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BASS NUMBERS OVER LOCAL RINGS VIA STABLE COHOMOLOGY

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To Jürgen Herzog on his 70th birthday.

ABSTRACT. For any non-zero finite module M of finite projective dimension over a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field k, it is proved that the natural map $\mathrm{Ext}_R(k,M) \to \mathrm{Ext}_R(k,M/\mathfrak{m}M)$ is non-zero when R is regular and is zero otherwise. A noteworthy aspect of the proof is the use of stable cohomology. Applications include computations of Bass series over certain local rings.

1. Introduction. Let $(R,\mathfrak{m},k)$ denote a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k; when R is not regular we say that it is *singular*.

This article revolves around the following result:

Theorem. *If* $(R,\mathfrak{m},k)$ *is a singular local ring and* M *an* R*-module of finite projective dimension, then* $\mathrm{Ext}_R(k,\pi^M)=0$ *for the canonical map* $\pi^M : M \to M/\mathfrak{m}M$.

Special cases, known for a long time, are surveyed at the end of Section 2. Even in those cases our proof is new. It utilizes a result of Martsinkovsky [**11**] through properties of Vogel's stable cohomology functors [**3, 6**] recalled in Section 1. It also suggests extensions to DG modules over certain commutative DG algebras; these will be discussed in [**2**]. Applications of the theorem include new criteria for regularity of local rings (in Section 2) and explicit computations of Bass numbers of modules (in Section 3).

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1. Stable cohomology. In this section we recall the construction of stable cohomology and basic results required in the sequel. The approach we adopt is based on a construction by Vogel and described in Goichot [**6**]; see also [**3**].

Let R be an associative ring, and let R^{c} denote its center. Given left R-modules L and M, choose projective resolutions P and Q of L and M, respectively. Recall that a homomorphism $P \to Q$ of degree n is a family $\beta = (\beta_i)_{i\in\mathbf{Z}}$ of R-linear maps $\beta_i : P_i \to Q_{i+n}$, that is, an element of the R^{c}-module

$$\operatorname{Hom}_R(P,Q)_n = \prod_{i\in\mathbf{Z}} \operatorname{Hom}_R(P_i, Q_{i+n}).$$

This module is the nth component of a complex $\operatorname{Hom}_R(P,Q)$, with differential

$$\partial_n(\beta)_i = \partial^Q_{i+n}\beta_i - (-1)^n\beta_{i-1}\partial^P_i .$$

The maps $\beta : P \to Q$ with $\beta_i = 0$ for $i \gg 0$ form a subcomplex with component

$$\overline{\operatorname{Hom}}_R(P,Q)_n = \coprod_{i\in\mathbf{Z}} \operatorname{Hom}_R(P_i, Q_{i+n}) \quad \text{for } n \in \mathbf{Z}.$$

We write $\widehat{\operatorname{Hom}}_R(P,Q)$ for the quotient complex. It is independent of the choices of P and Q up to R-linear homotopy, and so is the exact sequence of complexes

$$0 \longrightarrow \overline{\operatorname{Hom}}_R(P,Q) \longrightarrow \operatorname{Hom}_R(P,Q) \xrightarrow{\theta} \widehat{\operatorname{Hom}}_R(P,Q) \longrightarrow 0. \tag{1.0.1}$$

The *stable cohomology* of the pair (L,M) is the graded R^{c}-module $\widehat{\operatorname{Ext}}_R(L,M)$ with

$$\widehat{\operatorname{Ext}}{}^n_R(L,M) = \operatorname{H}^n(\widehat{\operatorname{Hom}}_R(P,Q)) \quad \text{for each } n \in \mathbf{Z}.$$

It is equipped with functorial homomorphisms of graded R^{c}-modules

$$\operatorname{Ext}^n_R(L,M) \xrightarrow{\eta^n(L,M)} \widehat{\operatorname{Ext}}{}^n_R(L,M) \quad \text{for all } n \in \mathbf{Z}. \tag{1.0.2}$$

1.1. If $\mathrm{pd}_R L$ or $\mathrm{pd}_R M$ is finite, then $\widehat{\mathrm{Ext}}^n_R(L,M)=0$ for all $n\in\mathbf{Z}$.

Indeed, in this case we may choose P or Q to be a bounded complex. The definitions then yield $\overline{\mathrm{Hom}}_R(P,Q) = \mathrm{Hom}_R(P,Q)$, and hence $\widehat{\mathrm{Hom}}_R(P,Q)=0$.

1.2. For a family $\{M_j\}_{j\in J}$ of R-modules and every integer n, the canonical inclusions $M_j \to \coprod_{j\in J} M_j$ induce, by functoriality, a commutative diagram of R^{c}-modules

$$\begin{array}{ccc}
\mathrm{Ext}^n_R\Big(L,\coprod_{j\in J} M_j\Big) & \xrightarrow{\ \eta^n(L,\coprod_{j\in J}M_j)\ } & \widehat{\mathrm{Ext}}^n_R\Big(L,\coprod_{j\in J} M_j\Big) \\
\uparrow & & \uparrow \\
\coprod_{j\in J}\mathrm{Ext}^n_R(L,M_j) & \xrightarrow[\ \coprod_{j\in J}\eta^n(L,M_j)\ ]{} & \coprod_{j\in J}\widehat{\mathrm{Ext}}^n_R(L,M_j)
\end{array} \tag{1.2.1}$$

Proposition 1.3. *Suppose L admits a resolution by finite projective R-modules.*

For every n, the vertical maps in (1.2.1) *are bijective. In particular, the map $\eta^n(L,\coprod_{j\in J}M_j)$ is injective or surjective for some n if and only if $\eta^n(L,M_j)$ has the corresponding property for every $j\in J$.*

Proof. Let P be a resolution of L by finite projective R-modules and Q_j a projective resolution of M_j. The complex $\coprod_{j\in J} Q_j$ is a projective resolution of $\coprod_{j\in J} M_j$, and we have a commutative diagram of morphisms of complexes of R^{c}-modules

$$\begin{array}{ccccccccc}
0 & \to & \overline{\mathrm{Hom}}_R(P,\coprod_{j\in J}Q_j) & \to & \mathrm{Hom}_R(P,\coprod_{j\in J}Q_j) & \to & \widehat{\mathrm{Hom}}_R(P,\coprod_{j\in J}Q_j) & \to & 0 \\
 & & \Big\uparrow{\scriptstyle \overline{\varkappa}\,\cong} & & \Big\uparrow{\scriptstyle \varkappa} & & \Big\uparrow{\scriptstyle \hat{\varkappa}} & & \\
0 & \to & \coprod_{j\in J}\overline{\mathrm{Hom}}_R(P,Q_j) & \to & \coprod_{j\in J}\mathrm{Hom}_R(P,Q_j) & \to & \coprod_{j\in J}\widehat{\mathrm{Hom}}_R(P,Q_j) & \to & 0
\end{array}$$

with natural vertical maps. The map $\mathrm{H}^n(\varkappa)$ is bijective, as it represents

$$\coprod_{j\in J} \mathrm{Ext}_R^n(L, M_j) \longrightarrow \mathrm{Ext}_R^n\Big(L, \coprod_{j\in J} M_j\Big),$$

which is bijective due to the hypothesis on L. As $\overline{\varkappa}$ is evidently bijective, $\mathrm{H}^n(\widehat{\varkappa})$ is an isomorphism. The right-hand square of the diagram above induces (1.2.1). □

2. Local rings. The next theorem is the main result of the paper. It concerns the maps

$$\mathrm{Ext}_R^n(k,\beta) : \mathrm{Ext}_R^n(k, M) \longrightarrow \mathrm{Ext}_R^n(k, V)$$

induced by some homomorphism $\beta : M \to V$, and is derived from a result of Martsinkovsky [**11**] by using properties of stable cohomology, described in Section 1.

Theorem 2.1. *Let $(R, \mathfrak{m}, k)$ be a local ring and V an R-module such that $\mathfrak{m}V = 0$.*

If R is singular and $\beta : M \to V$ is an R-linear map that factors through some module of finite projective dimension, then

$$\mathrm{Ext}_R^n(k,\beta) = 0 \quad \textit{for all } n \in \mathbf{Z}.$$

Proof. By hypothesis, there exists an R-module N of finite projective dimension such that β factors as $M \xrightarrow{\gamma} N \xrightarrow{\delta} V$. The following diagram

$$\begin{array}{ccccc}
\mathrm{Ext}_R^n(k,M) & \xrightarrow{\mathrm{Ext}_R^n(k,\beta)} & \mathrm{Ext}_R^n(k,V) & \xrightarrow{\eta^n(k,V)} & \widehat{\mathrm{Ext}}{}_R^n(k,V) \\
& \searrow{\scriptstyle \mathrm{Ext}_R^n(k,\gamma)} & \uparrow{\scriptstyle \mathrm{Ext}_R^n(k,\delta)} & & \uparrow{\scriptstyle \widehat{\mathrm{Ext}}{}_R^n(k,\delta)} \\
& & \mathrm{Ext}_R^n(k,N) & \xrightarrow[\eta^n(k,N)]{} & \widehat{\mathrm{Ext}}{}_R^n(k,N) = 0
\end{array}$$

commutes due to the naturality of the maps involved; the equality comes from 1.1. The map $\eta^n(k,k)$ is injective by [**11**, Theorem 6]. Proposition 1.3 shows that $\eta^n(k,V)$ is injective as well, so the diagram yields $\mathrm{Ext}_R^n(k,\beta) = 0$. □

Note that no finiteness condition on M is imposed in the theorem. This remark is used in the proof of the following corollary, which deals with the maps

$$\operatorname{Tor}^n_R(k,\alpha) : \operatorname{Tor}^R_n(k,V) \longrightarrow \operatorname{Tor}^R_n(k,M)$$

induced by some homomorphism $\alpha : V \to M$.

Corollary 2.2. *If R is singular and $\alpha : V \to M$ is an R-linear map that factors through some module of finite injective dimension, then*

$$\operatorname{Tor}^R_n(k,\alpha) = 0 \quad \textit{for all } n \in \mathbf{Z}.$$

Proof. Set $(-)^\vee = \operatorname{Hom}_R(-,E)$, where E is an injective envelope of the R-module k. Let $V \to L \to M$ be a factorization of α with L of finite injective dimension. By Ishikawa [**7**, 1.5], the module $L^\vee$ has finite flat dimension, so it has finite projective dimension by Jensen [**9**, 5.8]. As $\mathfrak{m}(V^\vee) = 0$ and $\alpha^\vee$ factors through $L^\vee$, Theorem 2.1 gives $\operatorname{Ext}^R_n(k,\alpha^\vee) = 0$. The natural isomorphism $\operatorname{Ext}^n_R(k,-^\vee) \cong \operatorname{Tor}^R_n(k,-)^\vee$ now yields $\operatorname{Tor}^R_n(k,\alpha)^\vee = 0$, whence we get $\operatorname{Tor}^R_n(k,\alpha) = 0$, as desired. □

Next we record an elementary observation, where $(-)^* = \operatorname{Hom}_R(-,R)$.

Lemma 2.3. *Let $(R,\mathfrak{m},k)$ be a local ring and $\chi : X \to Y$ an R-linear map. If* $\operatorname{Coker}(\chi)$ *has a non-zero free summand, then* $\operatorname{Ker}(\chi^*) \nsubseteq \mathfrak{m}Y^*$ *holds. When Y is free of finite rank the converse holds as well.*

Proof. The condition on $\operatorname{Coker}(\chi)$ holds if and only if there is an epimorphism $\operatorname{Coker}(\chi) \to R$, that is, an R-linear map $\upsilon : Y \to R$ with $\upsilon\chi = 0$ and $\upsilon(Y) \nsubseteq \mathfrak{m}$. When such an υ exists, it is in $\operatorname{Ker}(\chi^*)$, but not in $\mathfrak{m}Y^*$, for otherwise $\upsilon(Y) \subseteq \mathfrak{m}$.

When Y is finite free and $\operatorname{Ker}(\chi^*) \nsubseteq \mathfrak{m}Y^*$ holds, pick υ in $\operatorname{Ker}(\chi^*)$, but not in $\mathfrak{m}Y^*$. Since Y^* is finite free, υ can be extended to a basis of Y^*; hence, $\upsilon(Y) = R$. □

The theorem in the introduction is the crucial implication in the next result:

Theorem 2.4. *Let $(R, \mathfrak{m}, k)$ be a local ring, M an R-module, and*

$$\varepsilon_M^n = \operatorname{Ext}_R^n(k, \pi^M) : \operatorname{Ext}_R^n(k, M) \longrightarrow \operatorname{Ext}_R^n(k, M/\mathfrak{m}M)$$

the homomorphism induced by the natural map $\pi^M : M \to M/\mathfrak{m}M$.

The following conditions are equivalent.

(i) *R is regular.*

(ii) *$\varepsilon_R^n \neq 0$ for some n.*

(iii) *$\varepsilon_M^n \neq 0$ for an R-module M with $\operatorname{pd}_R M < \infty$ and some n.*

(iv) *$\varepsilon_M^d \neq 0$ for every finite R-module $M \neq 0$ and for $d = \dim R$.*

(v) $\operatorname{Coker}(\partial_n^F)$, *where F is a minimal free resolution of k over R, has a non-zero free direct summand for some n.*

Proof. Set $G = \operatorname{Hom}_R(F, R)$ with F as in (v).

From $\operatorname{Hom}_R(F, M) \cong G \otimes_R M$ and $\partial(G \otimes M) \subseteq \mathfrak{m}(G \otimes M)$ (by the minimality of F) we get a commutative diagram

$$\begin{array}{ccc}
\operatorname{Ext}_R^n(k, M) & \xrightarrow{\ \varepsilon_M^n\ } & \operatorname{Ext}_R^n(k, M/\mathfrak{m}M) \\
\cong \uparrow & & \uparrow \cong \\
\operatorname{H}_{-n}(G \otimes_R M) & \xrightarrow{\operatorname{H}_{-n}(G \otimes_R \pi^M)} & \operatorname{H}_{-n}(G \otimes_R (M/\mathfrak{m}M)) = G_{-n} \otimes (M/\mathfrak{m}M)
\end{array}$$

(i) $\implies$ (iv). As R is regular, F is the Koszul complex on a minimal generating set of $\mathfrak{m}$. This gives $G_d = R$, an isomorphism $\operatorname{H}_{-d}(G \otimes_R \pi^M)$, and an inequality $M/\mathfrak{m}M \neq 0$ by Nakayama's lemma; now the diagram yields $\varepsilon_M^d \neq 0$.

(iv) $\implies$ (ii) $\implies$ (iii). These implications are tautologies.

(iii) $\implies$ (i). This implication is a special case of Theorem 2.1.

(ii) $\iff$ (v). The preceding diagram shows that the condition $\varepsilon_R^n \neq 0$ is equivalent to $\operatorname{Ker}(\partial_{-n}^G) \nsubseteq \mathfrak{m}G_{-n}$. Thus, the desired assertion follows from Lemma 2.3. □

Notes 2.5. The equivalence of conditions (i) and (ii) in Theorem 2.4 was proved by Ivanov [**8**, Theorem 2] when R is Gorenstein and by Lescot [**10**, 1.4] in general.

The equivalence of (i) and (v) is due to Dutta [**4**, 1.3]. As shown above, it follows from Lescot's theorem via the elementary Lemma 2.3. Martsinkovsky deduced Dutta's theorem from [**11**, Theorem 6] and used the latter to prove regularity criteria different from (ii), (iii) and (iv) in Theorem 2.4, see [**11**, page 11].

3. Bass numbers of modules. The *nth Bass number* of a module M over a local ring $(R,\mathfrak{m},k)$ is the integer

$$\mu_R^n(M) = \operatorname{rank}_k \operatorname{Ext}_R^n(k,M).$$

Given a homomorphism $\beta : M \to N$ and an R-submodule $N' \subseteq N$, we let $M \cap N'$ denote the submodule $\beta^{-1}(N')$ of M.

Theorem 3.1. *Let $(R,\mathfrak{m},k)$ be a local ring, $M \to N$ an R-linear map, and set*

$$r = \operatorname{rank}_k(M/M \cap \mathfrak{m}N).$$

If R is singular and $\operatorname{pd}_R N$ is finite, then there is an equality

$$\mu_R^n(M \cap \mathfrak{m}N) = \mu_R^n(M) + r\mu_R^{n-1}(k), \quad \textit{for each } n \in \mathbf{Z}.$$

Proof. Set $\overline{M} = M/(M \cap \mathfrak{m}N)$ and $\overline{N} = N/\mathfrak{m}N$, and let $\pi : M \to \overline{M}$ and $\iota : \overline{M} \to \overline{N}$ be the induced maps. They appear in a commutative diagram with exact rows

$$\begin{CD}
@. M @>>> N \\
@. @V{\pi}VV @VVV \\
0 @>>> \overline{M} @>{\iota}>> \overline{N}
\end{CD}$$

Since ι is k-linear, it is split, so we get a commutative diagram with exact rows

$$\begin{CD}
@. \operatorname{Ext}_R(k,M) @>>> \operatorname{Ext}_R(k,N) \\
@. @V{\operatorname{Ext}_R(k,\pi)}VV @VV{0}V \\
0 @>>> \operatorname{Ext}_R(k,\overline{M}) @>>{\operatorname{Ext}_R(k,\iota)}> \operatorname{Ext}_R(k,\overline{N})
\end{CD}$$

and zero map due to Theorem 2.1. It implies $\mathrm{Ext}_R(k,\pi)=0$.

By definition, there exists an exact sequence of R-modules

$$0 \longrightarrow (M \cap \mathfrak{m}N) \longrightarrow M \xrightarrow{\pi} \overline{M} \longrightarrow 0.$$

As $\mathrm{Ext}_R(k,\pi)=0$, its cohomology sequence yields an exact sequence

$$0 \longrightarrow \mathrm{Ext}_R^{n-1}(k,\overline{M}) \longrightarrow \mathrm{Ext}_R^{n}(k,M\cap \mathfrak{m}N) \longrightarrow \mathrm{Ext}_R^{n}(k,M) \longrightarrow 0$$

of k-vector spaces for each integer n. Computing ranks over k and using the isomorphism $\mathrm{Ext}_R(k,\overline{M}) \cong \mathrm{Ext}_R(k,k)\otimes_k \overline{M}$, we obtain the desired equality. □

Recall that the nth *Betti number* of M is the integer

$$\beta_n^R(M) = \mathrm{rank}_k \mathrm{Ext}_R^n(M,k).$$

Corollary 3.2. *Assume R is singular, N is a finite R-module, $N \supseteq M \supseteq \mathfrak{m}N$ holds, and set*

$$s = \mathrm{rank}_k(N/M).$$

If $\mathrm{pd}_R N = p < \infty$ holds, then for each $n \in \mathbf{Z}$ there is an equality

$$\mu_R^n(M) = \sum_{i=0}^{p} \mu_R^{n+i}(R)\beta_i^R(N) + s\beta_{n-1}^R(k). \tag{3.2.1}$$

In particular, $\mu_R^n(\mathfrak{m}) = \mu_R^n(R) + \beta_{n-1}^R(k)$ for each $n \geq 0$.

Proof. The hypotheses give $M \cap \mathfrak{m}N = \mathfrak{m}N$ and $r = \mathrm{rank}_k(M/\mathfrak{m}N)$. Apply Theorem 3.1 to the inclusions $M \subseteq N$ and $\mathfrak{m}N \subseteq N$ to get

$$\begin{aligned}\mu_R^n(M) &= \mu_R^n(\mathfrak{m}N) - r\beta_{n-1}^R(k)\\ &= \mu_R^n(N) + \mathrm{rank}_k(N/\mathfrak{m}N)\beta_{n-1}^R(k) - r\beta_{n-1}^R(k)\\ &= \mu_R^n(N) + s\beta_{n-1}^R(k).\end{aligned}$$

As $\mathrm{pd}_R N$ is finite, Foxby [**5**, 4.3(2)] yields

$$\mu_R^n(N) = \sum_{i=0}^{p} \mu_R^{n+i}(R)\beta_i^R(N). \qquad \square$$

Remark 3.3. The hypothesis $p < \infty$ in the corollary is needed, as otherwise the sum in (3.2.1) is not defined. On the other hand, when R is regular, and so p is necessarily finite, formula (3.2.1) may fail. For instance, with $d = \dim R$, one has

$$\mu_R^n(\mathfrak{m}) = \begin{cases} \binom{d}{n-1} & \text{for } n = 1, \dots, d, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, $0 \to \mathfrak{m} \to R \to k \to 0$ induces a cohomology long exact sequence where $\mu_R^i(R) = 0$ for $n \neq d$, the map ε_R^d is bijective by Theorem 2.4, and $\mu_R^i(k) = \binom{d}{i}$.

3.4. Bass numbers are often described in terms of the generating formal power series $I_R^M(t) = \sum_{n\in\mathbf{Z}} \mu_R^n(M)t^n$. We also use the series $P_M^R(t) = \sum_{n\in\mathbf{Z}} \beta_n^R(M)t^n$.

In these terms, the formulas (3.2.1) can be restated as an equality

$$I_R^M(t) = I_R^R(t)P_N^R(t^{-1}) + stP_k^R(t). \tag{3.4.1}$$

3.5. Let $(S, \mathfrak{m}_S, k)$ and $(T, \mathfrak{m}_T, k)$ be local rings, and let

$$\varepsilon_S : S \to k \leftarrow T : \varepsilon_T$$

be the canonical maps. The *fiber product* is defined by the formula

$$S \times_k T := \{(s, t) \in S \times T \mid \varepsilon_S(s) = \varepsilon_T(t)\}.$$

This is a subring of $S \times T$, which is local with maximal ideal $\mathfrak{m} = \mathfrak{m}_S \oplus \mathfrak{m}_T$ and residue field k. Set $R = S \times_k T$.

Let N and P be finite modules over S and T, respectively. The canonical maps $S \leftarrow R \to T$ turn N and P into R-modules, and for them Lescot [**10**, 2.4] proved

$$\frac{I_R^N(t)}{P_k^R(t)} = \frac{I_S^N(t)}{P_k^S(t)} \quad \text{and} \quad \frac{I_R^P(t)}{P_k^R(t)} = \frac{I_T^P(t)}{P_k^T(t)}. \tag{3.5.1}$$

If $N/\mathfrak{n}_S N = V = P/\mathfrak{m}_T P$ holds for some k-module V, then

$$N \times_V P := \{(n,p) \in N \times P \mid \pi^N(n) = \pi^P(p)\}$$

has a natural structure of finite R-module.

Corollary 3.6. *With notation as in* 3.5*, set* $v = \operatorname{rank}_k V$ *and* $M = N \times_V P$*.*

If S *and* T *are singular and* $\operatorname{pd}_S N$ *and* $\operatorname{pd}_T P$ *are finite, then*

$$\frac{I_R^{\mathfrak{m}M}}{P_k^R(t)} = \frac{I_S^S(t) P_N^S(t^{-1})}{P_k^S(t)} + \frac{I_T^T(t) P_P^T(t^{-1})}{P_k^T(t)} + 2vt.$$

Proof. We have $\mathfrak{m}M \cong \mathfrak{m}_S N \oplus \mathfrak{m}_T P$ as R-modules, whence the first equality below:

$$\begin{aligned}\frac{I_R^{\mathfrak{m}M}(t)}{P_k^R(t)} &= \frac{I_R^{\mathfrak{m}_S N}(t)}{P_k^R(t)} + \frac{I_R^{\mathfrak{m}_T P}(t)}{P_k^R(t)} \\ &= \frac{I_S^S(t) P_N^S(t^{-1})}{P_k^S(t)} + vt + \frac{I_T^T(t) P_P^T(t^{-1})}{P_k^T(t)} + vt.\end{aligned}$$

The second one comes from (3.5.1) and (3.4.1), in this order. ◻

Notes 3.7. For all finite R-modules $N \supsetneq M \supseteq \mathfrak{m}N$, it is proved in [**1**, Theorem 4] that the Bass numbers of M and k *asymptotically* have the same size, measured on appropriate polynomial or exponential scales. The *closed formula* in Corollary 3.2 is a much more precise statement, but as noted in Remark 3.3 that formula may not hold when $\operatorname{pd}_R N$ is infinite or when R is regular.

The last conclusion in Corollary 3.2 is Lescot's result [**10**, 1.8(2)]. Combining it with the expression for $I_R^R(t)$ obtained from Corollary 3.6 by setting $N = S$, $P = T$ and $V = k$, one recovers [**10**, 3.2(1)]. The proof of Corollary 3.6 faithfully transposes Lescot's derivation of [**10**, 3.2(1)] from [**10**, 1.8(2)].

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