

## ON THE CHERN NUMBER OF $I$ -ADMISSIBLE FILTRATIONS OF IDEALS

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**ABSTRACT.** Let  $I$  be an  $\mathfrak{m}$ -primary ideal of a Noetherian local ring  $(R, \mathfrak{m})$  of positive dimension. The coefficient  $e_1(\mathcal{I})$  of the Hilbert polynomial of an  $I$ -admissible filtration  $\mathcal{I}$  is called the Chern number of  $\mathcal{I}$ . A formula for the Chern number has been derived involving the Euler characteristic of subcomplexes of a Koszul complex. Specific formulas for the Chern number have been given in local rings of dimension at most two. These have been used to provide new and unified proofs of several results about  $e_1(\mathcal{I})$ .

**Introduction.** Let  $(R, \mathfrak{m})$  be a Noetherian local ring of dimension  $d$ , and let  $I$  be an  $\mathfrak{m}$ -primary ideal. A sequence of ideals  $\mathcal{I} = \{I_n\}_{n \in \mathbf{Z}}$  is called an  $I$ -admissible filtration if a  $k \in \mathbf{N}$  exists such that, for all  $n, m \in \mathbf{Z}$ ,

$$I_{n+1} \subseteq I_n, \quad I_m I_n \subseteq I_{n+m} \quad \text{and} \quad I^n \subseteq I_n \subseteq I^{n-k}.$$

The Rees algebra  $\mathcal{R}(\mathcal{I})$  and the associated graded ring  $G(\mathcal{I})$  of the filtration  $\mathcal{I}$  are defined as

$$\mathcal{R}(\mathcal{I}) = \bigoplus_{n \in \mathbf{Z}} I_n t^n, \quad G(\mathcal{I}) = \bigoplus_{n \geq 0} I_n / I_{n+1}.$$

For the  $I$ -adic filtration  $\mathcal{I} = \{I^n\}$ , we put  $\mathcal{R}(\mathcal{I}) = \mathcal{R}(I)$  and  $G(\mathcal{I}) = G(I)$ . Note that  $\mathcal{I}$  is an  $I$ -admissible filtration if and only if  $\mathcal{R}(\mathcal{I})$  is a finitely generated  $\mathcal{R}(I)$ -module. Rees [10] proved that the integral closure filtration  $\{\overline{I^n}\}$  is an  $I$ -admissible filtration if and only if  $R$  is analytically unramified. Marley [9] showed that if  $\mathcal{I}$  is an  $I$ -admissible filtration, then the Hilbert function  $H_{\mathcal{I}}(n) = \lambda(R/I_n)$ , where  $\lambda$  denotes length as an  $R$ -module, coincides with a polynomial  $P_{\mathcal{I}}(x) \in \mathbf{Q}[x]$  of degree  $d$  for large  $n$ . This polynomial is written as

$$P_{\mathcal{I}}(x) = e_0(\mathcal{I}) \binom{x+d-1}{d} - e_1(\mathcal{I}) \binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{I}),$$

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Received by the editors on November 6, 2011, and in revised form on April 25, 2012.

DOI:10.1216/JCA-2012-4-4-577 Copyright ©2012 Rocky Mountain Mathematics Consortium

and it is called the *Hilbert polynomial* of the  $I$ -admissible filtration  $\mathcal{I}$  and the uniquely determined integers  $e_i(\mathcal{I})$  for  $i = 0, \dots, d$  are called the *Hilbert coefficients* of  $\mathcal{I}$ . The coefficient  $e_1(\mathcal{I})$  is called the *Chern number* associated with  $\mathcal{I}$ . A *reduction* of an  $I$ -admissible filtration  $\mathcal{I} = \{I_n\}_{n \in \mathbf{Z}}$  is an ideal  $J \subseteq I_1$  such that  $J I_n = I_{n+1}$  for all large  $n$ . Equivalently,  $J \subseteq I_1$  is a reduction of  $\mathcal{I}$  if and only if  $R(\mathcal{I})$  is a finite  $R(J)$ -module. A *minimal reduction* of  $\mathcal{I}$  is a reduction of  $\mathcal{I}$  minimal with respect to containment. Recall that minimal reductions of admissible filtration always exist and are generated by  $d$  elements if  $R/\mathfrak{m}$  is infinite. In this paper we assume that  $R/\mathfrak{m}$  is infinite.

The Chern number has traditionally been studied in Cohen-Macaulay local rings. The recent solutions by Goto et al. [2–4] of conjectures of Vasconcelos [13] for the Chern numbers of parameter ideals and integral closure filtrations require their understanding in arbitrary local rings. Therefore, it is useful to have versions of important formulas for the Chern numbers in general. We have chosen to find a general version of Huneke’s fundamental lemma [7] for this purpose. This lemma has played a crucial role in various studies of the Hilbert polynomial. This version expresses the Chern number in terms of Euler characteristics of a family of subcomplexes of the Koszul complex. Unexpectedly, we are able to apply this formula in dimension one to derive unified and simpler proofs of several results about the Hilbert polynomial in one-dimensional local rings.

The paper is organized as follows. In the first section we establish a formula for the Chern number in any dimension involving the Euler characteristic of subcomplexes of a Koszul complex. In the second section we derive concrete versions of this formula in dimension one and provide unified proofs of results of Rees, Sally, Lipman and Huneke. In the third section we unify several results for the Chern number in local rings of dimension two. Huneke’s fundamental lemma is deduced as a consequence.

**1. A formula for the Chern number  $e_1(\mathcal{I})$ .** In this section we give a formula for the Chern number of an  $I$ -admissible filtration in terms of the Euler characteristics of subcomplexes of a Koszul complex. Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional local ring and  $I$  an  $\mathfrak{m}$ -primary ideal. Let  $\{I_n\}_{n \in \mathbf{Z}}$  be an  $I$ -admissible filtration. Let  $\underline{x} = x_1, \dots, x_d \in I$  be a minimal reduction of  $\mathcal{I}$ . In order to prove the main theorem we define

for  $n \in \mathbf{Z}$  the complex  $K^{(n)}(\underline{x}, \mathcal{I})$  which is a sub-complex of the Koszul complex given by

$$0 \longrightarrow I_{n-d} \longrightarrow (I_{n-d+1})^{\binom{d}{1}} \longrightarrow \cdots \longrightarrow (I_{n-1})^{\binom{d}{d-1}} \longrightarrow I_n \longrightarrow 0.$$

The Euler characteristic of  $K^{(n)}(\underline{x}, \mathcal{I})$  is defined as

$$\chi(K^{(n)}(\underline{x}, \mathcal{I})) = \sum_{i=0}^d (-1)^i \lambda(H_i(K^{(n)}(\underline{x}, \mathcal{I}))).$$

For a numerical function  $f : \mathbf{Z} \rightarrow \mathbf{Z}$ , we put  $(\Delta f)(n) = f(n) - f(n-1)$ .

**Theorem 1.1.** *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional local ring. Let  $I$  be an  $\mathfrak{m}$ -primary ideal and  $\mathcal{I} = \{I_n\}_{n \in \mathbf{Z}}$  an  $I$ -admissible filtration. Let  $\underline{x} = (x_1, \dots, x_d) \subseteq I$  be a minimal reduction of  $\mathcal{I}$ . Then*

$$e_1(\mathcal{I}) = \sum_{n=1}^{\infty} \chi(K^{(n)}(\underline{x}, \mathcal{I})).$$

*Proof.* Consider for  $n \in \mathbf{Z}$  the modified Koszul complex  $C.(n, \mathcal{I})$  introduced in [6] as

$$0 \longrightarrow R/I_{n-d} \longrightarrow (R/I_{n-d+1})^{\binom{d}{1}} \longrightarrow \cdots \longrightarrow (R/I_n) \longrightarrow 0$$

with  $I_n = R$  for  $n \leq 0$ , and the differentials are induced by the differentials of the Koszul complex  $K. = K.(\underline{x})$ . Then we get

$$\begin{aligned} H_{\mathcal{I}}(n) - \binom{d}{1} H_{\mathcal{I}}(n-1) + \cdots + (-1)^d H_{\mathcal{I}}(n-d) \\ = \sum_{i=0}^d (-1)^i \lambda(H_i(C.(n, \mathcal{I}))). \end{aligned}$$

Notice that  $\Delta^d P_{\mathcal{I}}(n) = e_0(\mathcal{I})$  and

$$\Delta^d H_{\mathcal{I}}(n) = H_{\mathcal{I}}(n) - \binom{d}{1} H_{\mathcal{I}}(n-1) + \cdots + (-1)^d H_{\mathcal{I}}(n-d).$$

Therefore,

$$\Delta^d [P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)] = e_0(\mathcal{I}) - \chi(C.(n, \mathcal{I})).$$

By Serre’s theorem [1, Theorem 4.7.6], we have  $e_0(\mathcal{I}) = \chi(K.)$ . Therefore,

$$\Delta^d [P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)] = \chi(K.) - \chi(C.(n, \mathcal{I})).$$

Consider the exact sequence of complexes

$$(1) \quad 0 \longrightarrow K^{(n)}(\underline{x}, \mathcal{I}) \longrightarrow K.(\underline{x}) \longrightarrow C.(n, \mathcal{I}) \longrightarrow 0.$$

Then from the exact sequence (1) we have

$$\chi(K.) = \chi(K^{(n)}(\underline{x}, \mathcal{I})) + \chi(C.(n, \mathcal{I})).$$

Therefore,

$$(2) \quad \Delta^d [P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)] = \chi(K^{(n)}(\underline{x}, \mathcal{I})).$$

From [5, Proposition 2.9], it follows that

$$e_1(\mathcal{I}) = \sum_{n=1}^{\infty} \Delta^d [P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)] = \sum_{n=1}^{\infty} \chi(K^{(n)}(\underline{x}, \mathcal{I})). \quad \square$$

**2. Applications of the formula in dimension one.** In this section we explicitly write down the formula for the Chern number in one-dimensional local rings as consequences of Theorem 1.1. As applications we unify several results in dimension one. The next result was proved for Cohen-Macaulay local rings in [6].

**Theorem 2.1.** *Let  $(R, \mathfrak{m})$  be a one-dimensional local ring,  $I$  an  $\mathfrak{m}$ -primary ideal and  $\mathcal{I} = \{I_n\}_{n \in \mathbf{Z}}$  an  $I$ -admissible filtration. Let  $J = (x) \subseteq I_1$  be a minimal reduction. Then*

$$e_1(\mathcal{I}) = \sum_{n=1}^{\infty} [\lambda(I_n/JI_{n-1}) - \lambda((0 : x) \cap I_{n-1})].$$

In particular, if  $R$  is Cohen-Macaulay then

$$e_1(\mathcal{I}) = \sum_{n=1}^{\infty} \lambda(I_n/JI_{n-1}).$$

*Proof.* From Theorem 1.1, we have

$$e_1(\mathcal{I}) = \sum_{n=1}^{\infty} \chi(K^{(n)}(\mathcal{I}))$$

where the complex  $K^{(n)}(\mathcal{I})$  is given as

$$0 \longrightarrow I_{n-1} \xrightarrow{x} I_n \longrightarrow 0.$$

Note that

$$H_0(K^{(n)}(\mathcal{I})) = I_n/JI_{n-1}$$

and

$$H_1((K^{(n)}(\mathcal{I})) = (0 : x) \cap I_{n-1}.$$

Hence,

$$e_1(\mathcal{I}) = \sum_{n=1}^{\infty} [\lambda(I_n/JI_{n-1}) - \lambda((0 : x) \cap I_{n-1})].$$

If  $R$  is Cohen-Macaulay, then  $x$  is a regular element and so  $(0 : x) = 0$ . Hence, we have

$$e_1(\mathcal{I}) = \sum_{n=1}^{\infty} \lambda(I_n/JI_{n-1}). \quad \square$$

The next result is a deep theorem by Rees valid in any dimension. The proof given here in dimension one for admissible filtrations is new and straightforward.

**Theorem 2.2** [11]. *Let  $(R, \mathfrak{m})$  be a one-dimensional local ring. Let  $I$  be an  $\mathfrak{m}$ -primary ideal and  $\mathcal{I} = \{I_n\}_{n \in \mathbb{Z}}$  an  $I$ -admissible filtration. Let  $J \subseteq I_1$  be a parameter ideal such that  $e_0(J) = e_0(\mathcal{I})$ . Then  $J$  is a reduction of  $\mathcal{I}$ .*

*Proof.* Let  $J = (x) \subseteq I_1$ . Consider the complex

$$C.(n, J, \mathcal{I}) : 0 \longrightarrow I_{n-1}/J^{n-1} \xrightarrow{x} I_n/J^n \longrightarrow 0.$$

Let  $H_{\mathcal{I}/J}(n) = \lambda(I_n/J^n)$  be the Hilbert function and  $P_{\mathcal{I}/J}(n)$  denote the corresponding Hilbert polynomial. Note that

$$\begin{aligned} \Delta [P_{\mathcal{I}/J}(n) - H_{\mathcal{I}/J}(n)] \\ = e_0(\mathcal{I}/J) - \lambda(H_0(C.(n, J, \mathcal{I}))) + \lambda(H_1(C.(n, J, \mathcal{I}))). \end{aligned}$$

Since  $e_0(\mathcal{I}) = e_0(J)$ ,  $e_0(\mathcal{I}/J) = 0$ . Hence, for large  $n$ , we have

$$\lambda(H_1(C.(n, J, \mathcal{I}))) - \lambda(H_0(C.(n, J, \mathcal{I}))) = 0.$$

Notice that

$$H_0(C.(n, J, \mathcal{I})) = \frac{I_n}{JI_{n-1}}$$

and

$$H_1(C.(n, J, \mathcal{I})) = \frac{(J^n : x) \cap I_{n-1}}{J^{n-1}}.$$

Observe that  $(J^n : x) = J^{n-1} + (0 : x)$ . Hence, we have

$$\begin{aligned} \frac{(J^n : x) \cap I_{n-1}}{J^{n-1}} &= \frac{J^{n-1} + (0 : x) \cap I_{n-1}}{J^{n-1}} \\ &= \frac{(0 : x) \cap I_{n-1}}{(0 : x) \cap J^{n-1}}. \end{aligned}$$

By the Artin-Rees lemma,

$$(0 : x) \cap I_n \subseteq H_{\mathfrak{m}}^0(R) \cap I_n = I_{n-n_0}(I_{n_0} \cap H_{\mathfrak{m}}^0(R))$$

for some  $n_0$  and for all  $n \geq n_0$ . Since  $\lambda(H_{\mathfrak{m}}^0(R)) < \infty$ , for large  $n$  we have

$$I_{n-n_0}(I_{n_0} \cap H_{\mathfrak{m}}^0(R)) = 0.$$

Hence, for large  $n$ ,  $I_n = JI_{n-1}$ . Thus,  $J$  is a reduction of  $\mathcal{I}$ .  $\square$

**Theorem 2.3** [8, Theorem 1.9]. *Let  $(R, \mathfrak{m})$  be a one-dimensional Cohen-Macaulay ring,  $I$  an  $\mathfrak{m}$ -primary ideal and  $\mathcal{I} = \{I_n\}_{n \in \mathbf{Z}}$  an  $I$ -admissible filtration. Let  $J = (x)$  be a minimal reduction of  $\mathcal{I}$ . Then, for all  $n \geq 1$ ,*

$$\lambda(I_{n-1}/I_n) \leq e_0(\mathcal{I})$$

and

$$\lambda(I_{n-1}/I_n) = e_0(\mathcal{I}) \text{ if and only if } I_n = xI_{n-1}.$$

*Proof.* From equation (2), we have

$$\Delta[P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)] = \lambda(I_n/JI_{n-1}),$$

which implies

$$e_0(\mathcal{I}) - \lambda(R/I_n) + \lambda(R/I_{n-1}) = \lambda(I_n/JI_{n-1}).$$

Thus, we have  $\lambda(I_{n-1}/I_n) + \lambda(I_n/JI_{n-1}) = e_0(\mathcal{I})$ . Hence,  $\lambda(I_{n-1}/I_n) \leq e_0(\mathcal{I})$  for  $n \geq 1$ , and equality holds if and only if  $I_n = JI_{n-1}$ .  $\square$

**Theorem 2.4** [7, Theorem 2.1]. *Let  $(R, \mathfrak{m})$  be a one-dimensional Cohen-Macaulay ring,  $I$  an  $\mathfrak{m}$ -primary ideal and  $\mathcal{I} = \{I_n\}_{n \in \mathbf{Z}}$  an  $I$ -admissible filtration. Assume that*

$$e_1(\mathcal{I}) = e_0(\mathcal{I}) - \lambda(R/I_1).$$

Then

- (1)  $H_{\mathcal{I}}(n) = P_{\mathcal{I}}(n)$  for all  $n \geq 1$ .
- (2) If  $(x)$  is a minimal reduction of  $\mathcal{I}$ , then  $I_n = xI_{n-1}$  for all  $n \geq 2$ .

*Proof.* From Theorem 2.1, we have  $e_1(\mathcal{I}) = \sum_{n \geq 1} \lambda(I_n/JI_{n-1})$ . Since

$$e_1(\mathcal{I}) = e_0(\mathcal{I}) - \lambda(R/I_1) = \lambda(R/J) - \lambda(R/I_1) = \lambda(I_1/J),$$

we get  $\sum_{n \geq 2} \lambda(I_n/JI_{n-1}) = 0$ . Hence,  $I_n = JI_{n-1}$  for all  $n \geq 2$ . From equation (2), we have

$$\Delta[P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)] = 0$$

for  $n \geq 2$ . Hence,

$$P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n) = P_{\mathcal{I}}(n-1) - H_{\mathcal{I}}(n-1).$$

Notice that

$$P_{\mathcal{I}}(1) - H_{\mathcal{I}}(1) = e_0(\mathcal{I}) - e_1(\mathcal{I}) - \lambda(R/I_1) = 0.$$

Hence,  $H_{\mathcal{I}}(n) = P_{\mathcal{I}}(n)$  for all  $n \geq 1$ .  $\square$

**Theorem 2.5** [12, Proposition 3.2]. *Let  $(R, \mathfrak{m})$  be a one-dimensional Cohen-Macaulay ring and  $I$  an  $\mathfrak{m}$ -primary ideal. Assume that*

$$e_1(I) = e_0(I) - \lambda(R/I) + 1.$$

*Then*

- (1)  $H_I(n) = P_I(n)$  for all  $n > 1$ .
- (2) If  $(x)$  is a minimal reduction of  $I$ , then  $\lambda(I^2/xI) = 1$  and  $I^3 = xI^2$ .

*Proof.* From Theorem 2.1, we have  $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/JI^{n-1})$ . Since

$$e_1(I) = e_0(I) - \lambda(R/I) + 1 = \lambda(R/J) - \lambda(R/I) + 1 = \lambda(I/J) + 1,$$

we get

$$\sum_{n \geq 2} \lambda(I^n/JI^{n-1}) = 1 \quad \text{and} \quad \lambda(I^2/JI) = 1 \quad \text{and} \quad I^3 = JI^2.$$



From equation (2), we have  $\Delta[P_I(n) - H_I(n)] = 0$  for all  $n \geq 3$  and  $\lambda(I^2/xI) = 1$ . Note that

$$P_I(n) - H_I(n) = P_I(n - 1) - H_I(n - 1)$$

for all  $n \geq 3$  and

$$\begin{aligned} P_I(2) - H_I(2) &= P_I(1) - H_I(1) + 1 \\ &= e_0(I) - e_1(I) - \lambda(R/I) + 1 = 0. \end{aligned}$$

Hence,  $H_I(n) = P_I(n)$  for all  $n > 1$ .  $\square$

**3. Applications of the formula in dimension two.** In this section we derive the formula for the Chern number in dimension two as a consequence of Theorem 1.1 and as an application derive Huneke’s fundamental lemma.

**Proposition 3.1.** *Let  $(R, \mathfrak{m})$  be a two-dimensional local ring, and let  $J = (x, y)$  be a parameter ideal. Let  $\mathcal{J} = \{J_n\}$  be a  $J$ -admissible filtration with  $\text{depth } G(\mathcal{J}) \geq 1$  and  $x^* \in J_1/J_2$  be a nonzerodivisor in  $G(\mathcal{J})$ . Let  $K^{(n)}(\mathcal{J})$  be the subcomplex of the Koszul complex  $K(x, y)$  defined as:*

$$0 \longrightarrow J_{n-2} \xrightarrow{f_n = \begin{pmatrix} y & x \end{pmatrix}} (J_{n-1})^2 \xrightarrow{g_n = \begin{pmatrix} -x \\ y \end{pmatrix}} J_n \longrightarrow 0.$$

Then

$$H_1(K^{(n)}(\mathcal{J})) \cong \frac{(x : y) \cap J_{n-1}}{(x)J_{n-2}}.$$

*Proof.* In order to calculate the first homology of the above complex, we calculate  $\ker g_n$  and image  $f_n$ .

$$\begin{aligned} \ker g_n &= \{(a, b) \in (J_{n-1})^2 \mid -ax + by = 0\} \\ &= \{(a, b) \in (J_{n-1})^2 \mid ax = by\}. \end{aligned}$$

Observe that  $\ker g_n \cong (x : y) \cap J_{n-1}$  through the map  $(a, b) \mapsto b$ . Indeed, as  $x$  is a regular element, for each  $b \in (x : y)$ , there exists

a unique  $a \in R$  such that  $ax = by$ , and since  $x^*$  is  $G(\mathcal{J})$ -regular  $J_n : x = J_{n-1}$  for all  $n \geq 2$ , so  $a \in J_{n-1}$ . Thus,  $\ker g_n \cong (x : y) \cap J_{n-1}$ . Notice that  $\text{image } f_n = \{cy, cx \mid c \in J_{n-2}\}$ . Since  $\text{image } f_n \subseteq \ker g_n$ , we have  $\text{image } f_n \cong (x)J_{n-2}$ . Hence,

$$H_1(K^{(n)}(\mathcal{J})) \cong \frac{(x : y) \cap J_{n-1}}{(x)J_{n-2}}. \quad \square$$

**Theorem 3.2.** *Let  $(R, \mathfrak{m})$  be a two-dimensional local ring and  $J = (x, y)$  a parameter ideal. Let  $\mathcal{J} = \{J_n\}$  be a  $J$ -admissible filtration,  $\text{depth } G(\mathcal{J}) \geq 1$  and  $x^* \in J_1/J_2$  a nonzerodivisor in  $G(\mathcal{J})$ . Then*

$$e_1(\mathcal{J}) = \sum_{n \geq 1} \left[ \lambda \left( \frac{J_n}{JJ_{n-1}} \right) - \lambda \left( \frac{(x : y) \cap J_{n-1}}{(x)J_{n-2}} \right) \right].$$

*Proof.* By Theorem 1.1 we have

$$e_1(\mathcal{J}) = \sum_{n \geq 1} \chi(K^{(n)}(\mathcal{J})).$$

Notice that

$$H_0(K^{(n)}(\mathcal{J})) = J_n/JJ_{n-1}$$

and

$$H_2(K^{(n)}(\mathcal{J})) = (0 : J) \cap J_{n-2}.$$

Since  $\text{depth } G(\mathcal{J}) \geq 1$  and  $x^*$  is a nonzerodivisor so  $x$  is a nonzerodivisor, and hence  $H_2(K) = 0$ . Therefore,  $H_2(K^{(n)}(\mathcal{J})) = 0$  for all  $n$ . By Proposition 3.1, we have

$$H_1(K^{(n)}(\mathcal{J})) \cong \frac{(x : y) \cap J_{n-1}}{(x)J_{n-2}}.$$

Thus,

$$e_1(\mathcal{J}) = \sum_{n \geq 1} \left[ \lambda \left( \frac{J_n}{JJ_{n-1}} \right) - \lambda \left( \frac{(x : y) \cap J_{n-1}}{(x)J_{n-2}} \right) \right]. \quad \square$$

**Corollary 3.3.** *Let  $(R, \mathfrak{m})$  be a two-dimensional analytically unramified local ring and  $J = (x, y)$  a parameter ideal. Then*

$$\bar{e}_1(J) = \sum_{n \geq 1} \left[ \lambda \left( \frac{\overline{J^n}}{J \overline{J^{n-1}}} \right) - \lambda \left( \frac{(x : y) \cap \overline{J^{n-1}}}{(x) \overline{J^{n-2}}} \right) \right]$$

where  $\overline{J}$  denotes the integral closure of  $J$ .

*Proof.* Since  $R$  is analytically unramified,  $\mathcal{J} = \{\overline{J^n}\}$  is a  $J$ -admissible filtration and, by [9, Theorem 3.25],  $\text{depth } G(\mathcal{F}) \geq 1$ . Hence, by Theorem 3.2, we are done.  $\square$

Next we derive Huneke’s fundamental lemma for  $I$ -admissible filtrations. For this purpose, we need the following lemma.

**Lemma 3.4** [5, Lemma 3.2]. *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional local ring. Let  $I$  be an  $\mathfrak{m}$ -primary ideal and  $\mathcal{I} = \{I_n\}_{n \in \mathbf{Z}}$  an  $I$ -admissible filtration. Let  $x_1, \dots, x_d$  be elements of  $I_1$ . Consider the complex  $C.(n, \mathcal{I})$*

$$0 \longrightarrow R/I_{n-d} \longrightarrow (R/I_{n-d+1})^{(d)} \longrightarrow \dots \longrightarrow R/I_n \longrightarrow 0.$$

Then, for  $n \in \mathbf{Z}$ , the following hold:

- (1)  $H_d(C.(n, \mathcal{I})) = (I_{n-d+1} : (x_1, \dots, x_d))/I_{n-d}$ .
- (2)  $H_0(C.(n, \mathcal{I})) = R/(I_n + (x_1, \dots, x_d))$ .
- (3) If  $x_1, \dots, x_d$  is an  $R$ -regular sequence, then

$$H_1(C.(n, \mathcal{I})) \cong \frac{(x_1, \dots, x_d) \cap I_n}{(x_1, \dots, x_d)I_{n-1}}.$$

**Proposition 3.5** [7, Fundamental lemma 2.4]. *Let  $(R, \mathfrak{m})$  be a two-dimensional Cohen-Macaulay local ring,  $I$  an  $\mathfrak{m}$ -primary ideal and  $\mathcal{I} = \{I_n\}_{n \in \mathbf{Z}}$  an  $I$ -admissible filtration. Let  $J = (x, y)$  be a minimal reduction of  $\mathcal{I}$ . Then, for all  $n \geq 2$ ,*

$$(1) \Delta^2[P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)] = \lambda \left( \frac{I_n}{JI_{n-1}} \right) - \lambda \left( \frac{I_{n-1} : J}{I_{n-2}} \right).$$

$$(2) e_1(\mathcal{I}) = e_0(\mathcal{I}) - \lambda(R/I_1) + \sum_{n \geq 2} \left[ \lambda\left(\frac{I_n}{JI_{n-1}}\right) - \lambda\left(\frac{I_{n-1} : J}{I_{n-2}}\right) \right].$$

*Proof.* From equation (2), we have

$$\Delta^2[P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)] = \chi(K^{(n)}(\mathcal{I}))$$

where  $K^{(n)}(\mathcal{I})$  is the complex

$$0 \longrightarrow I_{n-2} \longrightarrow (I_{n-1})^2 \longrightarrow I_n \longrightarrow 0.$$

Note that  $H_0(K^{(n)}(\mathcal{I})) = I_n/JI_{n-1}$  and  $H_2(K^{(n)}(\mathcal{I})) = (0 : J) \cap I_{n-2} = 0$  as  $R$  is Cohen-Macaulay. From the exact sequence of complexes (1) we get the long exact sequence

$$\begin{aligned} 0 \longrightarrow H_2(K^{(n)}(\mathcal{I})) &\longrightarrow H_2(K.) \longrightarrow H_2(C.(n, \mathcal{I})) \longrightarrow \\ &\longrightarrow H_1(K^{(n)}(\mathcal{I})) \longrightarrow H_1(K.) \longrightarrow H_1(C.(n, \mathcal{I})) \longrightarrow . \end{aligned}$$

Since  $R$  is Cohen-Macaulay  $H_1(K.) = H_2(K.) = 0$ . Thus,

$$H_1(K^{(n)}(\mathcal{I})) \cong H_2(C.(n, \mathcal{I})) = \frac{I_{n-1} : J}{I_{n-2}} \quad (\text{by Lemma 3.4}).$$

Hence,

$$\Delta^2[P_{\mathcal{I}}(n) - H_{\mathcal{I}}(n)] = \lambda\left(\frac{I_n}{JI_{n-1}}\right) - \lambda\left(\frac{I_{n-1} : J}{I_{n-2}}\right).$$

Now, from Theorem 1.1, we have

$$e_1(\mathcal{I}) = e_0(\mathcal{I}) - \lambda(R/I_1) + \sum_{n \geq 2} \left[ \lambda\left(\frac{I_n}{JI_{n-1}}\right) - \lambda\left(\frac{I_{n-1} : J}{I_{n-2}}\right) \right]. \quad \square$$

### REFERENCES

1. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Revised edition, Cambridge University Press, Cambridge, 1998.
2. L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T.T.Phuong and W. Vasconcelos, *Cohen-Macaulayness versus the vanishing of the first Hilbert coefficient of parameter ideal*, J. Lond. Math. Soc. **81** (2010), 679–695.

3. L. Ghezzi, J. Hong and W. Vasconcelos, *The signature of the Chern coefficients of local rings*, Math. Res. Lett **16** (2009), 279–289.
4. S. Goto, M. Mandal and J. Hong, *The positivity of the first coefficient of normal Hilbert polynomials*, Proc. Amer. Math. Soc. **139** (2011), 2399–2406.
5. S. Huckaba, *A  $d$ -dimensional extension of a lemma of Huneke’s and formulas for the Hilbert coefficient*, Proc. Amer. Math. Soc. **124** (1996), 1393–1401.
6. S. Huckaba and T. Marley, *Hilbert coefficients and the depths of associated graded rings*, J. London Math. Soc. **56** (1997), 64–76.
7. C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987), 293–318.
8. J. Lipman, *Stable ideals and Arf rings*, Amer. J. Math. **93** (1971), 649–685.
9. T. Marley, *Hilbert functions of ideals in Cohen-Macaulay rings*, Ph.D. Thesis, Purdue University, Lafayette, 1989.
10. D. Rees, *A note on analytically unramified local rings*, J. London Math. Soc. **36** (1961), 24–28.
11. ———,  *$A$ -transforms of local rings and a theorem on multiplicities of ideals*, Proc. Cambridge Philo. Soc. **57** (1961), 8–17.
12. J.D. Sally, *Hilbert coefficients and reduction number 2*, J. Algebraic Geom **1** (1992), 325–333.
13. W. Vasconcelos, *The Chern coefficients of local rings*, Michigan Math. J. **57** (2008), 725–743.

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