_1 \cdots m_k M_{k+1} \cdots M_s}{(s+l)!} \le e_l(S)$$

$$\le f_l(M_1, \ldots, M_k, m_{k+1}, \ldots, m_s)\frac{M_1 \cdots M_k m_{k+1} \cdots m_s}{(s+l)!}.$$

Remark 4.3. (1) $M_s = m_s = c$.

(2) These bounds are strictly stronger than the bounds in the conjecture found by Herzog and Zheng in [**5**].

Example 4.4. Let $R = k[x, y, z, w, t]$ and $I = (x^2, y^3 - t^3, z^3 - w^3, yzw)$. The minimal free resolution of R/I is given by

$$0 \longrightarrow R(-11) \longrightarrow R(-9) \oplus R(-8)^3 \longrightarrow R(-6)^3 \oplus R(-5)^3$$
$$\longrightarrow R(-3)^3 \oplus R(-2) \longrightarrow R,$$

$n = 5$ and $s = 4$. So we have in our case:

$$m_1 m_2 M_3 M_4 \leq 4!e(R/I) \leq M_1 M_2 m_3 m_4$$

and

$$[m_1 + m_2 + M_3 + M_4 - 10]m_1 m_2 M_3 M_4 \leq 4!e_1(R/I)$$
$$\leq [M_1 + M_2 + m_3 + m_4 - 10]M_1 M_2 m_3 m_4,$$

which implies

$$2 \cdot 5 \cdot 9 \cdot 11 \leq 4!e(R/I) \leq 3 \cdot 6 \cdot 8 \cdot 11$$

and

$$17 \cdot 2 \cdot 5 \cdot 9 \cdot 11 \leq 4!e_1(R/I) \leq 18 \cdot 3 \cdot 6 \cdot 8 \cdot 11.$$

Both of these are stronger than the bounds that come from Herzog and Zheng in [**5**], which are

$$2 \cdot 5 \cdot 8 \cdot 11 \leq 4!e(R/I) \leq 3 \cdot 6 \cdot 9 \cdot 11$$

and

$$16 \cdot 2 \cdot 5 \cdot 8 \cdot 11 \leq 4!e_1(R/I) \leq 19 \cdot 3 \cdot 6 \cdot 9 \cdot 11.$$

Let $S = R/I$. Then S has an R-free resolution of length $s = 2k + 1$ or $s = 2k$. The first half of the resolution is given below.

$$(4) \quad \sum_{j=1}^{b_k} R(-a_{kj}) \longrightarrow \cdots \longrightarrow \sum_{j=1}^{b_1} R(-a_{1j}) \longrightarrow R, \quad s = 2k+1$$

and

$$\sum_{j=1}^{b_k/2=r_k} R(-(c-a_{kj})) \oplus \sum_{j=1}^{b_k/2=r_k} R(-a_{kj}) \longrightarrow \cdots$$

$$\longrightarrow \sum_{j=1}^{b_1} R(-a_{1j}) \longrightarrow R, \ s = 2k.$$

Thus, we let $r_i = b_i$, $i \neq k$, and $r_k = b_k$ if s is odd and $r_k = b_k/2$ if s is even.

Without loss of generality we may take $a_{i1} \leq a_{i2} \leq \cdots a_{ib_i}$ for all i.

If $s = 2k$, we pick a_{kj} so that $c - a_{kr_k} \geq a_{kr_k}$. The symmetry of the resolution and the exactness criterion force b_k to be even. Srinivasan showed in [**9**, Lemma 4], that

Lemma 4.5. *If S is Gorenstein with a quasi-pure resolution, then $c \geq 2a_{ij}$ for all i, j.*

Proof. Since quasi-purity means the a_{ij} increase with i, we just need to check $c \geq 2a_{kr_k}$. If $s = 2k+1$, then $c - a_{kr_k} = m_{k+1} \geq M_k = a_{kr_k}$. So $c \geq 2a_{kr_k}$. If $s = 2k$, then $c \geq 2a_{kr_k}$ by choice, and hence the result. □

To be able to prove Theorem 4.2, we need to consider two different determinants depending on whether s is even or odd.

Suppose s is odd. Let:

$$
M_t = \begin{vmatrix}
\displaystyle\sum_{j=1}^{b_1} \alpha_{1j}(c-2a_{1j}) & \cdots & \displaystyle\sum_{j=1}^{b_i} \alpha_{ij}(c-2a_{ij}) & \cdots & \displaystyle\sum_{j=1}^{b_k} \alpha_{kj}(c-2a_{kj}) \\
\displaystyle\sum_{j=1}^{b_1} \alpha_{1j}^2(c-2a_{1j}) & \cdots & \displaystyle\sum_{j=1}^{b_i} \alpha_{ij}^2(c-2a_{ij}) & \cdots & \displaystyle\sum_{j=1}^{b_k} \alpha_{kj}^2(c-2a_{kj}) \\
\vdots & & \vdots & & \vdots \\
\displaystyle\sum_{j=1}^{b_1} \alpha_{1j}^{k-1}(c-2a_{1j}) & \cdots & \displaystyle\sum_{j=1}^{b_i} \alpha_{ij}^{k-1}(c-2a_{ij}) & \cdots & \displaystyle\sum_{j=1}^{b_k} \alpha_{kj}^{k-1}(c-2a_{kj}) \\
\displaystyle\sum_{j=1}^{b_1} \alpha_{1j}^{k+t}(c-2a_{1j}) & \cdots & \displaystyle\sum_{j=1}^{b_i} \alpha_{ij}^{k+t}(c-2a_{ij}) & \cdots & \displaystyle\sum_{j=1}^{b_k} \alpha_{kj}^{k+t}(c-2a_{kj})
\end{vmatrix},
$$

where $\alpha_{ij} = a_{ij}(c - a_{ij})$. Then,

$$
M_t = \sum_{1\le j_i \le b_i} \prod_{i=1}^{k} \alpha_{ij_i}(c-2a_{ij_i}) V_t(\alpha_{1j_1}, \ldots, \alpha_{kj_k}).
$$

Now consider

$$
\sum_{r=0}^{l} (-1)^{l-r}\nu_{l-r} \sum_{t=0}^{[r/2]} (-1)^t \binom{k+r-t}{k+t} c^{r-2t} M_t.
$$

It is equal to

$$
\sum_{1\le j_i \le b_i} \prod_{i=1}^{k} \alpha_{ij_i}(c-2a_{ij_i}) V(\alpha_{1j_1}, \ldots, \alpha_{kj_k})\cdot
$$

$$
\sum_{r=0}^{l} (-1)^{l-r}\nu_{l-r} \sum_{t=0}^{[r/2]} (-1)^t \binom{k+r-t}{k+t} c^{r-2t} \sum_{\sum \beta_i = t} \prod_{i=1}^{k} \alpha_{ij_i}^{\beta_i}.
$$

Lemma 4.6. *We have, for all $c, a_i > 0$ and $\alpha_i = a_i(c - a_i)$,*

$$
\sum_{t=0}^{[r/2]} (-1)^t \binom{k+r-t}{k+t} c^{r-2t} \sum_{\sum \beta_i = t} \prod_{i=1}^{k} \alpha_i^{\beta_i}
$$

$$
= \sum_{\beta_1 + \cdots + \beta_{2k+1} = r} \prod_{i=1}^{k} a_i^{\beta_i}(c-a_i)^{\beta_{k+i}} c^{\beta_{2k+1}}.
$$

Proof. $\sum_{\beta_1+\cdots+\beta_{2k+1}=r} \prod_{i=1}^{k} a_i^{\beta_i}(c-a_i)^{\beta_{k+i}} c^{\beta_{2k+1}}$ is the coefficient of x^r in

$$\frac{1}{1-cx} \prod_{i=1}^{k} \frac{1}{1-a_ix} \frac{1}{1-(c-a_i)x}$$

$$= \frac{1}{1-cx} \prod_{i=1}^{k} \frac{1}{1-(cx-a_i(c-a_i)x^2)}$$

$$= \sum_{\gamma_1,\ldots,\gamma_{k+1}} \prod_{i=1}^{k} \left(cx-a_i(c-a_i)x^2\right)^{\gamma_i} (cx)^{\gamma_{k+1}} ;$$

the coefficient of x^r is in this last expression is:

$$\sum_{\sum_{i=1}^{k}(2\beta_i+\gamma_i-\beta_i)+\gamma_{k+1}=r} \prod_{i=1}^{k} \binom{\gamma_i}{\beta_i} c^{\gamma_i-\beta_i}(-1)^{\beta_i} \left(a_i(c-a_i)\right)^{\beta_i} c^{\gamma_{k+1}}$$

$$= \sum_{\sum_{i=1}^{k}(\beta_i+\gamma_i)+\gamma_{k+1}=r} \prod_{i=1}^{k} \binom{\gamma_i}{\beta_i} c^{\gamma_i-\beta_i}(-1)^{\beta_i} \left(a_i(c-a_i)\right)^{\beta_i} c^{\gamma_{k+1}}$$

$$= \sum_{t\geq 0} c^{r-2t}(-1)^t \sum_{\sum_{i=1}^{k+1}\gamma_i=r-t} \sum_{\beta_1+\cdots+\beta_k=t} \prod_{i=1}^{k} \left(a_i(c-a_i)\right)^{\beta_i} \binom{\gamma_i}{\beta_i}$$

$$= \sum_{t\geq 0} c^{r-2t}(-1)^t \sum_{\beta_1+\cdots+\beta_k=t} \prod_{i=1}^{k} \left(a_i(c-a_i)\right)^{\beta_i} \sum_{\sum_{i=1}^{k+1}\gamma_i=r-t} \prod_{i=1}^{k} \binom{\gamma_i}{\beta_i}.$$

It remains to show that

$$\sum_{\sum_{i=1}^{k+1}\gamma_i=r-t} \prod_{i=1}^{k} \binom{\gamma_i}{\beta_i} = \binom{r-t+k}{t+k}.$$

This can be proved by induction on n, t, k. Alternatively, consider the negative binomial theorem

$$(1-x)^{-(\beta_i+1)} = \sum_{n} \binom{n+\beta_i}{\beta_i} x^n.$$

$$x^{\beta_i}(1-x)^{-(\beta_i+1)} = \sum_n \binom{n+\beta_i}{\beta_i} x^{n+\beta_i} = \sum_{\gamma_i} \binom{\gamma_i}{\beta_i} x^{\gamma_i},$$

so that $\sum_{\gamma_1+\cdots+\gamma_{k+1}=r-t} \prod_{i=1}^k \binom{\gamma_i}{\beta_i}$ is the coefficient of $x^{r-t-\gamma_{k+1}}$ in

$$\prod_{i=1}^k \frac{x^{\beta_i}}{(1-x)^{\beta_i+1}} = \frac{x^t}{(1-x)^{t+k}}.$$

This in turn, is equal to the coefficient of $x^{r-2t-\gamma_{k+1}}$ in $1/(1-x)^{t+k}$. So,

$$\sum_{\gamma_1+\cdots+\gamma_{k+1}=r-t} \prod_{i=1}^k \binom{\gamma_i}{\beta_i} = \sum_{\gamma_{k+1}=0}^{r-t} \binom{(r-\gamma_{k+1}-2t)+(t+k-1)}{t+k-1}$$

$$= \sum_{\gamma_{k+1}=0}^{r-t} \binom{(r-t+k-1-\gamma_{k+1})}{t+k-1}$$

$$= \binom{r-t+k}{t+k}. \qquad \square$$

Now, when s is even we consider the following determinant:

$$N_t = \begin{vmatrix} \sum_{j=1}^{r_1} \alpha_{1j} & \cdots & \sum_{j=1}^{r_i} \alpha_{ij} & \cdots & \sum_{j=1}^{r_k} \alpha_{kj} \\ \sum_{j=1}^{r_1} \alpha_{1j}^2 & \cdots & \sum_{j=1}^{r_i} \alpha_{ij}^2 & \cdots & \sum_{j=1}^{r_k} \alpha_{kj}^2 \\ \vdots & & \vdots & & \vdots \\ \sum_{j=1}^{r_1} \alpha_{1j}^{k-1} & \cdots & \sum_{j=1}^{r_i} \alpha_{ij}^{k-1} & \cdots & \sum_{j=1}^{r_k} \alpha_{kj}^{k-1} \\ \sum_{j=1}^{r_1} \alpha_{1j}^{k+t} & \cdots & \sum_{j=1}^{r_i} \alpha_{ij}^{k+t} & \cdots & \sum_{j=1}^{r_k} \alpha_{kj}^{k+t} \end{vmatrix}$$

with $\alpha_{ij} = a_{ij}(c - a_{ij})$, $r_i = b_i$ for $i = 1, \ldots, k-1$ and $r_k = b_k/2$. Then,

$$N_t = \sum_{1 \le j_i \le b_i} \prod_{i=1}^k \alpha_{ij_i} V_t(\alpha_{1j_1}, \ldots, \alpha_{kj_k}).$$

Now consider

$$\sum_{0\le r\le l}(-1)^{l-r}\nu_{l-r}\sum_{t=0}^{[r/2]}(-1)^t\left[\binom{k+r-t}{k+t}+\binom{k+r-t-1}{k+t-1}\right]c^{r-2t}N_t.$$

It is equal to

$$\sum_{j_i}\prod_{i=1}^{k}\alpha_{ij_i}V(\alpha_{1j_1},\dots,\alpha_{kj_k})$$

$$\cdot\sum_{0\le r\le l}(-1)^{l-r}\nu_{l-r}\sum_{t=0}^{[r/2]}(-1)^t\left[\binom{k+r-t}{k+t}+\binom{k+r-t-1}{k+t-1}\right]$$

$$\times c^{r-2t}\sum_{\sum\beta_i=t}\prod_{i=1}^{k}\alpha_{ij_i}^{\beta_i}.$$

The following is the version of Lemma 4.6 for the case where s is even.

Lemma 4.7. *For all $c, a_i > 0$ and $\alpha_i = a_i(c - a_i)$, we have*

$$\sum_{t=0}^{[r/2]}(-1)^t\left[\binom{k+r-t}{k+t}+\binom{k+r-t-1}{k+t-1}\right]c^{r-2t}\sum_{\sum\beta_i=t}\prod_{i=1}^{k}\alpha_i^{\beta_i}$$

$$=\sum_{\beta_1+\cdots+\beta_s=r}\prod_{i=1}^{k-1}a_i^{\beta_i}(c-a_i)^{\beta_{k+i}}c^{\beta_{2k}}(a_k^{\beta_k}+(c-a_k)^{\beta_k}).$$

Proof. The proof goes along the same lines as the odd case.

$$\sum_{\beta_1+\cdots+\beta_s=r}\prod_{i=1}^{k-1}a_i^{\beta_i}(c-a_i)^{\beta_{k+i}}c^{\beta_{2k}}\left(a_k^{\beta_k}+(c-a_k)^{\beta_k}\right)$$

is the coefficient of x^r in

$$\frac{1}{1-cx}\prod_{i=1}^{k-1}\frac{1}{1-a_ix}\frac{1}{1-(c-a_i)x}\left(\frac{1}{1-a_kx}+\frac{1}{1-(c-a_k)x}\right)$$

$$= \frac{1}{1-cx} \prod_{i=1}^{k} \frac{1}{1-a_i x} \frac{1}{1-(c-a_i)x} \left[1 - (c-a_k)x + 1 - a_k x\right]$$

$$= \frac{1}{1-cx} \prod_{i=1}^{k} \frac{1}{1-a_i x} \frac{1}{1-(c-a_i)x} \left[2 - cx\right]$$

$$= \prod_{i=1}^{k} \frac{1}{1-a_i x} \frac{1}{1-(c-a_i)x} \left[1 + \frac{1}{1-cx}\right].$$

By an argument similar to the proof of Lemma 4.6 applied to the two sums separately, we see that this is the coefficient of x^r in

$$\sum_{t=0}^{[r/2]} (-1)^t \binom{k-1+r-t}{k-1+t} c^{r-2t} \sum_{\sum \beta_i = t} \prod_{i=1}^{k} \alpha_i^{\beta_i}$$

$$+ \sum_{t=0}^{[r/2]} (-1)^t \binom{k+r-t}{k+t} c^{r-2t} \sum_{\sum \beta_i = t} \prod_{i=1}^{k} \alpha_i^{\beta_i}$$

$$= \sum_{t=0}^{[r/2]} (-1)^t \left[\binom{k+r-t}{k+t} + \binom{k+r-t-1}{k+t-1}\right] c^{r-2t} \sum_{\sum \beta_i = t} \prod_{i=1}^{k} \alpha_i^{\beta_i}. \quad \square$$

Now in both cases whether s is even or odd we have

Lemma 4.8.

$$\sum_{0 \le r \le l} (-1)^{l-r} \nu_{l-r} \sum_{\beta_1 + \cdots + \beta_{2k+1} = r} \prod_{i=1}^{k} a_{ij_i}^{\beta_i} (c - a_{ij_i})^{\beta_{k+i}} c^{\beta_{2k+1}}$$

and

$$\sum_{0 \le r \le l} (-1)^{l-r} \nu_{l-r} \sum_{\beta_1 + \cdots + \beta_s = r} \prod_{i=1}^{k-1} a_{ij_i}^{\beta_i} (c - a_{ij_i})^{\beta_{k+i}} c^{\beta_{2k}} (a_{kj_k}^{\beta_k} + (c - a_{kj_k})^{\beta_k})$$

are both equal to

$$\sum_{1 \le i_1 \le \cdots \le i_l \le s} \left(\prod_{t=1}^{l} d_{i_t j_{i_t}} - (i_t + t - 1)\right).$$

Proof. For any given r-tuples $1 \leq \alpha_1 \leq \cdots \leq \alpha_r \leq s$ with $0 \leq r \leq l$, we will have $1 \leq \beta_1 \leq \cdots \leq \beta_{l-r} \leq s$ such that $\{\alpha_1, \ldots, \alpha_r\} \cup \{\beta_1, \ldots, \beta_{l-r}\}$ is equal to $\{i_1, \ldots, i_l\}$. In the product

$$\sum_{1 \leq i_1 \leq \cdots \leq i_l \leq s} \left(\prod_{t=1}^{l} d_{i_t j_{i_t}} - (i_t + t - 1) \right),$$

the coefficient of

$$\prod_{1 \leq \alpha_1 \leq \cdots \leq \alpha_r \leq s} d_{\alpha_t j_{\alpha_t}}$$

is

$$\sum_r (-1)^{l-r} \prod_{1 \leq \beta_1 < \cdots < \beta_{l-r} \leq s+l-1} \beta_1 \cdots \beta_{l-r} = \sum_r (-1)^{l-r} \nu_{l-r},$$

since $i_t + t - 1$ is strictly increasing until $i_l + l - 1$. Further, if s is odd $d_{ij} = a_{ij}$, $i \leq k$, and $d_{ij} = c - a_{(2k+1-i),j}$, $i > k$, and if s is even $d_{ij} = a_{ij}$, $i < k$, $d_{ij} = c - a_{(2k-i),j}$, $i > k$ and $d_{kj} = a_{kj}$ or $c - a_{kj}$. Hence,

$$\sum_{1 \leq i_1 \leq \cdots \leq i_l \leq s} \left(\prod_{t=1}^{l} d_{i_t j_{i_t}} - (i_t + t - 1) \right)$$

$$= \begin{cases} \sum_{0 \leq r \leq l} (-1)^{l-r} \nu_{l-r} \sum_{\beta_1 + \cdots + \beta_{2k+1} = r} \\ \prod_{i=1}^{k} a_{ij_i}^{\beta_i} (c - a_{ij_i})^{\beta_{k+i}} c^{\beta_{2k+1}} & \text{when } s = 2k+1, \\ \sum_{0 \leq r \leq l} (-1)^{l-r} \nu_{l-r} \sum_{\beta_1 + \cdots + \beta_s = r} \\ \prod_{i=1}^{k-1} a_{ij_i}^{\beta_i} (c - a_{ij_i})^{\beta_{k+i}} c^{\beta_{2k}} \\ (a_{kj_k}^{\beta_k} + (c - a_{kj_k})^{\beta_k}) & \text{when } s = 2k. \end{cases}$$

This completes the proof. $\quad\square$

Remark 4.9.

$$\sum_{1 \leq i_1 \leq \cdots \leq i_l \leq s} \left(\prod_{t=1}^{l} d_{i_t j_{i_t}} - (i_t + t - 1) \right) = f_l(a_{1j_1}, \ldots, c),$$

when $s = 2k + 1$ is odd. However, when $s = 2k$, d_{kj} can equal either a_{kj} or $(c - a_{kj})$, and hence

$$\sum_{1 \leq i_1 \leq \cdots \leq i_l \leq s} \left(\prod_{t=1}^{l} d_{i_t j_{i_t}} - (i_t + t - 1) \right)$$
$$= f_l(a_{1j_1}, \ldots, a_{kj_k}, \ldots, c) + f_l(a_{1j_1}, \ldots, c - a_{kj_k}, \ldots, c).$$

Thus, when $s = 2k + 1$,

$$\sum_{0 \leq r \leq l} (-1)^{l-r} \nu_{l-r} \sum_{\beta_1 + \cdots + \beta_{2k+1} = r} \prod_{i=1}^{k} a_{ij_i}^{\beta_i} (c - a_{ij_i})^{\beta_{k+i}} c^{\beta_{2k+1}} = f_l(a_{1j_1}, \ldots, c)$$

and, when $s = 2k$,

$$\sum_{r=0}^{l} \sum_{\beta_1 + \cdots + \beta_s = r} (-1)^{l-r} \nu_{l-r} \prod_{i=1}^{k-1} a_{ij_i}^{\beta_i} (c - a_{ij_i})^{\beta_{k+i}} c^{\beta_{2k}} \left(a_{kj_k}^{\beta_k} + (c - a_{kj_k})^{\beta_k} \right)$$
$$= f_l(a_{1j_1}, \ldots, a_{kj_k}, \ldots, c) + f_l(a_{1j_1}, \ldots, c - a_{kj_k}, \ldots, c).$$

Remark 4.10. Suppose R/I has a quasi-pure resolution. Since the Hilbert function of R/I is unaltered by any cancelations, we may assume $d_{ij} > d_{i-1j}$ for all i. Since $d_{ij} \geq i$ for all i, we get $d_{ij} - i \geq d_{i-1j} - (i-1)$, and hence $d_{pj} - p \geq d_{qj} - q$ for all $p \geq q$. Now assume that not all factors in $\prod_{t=1}^{l} (d_{i_t j_{i_t}} - (i_t + t - 1))$ are positive, and let p be the smallest integer with $d_{i_p j_{i_p}} - (i_p + p - 1) < 0$. Then $p > 1$ and $d_{i_{p-1} j_{i_{p-1}}} - (i_{p-1} + p - 2) > 0$. It follows that

$$d_{i_{p-1} j_{i_{p-1}}} - (i_{p-1} + p - 2) - d_{i_p j_{i_p}} - (i_p + p - 1) \geq 2,$$

or equivalently,

$$i_p - i_{p-1} \geq d_{i_p j_{i_p}} - d_{i_{p-1} j_{i_{p-1}}} + 1$$
$$d_{i_{p-1} j_{i_{p-1}}} - i_{p-1} \geq d_{i_p j_{i_p}} - i_p + 1 > d_{i_p j_{i_p}} - i_p,$$

which is a contradiction, and we get that

$$\prod_{t=1}^{l} (d_{i_t j_{i_t}} - (i_t + t - 1)) \geq 0.$$

Thus, $f_l(d_{1j_1}, \ldots, d_{sj_s}) \geq 0$, for all s-tuples $(d_{1j_1}, \ldots, d_{sj_s})$, provided the resolution is quasi-pure.

Proof of Theorem 4.2. We have $\alpha_{ij} = a_{ij}(c - a_{ij})$ and $\alpha_{ij} - \alpha'_{ij} = (a_{ij} - a'_{ij'})(c - a_{ij} - a'_{ij'})$. By the quasi-purity of the resolution of S and Lemma 4.5, $c \geq 2a_{ij}$ for all i, j. So $a_{i1} \leq a_{i2} \leq \cdots \leq a_{ib_i}$ implies that $\alpha_{i1} \leq \alpha_{i2} \leq \cdots \leq \alpha_{ib_i}$.

We also have, for all $1 \leq i \leq k$, $p_i = \min_j \alpha_{ij} = m_i M_{s-i}$ and $P_i = \max_j \alpha_{ij} = M_i m_{s-i}$.

We treat the even and odd cases separately.

Case 1. $s = 2k + 1$ is odd. The resolution starts as in (4) and, by Lemmas 4.6, 4.8 and Remark 4.9, we have that:

$$\sum_{1 \leq j_i \leq b_i} \prod_{i=1}^{k} \alpha_{ij_i}(c - 2a_{ij_i}) V(\alpha_{1j_1}, \ldots, \alpha_{kj_k})$$

$$\cdot \sum_{r=0}^{l} (-1)^{l-r} \nu_{l-r} \sum_{t=0}^{[r/2]} (-1)^t \binom{k+r-t}{k+t} c^{r-2t} \sum_{\sum \beta_i = t} \prod_{i=1}^{k} \alpha_{ij_i}^{\beta_i}$$

$$= \sum_{1 \leq j_i \leq b_i} \prod_{i=1}^{k} \alpha_{ij_i}(c - 2a_{ij_i}) V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) f_l(a_{1j_1}, \ldots, c).$$

By Remark 4.10, $f_l(d_{1j_1}, \ldots, d_{sj_s}) \geq 0$. Further, $c \geq 2a_{ij_i}$ by Lemma 4.5 and $V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) \geq 0$, for $\alpha_{1j_1} \leq \cdots \leq \alpha_{kj_k}$. So

$$\sum_{j_i} \prod_{i=1}^{k} p_i(c - 2a_{ij_i}) V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) f_l(m_1, \ldots, m_k, M_{k+1}, \ldots, M_s)$$

$$\leq \sum_{j_i} \prod_{i=1}^{k} \alpha_{ij_i}(c - 2a_{ij_i}) V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) f_l(a_{1j_1}, \ldots, c)$$

$$\leq \sum_{j_i} \prod_{i=1}^{k} P_i(c - 2a_{ij_i}) V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) f_l(M_1, \ldots, M_k, m_{k+1}, \ldots, m_s)$$

which is the same as

$$f_l(m_1,\dots,m_k,M_{k+1},\dots,M_s)\prod_{i=1}^{k}p_i\,\det(Q)$$

$$\leq \sum_{j_i}\prod_{i=1}^{k}\alpha_{ij_i}(c-2a_{ij_i})V(\alpha_{1j_1},\dots,\alpha_{kj_k})f_l(a_{1j_1},\dots,c)$$

$$\leq f_l(M_1,\dots,M_k,m_{k+1},\dots,m_s)\prod_{i=1}^{k}P_i\,\det(Q)$$

where

$$Q=\begin{pmatrix}
\displaystyle\sum_{j=1}^{b_1}(c-2a_{1j}) & \cdots & \displaystyle\sum_{j=1}^{b_i}(c-2a_{ij}) & \cdots & \displaystyle\sum_{j=1}^{b_k}(c-2a_{kj}) \\[2ex]
\displaystyle\sum_{j=1}^{b_1}\alpha_{1j}(c-2a_{1j}) & \cdots & \displaystyle\sum_{j=1}^{b_i}\alpha_{ij}(c-2a_{ij}) & \cdots & \displaystyle\sum_{j=1}^{b_k}\alpha_{kj}(c-2a_{kj}) \\[2ex]
\vdots & & \vdots & & \vdots \\[2ex]
\displaystyle\sum_{j=1}^{b_1}\alpha_{1j}^{k-1}(c-2a_{1j}) & \cdots & \displaystyle\sum_{j=1}^{b_i}\alpha_{ij}^{k-1}(c-2a_{ij}) & \cdots & \displaystyle\sum_{j=1}^{b_k}\alpha_{kj}^{k-1}(c-2a_{kj})
\end{pmatrix},$$

$\det(Q)>0$, since at least one of the $V(\alpha_{1j_1},\dots,\alpha_{kj_k})>0$ and $\prod_{i=1}^{k}(c-2a_{ij_i})>0$. Replacing the last column by the alternating sums of columns in Q and using Theorems 3.3 and 3.7, we get

$$\det(Q)=\det\begin{pmatrix}\cdots & -c \\ L & 0\end{pmatrix}=c\cdot\det(L).$$

So

$$f_l(m_1,\dots,m_k,M_{k+1},\dots,M_s)\prod_{i=1}^{k}p_i c\,\det(L)$$

$$\leq \sum_{j_i}\prod_{i=1}^{k}\alpha_{ij_i}(c-2a_{ij_i})V(\alpha_{1j_1},\dots,\alpha_{kj_k})f_l(a_{1j_1},\dots,c)$$

$$\leq f_l(M_1,\dots,M_k,m_{k+1},\dots,m_s)\prod_{i=1}^{k}P_i c\,\det(L)$$

On the other hand, we start with M_t again. Replacing the last column of M_t by alternating sums of the columns and using Theorem 3.5, we get

$$
M_t = \begin{vmatrix} L & & 0 \\ \dots & \displaystyle\sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^{k+t}(c-a_{ij})^{k+t}(c-2a_{ij}) \end{vmatrix}.
$$

So

$$
\sum_{j_i}\prod_{i=1}^{k}\alpha_{ij_i}(c-2a_{ij_i})V(\alpha_{1j_1},\ldots,\alpha_{kj_k})f_l(d_{1j_1},\ldots!,d_{sj_s})=(s+l)!e_l\det(L),
$$

and hence the result.

Case 2. $s=2k$ is even. Now the resolution starts as in (5) and, by Lemmas 4.7, 4.8 and Remark 4.9, we obtain:

$$
\sum_{j_i}\prod_{i=1}^{k}\alpha_{ij_i}V(\alpha_{1j_1},\ldots,\alpha_{kj_k})
$$

$$
\cdot\sum_{0\le r\le l}(-1)^{l-r}\nu_{l-r}\sum_{t=0}^{[r/2]}(-1)^t\left[\binom{k+r-t}{k+t}+\binom{k+r-t-1}{k+t-1}\right]
$$

$$
\cdot c^{r-2t}\sum_{\sum\beta_i=t}\prod_{i=1}^{k}\alpha_{ij_i}^{\beta_i}
$$

$$
=\sum_{1\le j_i\le b_i}\prod_{i=1}^{k}\alpha_{ij_i}V(\alpha_{1j_1},\ldots,\alpha_{kj_k})
$$

$$
\cdot\left(f_l(a_{1j_1},\ldots,a_{kj_k},\ldots,c)+f_l(a_{1j_1},\ldots,c-a_{kj_k},\ldots,c)\right).
$$

Since $f_l(d_{1j_1}, \ldots, d_{sj_s}) \geq 0$ and $V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) \geq 0$, we have:

$$
\begin{aligned}
\sum_{j_i} \prod_{i=1}^k & p_i V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) \left(f_l\left(m_1, \ldots, m_k, M_{k+1}, \ldots, M_s\right)\right. \\
& \left. + f_l(m_1, \ldots, m_{k-1}, M_k, \ldots, M_s)\right) \\
\leq \sum_{1 \leq j_i \leq b_i} \prod_{i=1}^k & \alpha_{ij_i} V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) \left(f_l(a_{1j_1}, \ldots, a_{kj_k}, \ldots, c)\right. \\
& \left. + f_l(a_{1j_1}, \ldots, c - a_{kj_k}, \ldots, c)\right) \\
\leq \sum_{j_i} \prod_{i=1}^k & P_i V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) \left(f_l\left(M_1, \ldots, M_k, m_{k+1}, \ldots, m_s\right)\right. \\
& \left. + f_l(M_1, \ldots, M_{k-1}, m_k, \ldots, m_s)\right).
\end{aligned}
$$

But,

$$
f_l\left(m_1, \ldots, m_k, M_{k+1}, \ldots, M_s\right) \leq f_l(m_1, \ldots, m_{k-1}, M_k, \ldots, M_s)
$$

and

$$
f_l(M_1, \ldots, M_{k-1}, m_k, \ldots, m_s) \leq f_l\left(M_1, \ldots, M_k, m_{k+1}, \ldots, m_s\right).
$$

Therefore,

$$
\begin{aligned}
\sum_{j_i} \prod_{i=1}^k & p_i V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) \cdot 2 f_l\left(m_1, \ldots, m_k, M_{k+1}, \ldots, M_s\right) \\
\leq \sum_{1 \leq j_i \leq b_i} \prod_{i=1}^k & \alpha_{ij_i} V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) \left(f_l(a_{1j_1}, \ldots, a_{kj_k}, \ldots, c)\right. \\
& \left. + f_l(a_{1j_1}, \ldots, c - a_{kj_k}, \ldots, c)\right) \\
\leq \sum_{j_i} \prod_{i=1}^k & P_i V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) \cdot 2 f_l\left(M_1, \ldots, M_k, m_{k+1}, \ldots, m_s\right),
\end{aligned}
$$

which is the same as

$$2f_l\left(m_1, \ldots, m_k, M_{k+1}, \ldots, M_s\right) \prod_{i=1}^{k} p_i \, \det\left(Q'\right)$$

$$\leq \sum_{1 \leq j_i \leq b_i} \prod_{i=1}^{k} \alpha_{ij_i} V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) \, (f_l(a_{1j_1}, \ldots, a_{kj_k}, \ldots, c)$$

$$+ f_l(a_{1j_1}, \ldots, c - a_{kj_k}, \ldots, c))$$

$$\leq 2f_l\left(M_1, \ldots, M_k, m_{k+1}, \ldots, m_s\right) \prod_{i=1}^{k} P_i \, \det\left(Q'\right)$$

where

$$Q' = \begin{pmatrix} \overset{r_1}{\underset{j=1}{\sum}} \alpha_{1j} & \cdots & \overset{r_i}{\underset{j=1}{\sum}} \alpha_{ij} & \cdots & \overset{r_k}{\underset{j=1}{\sum}} \alpha_{kj} \\ \vdots & & \vdots & & \vdots \\ \overset{r_1}{\underset{j=1}{\sum}} \alpha_{1j}^{k-1} & \cdots & \overset{r_i}{\underset{j=1}{\sum}} \alpha_{ij}^{k-1} & \cdots & \overset{r_k}{\underset{j=1}{\sum}} \alpha_{kj}^{k-1} \end{pmatrix},$$

$\det\left(Q'\right) > 0$ since at least one of the $V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) > 0$. Replacing the last column by the alternating sums of columns and using Theorems 3.4 and 3.7, we get

$$\det\left(Q'\right) = \det \begin{pmatrix} \cdots & -1 \\ L' & 0 \end{pmatrix} = \det\left(L'\right).$$

Then

$$2f_l\left(m_1, \ldots, m_k, M_{k+1}, \ldots, M_s\right) \prod_{i=1}^{k} p_i \, \det\left(L'\right)$$

$$\leq \sum_{1 \leq j_i \leq b_i} \prod_{i=1}^{k} \alpha_{ij_i} V(\alpha_{1j_1}, \ldots, \alpha_{kj_k}) \, (f_l(a_{1j_1}, \ldots, a_{kj_k}, \ldots, c)$$

$$+ f_l(a_{1j_1}, \ldots, c - a_{kj_k}, \ldots, c))$$

$$\leq 2f_l\left(M_1, \ldots, M_k, m_{k+1}, \ldots, m_s\right) \prod_{i=1}^{k} P_i \, \det\left(L'\right).$$

On the other hand, we start with N_t again. Replacing the last column of N_t by alternating sums of the columns and using Theorem 3.7, we get:

$$N_t = \begin{vmatrix} L' & 0 \\ \dots & \displaystyle\sum_{i=1}^{k}\sum_{j=1}^{b_i}(-1)^i a_{ij}^{k+t}(c - a_{ij})^{k+t} \end{vmatrix}.$$

So

$$(s + l)!e_l \det (L') = \sum_{1 \le j_i \le b_i} \prod_{i=1}^{k} \alpha_{ij_i} V(\alpha_{1j_1}, \dots, \alpha_{kj_k})$$
$$(f_l(a_{1j_1}, \dots, a_{kj_k}, \dots, c) + f_l(a_{1j_1}, \dots, c - a_{kj_k}, \dots, c)).$$

We get:

$$2f_l(m_1, \dots, m_k, M_{k+1}, \dots, M_s) \prod_{i=1}^{k} p_i \le (s + l)!e_l$$

$$\le 2f_l(M_1, \dots, M_k, m_{k+1}, \dots, m_s) \prod_{i=1}^{k} P_i,$$

$$2f_l(m_1, \dots, m_k, M_{k+1}, \dots, M_s) m_1 \cdots m_k M_k M_{k+1} \cdots M_{2k-1}$$
$$\le (s + l)!e_l$$
$$\le 2f_l(M_1, \dots, M_k, m_{k+1}, \dots, m_s) M_1 \cdots M_k m_k m_{k+1} \cdots m_{2k-1},$$

$$f_l(m_1, \dots, m_k, M_{k+1}, \dots, M_s) m_1 \cdots m_k M_{k+1} \cdots M_{2k-1}(2M_k)$$
$$\le (s + l)!e_l$$
$$\le f_l(M_1, \dots, M_k, m_{k+1}, \dots, m_s) M_1 \cdots M_k m_{k+1} \cdots m_{2k-1}(2m_k).$$

Now $M_k = c - a_{k1}$ and $m_k = a_{k1}$. Clearly, $c \ge 2a_{k1} = 2m_k$. Also, $2(c - a_{k1}) = c + (c - 2a_{k1}) \ge c$. So $M_{2k} = c \le 2M_k$ and $2m_k \le c = m_{2k}$, and hence,

$$f_l(m_1, \dots, m_k, M_{k+1}, \dots, M_s) m_1 \cdots m_k M_{k+1} \cdots M_{2k-1} M_{2k}$$
$$\le (s + l)!e_l$$
$$\le f_l(M_1, \dots, M_k, m_{k+1}, \dots, m_s) M_1 \cdots M_k m_{k+1} \cdots m_{2k-1} m_{2k}.$$

This completes the proof. $\square$

Let $R = k[x_1, \dots, x_n]$ and $S = R/I$ where I is a homogeneous ideal of height s.

Corollary 4.11. *Let $S = R/I$ be as above. Suppose the Betti diagram of S is quasi-pure symmetric, that is, $\beta_{ij} = \beta_{s-i,c-j}$, $0 \leq i \leq s$, and $\beta_{sj} = 0, j \neq c$. Then, the Hilbert coefficients will satisfy the same bounds as in Theorem 4.2.*

Corollary 4.12. *Let $S = R/I$ be as above. Suppose the Betti diagram of S is a positive rational linear combinations of quasi-pure Gorenstein (symmetric) Betti diagrams. Then, the Hilbert coefficients of S satisfy the same bounds as in Theorem 4.2.*

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